THE FUNCTION $(b^x - a^x)/x$: LOGARITHMIC CONVEXITY

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Abstract. In the article, logarithmically convex properties of the function $\frac{b^x-a^x}{x}$ for $x \ne 0$ and positive numbers a and b are investigated, several equalities and inequalities of Bernoulli's numbers and their generalizations are established, and some related problems and applications are introduced.

This article provides supplements to the papers [46, 47].

1. Introduction

1.1. For $t > 0$ and $\lambda \in \mathbb{R}$, let

$$f(t) = \frac{1}{t^2} - \frac{e^{-t}}{(1-e^{-t})^2}, \quad h(t) = \frac{1}{t} - \frac{1}{e^t - 1} \quad \text{and} \quad F_\lambda(t) = \frac{t}{e^{\lambda t} - e^{(\lambda-1)t}}. \tag{1}$$

Several references such as [5, 9, 12, 15, 25, 43, 50], [7, p. 217] and [14, p. 295] have been devoted to studying inequalities, monotonicity, logarithmic convexity, history, background, origin and applications of these three functions.

In [15, 25], the function $f(t)$ was proved to be strictly decreasing. In [15], the function $F_\lambda(t)$ was proved to be logarithmically concave and, when $\lambda \ge \frac{1}{2}$, to be decreasing. The properties of $h(t)$ were applied in [12, 43] to study completely monotonic properties of remainders of Binet's formula and the psi function.

These three functions have the following relationships

$$f(t) = h'(t) \quad \text{and} \quad h(t) = [\ln F_\lambda(t)]' + \lambda \tag{2}$$

which were not aware of before [9] seemingly, in which many known properties of these functions were extended and some new results such as 3-log-convex and 3-log-concave properties of $F_\lambda(t)$, which means $[\ln f(t)]''' \ge 0$ and $[\ln f(t)]''' \le 0$, were presented by using the power series expansion of e^t at $t = 0$ and the well-known Hermite-Hadamard's integral inequality [45, 49] for convex functions.

1.2. For real numbers a and b with $b > a$, let

$$F_{a,b}(t) = \begin{cases} \dfrac{t}{e^{bt} - e^{at}}, & t \ne 0; \\ \dfrac{1}{b-a}, & t = 0. \end{cases} \tag{3}$$

It is easy to see that

$$F_\lambda(t) = F_{\lambda-1,\lambda}(t). \tag{4}$$

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In [9, 10, 16, 17, 18, 19, 20, 21, 39], the function $F_{\ln a,\ln b}(t)$ had been used to generalize Bernoulli's and Euler's numbers and polynomials.

The first aim of this paper is to research properties of the function $F_{a,b}(t)$ simply and elegantly, from which most properties of $h(t)$, $f(t)$ and $F_\lambda(t)$ obtained in [5, 7, 12, 15, 25, 50] can be deduced easily and straightforwardly.

Theorem 1. *For real numbers a and b with $b>a$, the function $F_{a,b}(t)$ is 3-log-concave on $(-\infty,0)$, 3-log-convex on $(0,\infty)$, logarithmic concave on $(-\infty,\infty)$, and the function*

$$H_{a,b}(t)=\begin{cases}\dfrac1t-\dfrac{be^{bt}-ae^{at}}{e^{bt}-e^{at}}, & t\ne0\\[2mm] -\dfrac{a+b}{2}, & t=0\end{cases}\tag{5}$$

is decreasing on $(-\infty,\infty)$ with

$$\lim_{t\to-\infty}H_{a,b}(t)=-a\quad\text{and}\quad\lim_{t\to\infty}H_{a,b}(t)=-b.\tag{6}$$

Remark 1. The 3-log-convex properties of $F_{a,b}(t)$ in Theorem 1 can be applied to prove simply and elegantly the logarithmic convexity of extended mean values $E(r,s;x,y)$. For detailed information, please refer to [28].

Remark 2. From Theorem 1, it is easy to deduce that the function $F_\lambda(t)$ for $\lambda\in\mathbb{R}$ is logarithmically concave in $t\in(0,\infty)$. This provides an alternative and elegant proof for all results in [15] and related references therein.

1.3. For positive numbers x and y with $y>x$, set

$$g_{x,y}(t)=\int_x^y u^{t-1}\,\mathrm{d}u=\begin{cases}\dfrac{y^t-x^t}{t}, & t\ne0;\\ \ln y-\ln x, & t=0.\end{cases}\tag{7}$$

It is clear that

$$F_{a,b}(t)=\frac{1}{g_{e^b,e^a}(t)}\quad\text{and}\quad g_{x,y}(t)=\frac{1}{F_{\ln x,\ln y}(t)}.\tag{8}$$

In [4, 6, 8, 27, 29, 30, 32, 33, 35, 36, 37, 40, 41, 42, 44, 47, 48, 51] and related references therein, the function $g_{x,y}(t)$ were used to investigate some properties of extended mean values $E(r,s;x,y)$ and other things.

From relationships in (8) and Theorem 1, some properties of $g_{x,y}(t)$ can be deduced directly as follows.

