An equivalent form of fundamental triangle inequality and its applications

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Abstract: An equivalent form of the fundamental triangle inequality is given. The result is then used to obtain an improvement of the Leuenberger's inequality and a new proof of Garfunkel-Bankoff inequality.

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1 Introduction and main results

In what follows, we denote by A, B, C the angles of triangle ABC, let a, b, c denote the lengths of its corresponding sides, and let s, R and r denote respectively the semi-perimeter, circumradius and inradius of a triangle. We will customarily use the symbol of cyclic sum such as

$$\sum f(a) = f(a) + f(b) + f(c), \quad \sum f(a,b) = f(a,b) + f(b,c) + f(c,a).$$

The fundamental triangle inequality is one of the cornerstones of geometric inequalities for triangle. It reads as follows:

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}. \tag{1}$$

The equality holds in left (or right) inequality of (1) if and only if the triangle is isosceles.

As is well known, the inequality (1) is a necessary and sufficient condition for the existence of a triangle with elements R, r and s. This classical inequality has many important applications in the theory of geometric inequality and has received a lot of attention. There exist a large number of papers that have been written about applying the inequality (1) to establish and prove the geometric inequalities for triangle, e.g., see [1-9] and the references cited in them.

The objective of this paper is to give an equivalent form of the fundamental triangle inequality. As applications, we shall apply our results to a new proof of Garfunkel-Bankoff inequality and the improvement of the Leuenberger's inequality, it will be shown that our new inequality can efficaciously reduce computational complexity in the proof of certain inequalities for triangle. We state the main result in the following theorem:

Theorem 1. *For any triangle ABC the following inequalities hold true:*

$$\frac{1}{4}\delta(4-\delta)^3 \leqslant \frac{s^2}{R^2} \leqslant \frac{1}{4}(2-\delta)(2+\delta)^3, \tag{2}$$

where $\delta = 1 - \sqrt{1-\frac{2r}{R}}$*. Furthermore, the equality holds in left (or right) inequality of (2) if and only if the triangle is isosceles.*

2 Proof of Theorem 1

We write the fundamental triangle inequality (1) as

$$2 + \frac{10r}{R} - \frac{r^2}{R^2} - 2\left(1-\frac{2r}{R}\right)\sqrt{1-\frac{2r}{R}} \leqslant \frac{s^2}{R^2} \leqslant 2 + \frac{10r}{R} - \frac{r^2}{R^2} + 2\left(1-\frac{2r}{R}\right)\sqrt{1-\frac{2r}{R}}. \tag{3}$$

By the Euler's inequality $R \geqslant 2r$ (see [1]), we observe that

$$0 \leqslant 1 - \frac{2r}{R} < 1.$$

Let

$$\sqrt{1-\frac{2r}{R}} = 1-\delta, \quad 0 < \delta \leqslant 1. \tag{4}$$

Also, the identity (4) is equivalent to the following identity:

$$\frac{r}{R} = \delta - \frac{1}{2}\delta^2. \tag{5}$$

By applying identities (4) and (5) to inequality (3), we obtain that

$$\begin{aligned} 2 + 10\left(\delta - \frac{1}{2}\delta^2\right) - \left(\delta - \frac{1}{2}\delta^2\right)^2 - 2\left[1 - 2\left(\delta - \frac{1}{2}\delta^2\right)\right](1-\delta) \leqslant \frac{s^2}{R^2} \\ \leqslant 2 + 10\left(\delta - \frac{1}{2}\delta^2\right) - \left(\delta - \frac{1}{2}\delta^2\right)^2 + 2\left[1 - 2\left(\delta - \frac{1}{2}\delta^2\right)\right](1-\delta), \end{aligned}$$

that is

$$16\delta - 12\delta^2 + 3\delta^3 - \frac{1}{4}\delta^4 \leqslant \frac{s^2}{R^2} \leqslant 4 + 4\delta - \delta^3 - \frac{1}{4}\delta^4.$$

After factoring out common factors, the above inequality can be transformed into the desired inequality (2). This completes the proof of Theorem 1.

3 Application to new proof of Garfunkel-Bankoff inequality

Theorem 2. *Let A, B, C be angles of an arbitrary triangle, then we have the inequality*

$$\sum \tan^2 \frac{A}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{6}$$

The equality holds in (6) if and only if the triangle ABC is equilateral.

Inequality (6) was proposed by Garfunkel as a conjecture in [10], it was first proved by Bankoff in [11]. In this section, we give a simplified proof of Garfunkel-Bankoff inequality by means of the equivalent form of fundamental triangle inequality.

Proof. From the identities for triangle (see [2]):

$$\sum \tan^2 \frac{A}{2} = \frac{(4R+r)^2}{s^2} - 2, \tag{7}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}, \tag{8}$$

it is easy to see that the Garfunkel-Bankoff inequality is equivalent to the following inequality:

$$\left(4 - \frac{2r}{R}\right)\frac{s^2}{R^2} - \left(4 + \frac{r}{R}\right)^2 \leqslant 0. \tag{9}$$

By applying inequality (2) and identity (5) to inequality (9), it follows that

$$\begin{aligned}
\left(4 - \frac{2r}{R}\right)\frac{s^2}{R^2} - \left(4 + \frac{r}{R}\right)^2 &\leqslant \frac{1}{4}(4 - 2\delta + \delta^2)(2-\delta)(2+\delta)^3 - \frac{1}{4}(4-\delta)^2(2+\delta)^2 \\
&= -\frac{1}{4}\delta^2(2+\delta)^2(1-\delta)^2 \\
&\leqslant 0,
\end{aligned}$$

which implies the required inequality (9). The Garfunkel-Bankoff inequality is proved.