Theorem 2. *For positive numbers x and y with $y>x$, the function $g_{x,y}(t)$ is 3-log-convex on $(-\infty,0)$, 3-log-concave on $(0,\infty)$, logarithmic convex on $(-\infty,\infty)$, and the function*

$$h_{x,y}(t)=\begin{cases}\dfrac{y^t\ln y-x^t\ln x}{y^t-x^t}-\dfrac1t, & t\ne0\\ \ln\sqrt{xy}\,, & t=0\end{cases}\tag{9}$$

is increasing on $(-\infty,\infty)$ with

$$\lim_{t\to-\infty}h_{x,y}(t)=\ln x\quad\text{and}\quad\lim_{t\to\infty}h_{x,y}(t)=\ln y.\tag{10}$$

For completeness, a simple and elegant proof of Theorem 2, which is slightly different from that of Theorem 1, will be also given in next section.

Remark 3. The 3-log-convex properties of $F_{a,b}(t)$ in Theorem 2 can be applied to verify directly and self-containedly the Schur-convexity and the logarithmic convexity of extended mean values $E(r,s;x,y)$. For more information, please refer to [23, 26] and related references therein.

Remark 4. It is worthwhile to remark that the monotonicity and the logarithmic convexity of a special case of the ratio of $F_{a,b}(t)$ or $g_{x,y}(t)$ had been investigated in [22, 31, 34, 41] and related references therein.

Remark 5. By the way, it is remarked that some inequalities for the exponential function e^x were established in [24] and that some inequalities for several power-exponential functions were presented in [11, 13, 38] and the references therein.

1.4. It is well-known that Bernoulli's numbers B_n can be defined [2] as

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty \frac{B_n}{n!}x^n, \quad |x|<2\pi. \tag{11}$$

In [10, 16, 17, 20, 39], Bernoulli's numbers B_n were generalized as $B_n(a,b)$ for positive numbers a and b by

$$\frac{x}{b^x-a^x}=\sum_{n=0}^\infty \frac{B_n(a,b)}{n!}x^n, \quad |x|<\frac{2\pi}{\ln b-\ln a}. \tag{12}$$

It is clear that $B_n(1,e)=B_n$.

The third aim of this paper is to construct some inequalities and equalities of Bernoulli's numbers and their generalizations.

Theorem 3. *Let n and i be two nonnegative integers with $n\ge i$. Then*

$$\frac{1}{2^{n-2}}\sum_{k=0}^n \binom{n}{k}B_k(e^a,e^b)B_{n-k}(e^a,e^b)=\sum_{k=0}^n \binom{n}{k}B_k(e^{a/2},e^{b/2})B_{n-k}(e^{a/2},e^{b/2}) \tag{13}$$

and

$$B_i(e^a,e^b)B_{n-i}(e^a,e^b)\le \frac{1}{4}\sum_{k=0}^n \binom{n}{k}B_k(e^{a/2},e^{b/2})B_{n-k}(e^{a/2},e^{b/2}). \tag{14}$$

2. Proofs of theorems

Proof of Theorem 1. Straightforward calculation gives

$$\begin{aligned}
\ln F_{a,b}(t)&=\ln|t|-\ln\left|e^{bt}-e^{at}\right|,\\
[\ln F_{a,b}(t)]'&=\frac{1}{t}-\frac{be^{bt}-ae^{at}}{e^{bt}-e^{at}},\\
[\ln F_{a,b}(t)]''&=\frac{(a-b)^2e^{(a+b)t}}{(e^{at}-e^{bt})^2}-\frac{1}{t^2}\\
&=\left(\frac{a-b}{e^{at}-e^{bt}}\right)^2\left[e^{(a+b)t}-\left(\frac{e^{at}-e^{bt}}{at-bt}\right)^2\right].
\end{aligned} \tag{15}$$

The well-known Hermite-Hadamard's integral inequality [45, 49] states that if $f(x)$ is a convex function on the closed interval $[a,b]$ then

$$0\le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-f\left(\frac{a+b}{2}\right)\le \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t. \tag{16}$$

The left hand side inequality in (16) applying to $f(t)=e^t$ implies that the factor in the bracket of (15), and so the second derivative $[\ln F_{a,b}(t)]''$, is non-positive in $t\in\mathbb{R}$ for real numbers a and b with $b>a$. Hence, the function $F_{a,b}(t)$ is logarithmically concave and the function $[\ln F_{a,b}(t)]'$ is decreasing on $\mathbb{R}$ in $t\in\mathbb{R}$ for all real numbers a and b with $b>a$.