4 Application to the improvement of Leuenberger's inequality

In 1960, Leuenberger presented the following inequality concerning the sides and the circumradius of a triangle (see [1])

$$\sum \frac{1}{a} \geqslant \frac{\sqrt{3}}{R}. \tag{10}$$

After three years, Steining sharpened inequality (10) in the following form (see also [1]):

$$\sum \frac{1}{a} \geqslant \frac{3\sqrt{3}}{2(R+r)}. \tag{11}$$

Mitrinović et al [2, p.173] showed another sharpened form of (10), as follows

$$\sum \frac{1}{a} \geqslant \frac{5R - r}{2R^2 + (3\sqrt{3}-4)Rr}. \tag{12}$$

Recently, a unified improvement of the inequalities (11) and (12) was given by Wu [12], that is,

$$\sum \frac{1}{a} \geqslant \frac{11\sqrt{3}}{5R + 12r + k_0(2r - R)}. \tag{13}$$

where $k_0 = 0.02206078402\cdots$, it is the root on the interval $(0, \frac{1}{15})$ of the following equation

$$405k^5 + 6705k^4 + 129586k^3 + 1050976k^2 + 2795373k - 62181 = 0.$$

We show here a new improvement of the inequalities (11) and (12), which is stated in the Theorem 3 below.

Theorem 3. *For any triangle ABC the following inequality holds true:*

$$\sum \frac{1}{a} \geqslant \frac{\sqrt{25Rr - 2r^2}}{4Rr}, \tag{14}$$

with equality holding if and only if the triangle ABC is equilateral.

Proof. By using the identities for triangle (see [2]):

$$\sum ab = s^2 + 4Rr + r^2, \quad abc = 4sRr, \tag{15}$$

and the identity (5), we have

$$\begin{aligned} \left(\sum \frac{1}{a}\right)^2 - \frac{25Rr - 2r^2}{16R^2r^2} &= \frac{(s^2 + 4Rr + r^2)^2}{16s^2R^2r^2} - \frac{25Rr - 2r^2}{16R^2r^2}, \\ &= \frac{1}{16R^6s^2r^2}\left[\frac{s^4}{R^4} - \left(\frac{17r}{R} - \frac{4r^2}{R^2}\right)\frac{s^2}{R^2} + \left(\frac{4r}{R} + \frac{r^2}{R^2}\right)^2\right] \end{aligned}$$

$$= \frac{1}{16R^6s^2r^2}\left[\frac{s^4}{R^4} - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right)\frac{s^2}{R^2} + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2\right]. \tag{16}$$

Note that the inequality (2):

$$\frac{s^2}{R^2} \geqslant \frac{1}{4}\delta(4-\delta)^3$$

and

$$\frac{1}{4}\delta(4-\delta)^3 - \frac{1}{2}\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right) = \delta(\frac{15}{2} - \frac{23}{4}\delta) + \delta^3 + \frac{1}{4}\delta^4 > 0,$$

then, from the monotonicity of the function $f(x) = x^2 - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right)x + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2$, we deduce that

$$\begin{aligned} &\frac{s^4}{R^4} - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right)\frac{s^2}{R^2} + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2 \\ \geqslant\ & \frac{1}{16}\delta^2(4-\delta)^6 - \frac{1}{4}\delta(4-\delta)^3\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right) + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2 \\ =\ & \frac{1}{8}(4-\delta)^2(1-\delta)^2(6-\delta)\delta^3 \\ \geqslant\ & 0. \end{aligned}$$

The above inequality with the identity (16) lead us to that

$$\left(\sum \frac{1}{a}\right)^2 - \frac{25Rr - 2r^2}{16R^2r^2} \geqslant 0.$$

The Theorem 3 is thus proved.

Remark. The inequality (14) is stronger than the inequalities (10), (11) and (12) because from the Euler's inequality $R \geqslant 2r$ it is easy to verify that the following inequalities hold for any triangle.

$$\sum \frac{1}{a} \geqslant \frac{\sqrt{25Rr-2r^2}}{4Rr} \geqslant \frac{5R-r}{2R^2+(3\sqrt{3}-4)Rr} \geqslant \frac{\sqrt{3}}{R}, \tag{17}$$

$$\sum \frac{1}{a} \geqslant \frac{\sqrt{25Rr-2r^2}}{4Rr} \geqslant \frac{3\sqrt{3}}{2(R+r)} \geqslant \frac{\sqrt{3}}{R}. \tag{18}$$

In addition, it is worth noticing that the inequality (13) and the inequality (14) are incomparable in general, which can be observed by the following fact.

Letting $a=\sqrt{3},\ b=1,\ c=1$, then $R=1,\ r=\sqrt{3}-\frac{3}{2}$, direct calculating gives

$$\frac{\sqrt{25Rr-2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R+12r+k_0(2r-R)} > \frac{\sqrt{25Rr-2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R+12r+0.023(2r-R)} = 0.11934\cdots > 0.$$

Letting $a=2,\ b=\sqrt{2},\ c=\sqrt{2}$, then $R=1,\ r=\sqrt{2}-1$, direct calculating gives

$$\frac{\sqrt{25Rr-2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R+12r+k_0(2r-R)} < \frac{\sqrt{25Rr-2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R+12r+0.022(2r-R)} = -0.00183\cdots < 0.$$

As a further improvement of the inequality (14), we propose the following significant problem.

Open Problem. *Determine the best constant k for which the inequality below holds*

$$\sum \frac{1}{a} \geqslant \frac{1}{4Rr}\sqrt{25Rr-2r^2+k\left(1-\frac{2r}{R}\right)\frac{r^3}{R}}. \tag{19}$$

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