Further calculating yields

$$
\begin{aligned}
[\ln F_{a,b}(t)]''' &= \frac{2}{t^3}-\frac{(a-b)^3e^{(a+b)t}(e^{at}+e^{bt})}{(e^{at}-e^{bt})^3}\\
&= \frac{2}{t^3}\biggl(\frac{at-bt}{e^{at}-e^{bt}}\biggr)^3\biggl[\biggl(\frac{e^{at}-e^{bt}}{at-bt}\biggr)^3-\frac{e^{(a+b)t}(e^{at}+e^{bt})}{2}\biggr]\\
&= \frac{2e^{3(a+b)t/2}}{t^3}\biggl(\frac{at-bt}{e^{at}-e^{bt}}\biggr)^3\biggl\{\biggl[\frac{e^{(a-b)t/2}-e^{(b-a)t/2}}{(a-b)t}\biggr]^3-\frac{e^{(a-b)t/2}+e^{(b-a)t/2}}{2}\biggr\}\\
&\triangleq \frac{2e^{3(a+b)t/2}}{t^3}\biggl(\frac{at-bt}{e^{at}-e^{bt}}\biggr)^3 Q\biggl(\frac{a-b}{2}t\biggr).
\end{aligned}
$$

Lazarević's inequality in [14, p. 300] and [3, p. 131] tells us that

$$
Q(t)=\biggl(\frac{e^t-e^{-t}}{2t}\biggr)^3-\frac{e^{-t}+e^t}{2}=\biggl(\frac{\sinh t}{t}\biggr)^3-\cosh t>0 \tag{17}
$$

for $t\in\mathbb{R}$ with $t\ne0$. Hence $[\ln F_{a,b}(t)]'''>0$ in $(0,\infty)$ and $[\ln F_{a,b}(t)]'''<0$ in $(-\infty,0)$.

From

$$
\frac{be^{bt}-ae^{at}}{e^{bt}-e^{at}}=\frac{be^{(b-a)t}-a}{e^{(b-a)t}-1}=\frac{b-ae^{(a-b)t}}{1-e^{(a-b)t}},
$$

it follows easily that

$$
\lim_{t\to-\infty}[\ln F_{a,b}(t)]'=-a \quad\text{and}\quad \lim_{t\to\infty}[\ln F_{a,b}(t)]'=-b.
$$

L'Hôspital's rule gives

$$
\begin{aligned}
\lim_{t\to0}\{[\ln F_{a,b}(t)]'\} &= \lim_{t\to0}\frac{e^{bt}-e^{at}-t(be^{bt}-ae^{at})}{t(e^{bt}-e^{at})}\\
&= \lim_{t\to0}\frac{a^2e^{at}-b^2e^{bt}}{(be^{bt}-ae^{at})+(e^{bt}-e^{at})/t}\\
&= -\frac{a+b}{2}.
\end{aligned}
$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. For $t\ne0$, taking the logarithm of $g_{x,y}(t)$ and differentiating straightforwardly yields

$$
\begin{aligned}
\ln g_{x,y}(t) &= \ln\left|y^t-x^t\right|-\ln|t|,\\
[\ln g_{x,y}(t)]' &= \frac{y^t\ln y-x^t\ln x}{y^t-x^t}-\frac{1}{t},\\
[\ln g_{x,y}(t)]'' &= \frac{1}{t^2}-\frac{x^ty^t(\ln x-\ln y)^2}{(x^t-y^t)^2},\\
[\ln g_{x,y}(t)]''' &= \frac{x^ty^t(x^t+y^t)(\ln x-\ln y)^3}{(x^t-y^t)^3}-\frac{2}{t^3}
\end{aligned}
$$

$$= \frac{2(xy)^{3t/2}}{t^3}\left(\frac{t\ln x - t\ln y}{x^t - y^t}\right)^3\left\{\frac{(x/y)^{t/2}+(y/x)^{t/2}}{2} - \left[\frac{(x/y)^{t/2}-(y/x)^{t/2}}{(\ln x - \ln y)t}\right]^3\right\}$$

$$\triangleq \frac{2(xy)^{3t/2}}{t^3}\left(\frac{t\ln x - t\ln y}{x^t - y^t}\right)^3 Q_{x,y}(t)$$

and

$$Q_{x,y}\left(\frac{2t}{\ln x - \ln y}\right) = \frac{e^{-t}+e^t}{2} - \left(\frac{e^t - e^{-t}}{2t}\right)^3 = \cosh t - \left(\frac{\sinh t}{t}\right)^3 < 0$$

by using Lazarević's inequality in [3, p. 131] and [14, p. 300]. Consequently,

$$[\ln g_{x,y}(t)]''' = \left[\frac{g'_{x,y}(t)}{g_{x,y}(t)}\right]'' \begin{cases} \ge 0 & t \in (-\infty, 0) \\ \le 0 & t \in (0, \infty) \end{cases} \tag{18}$$

and so the function $[\ln g_{x,y}(t)]''$ is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$. Since

$$\lim_{t\to-\infty}\frac{x^t y^t}{(x^t - y^t)^2} = \lim_{t\to-\infty}\frac{(x/y)^t}{[(x/y)^t - 1]^2} = \lim_{t\to-\infty}\frac{(y/x)^t}{[(y/x)^t - 1]^2} = 0$$

and the function $[\ln g_{x,y}(t)]''$ is even on $\mathbb{R}$, then $[\ln g_{x,y}(t)]'' > 0$, and then the function $[\ln g_{x,y}(t)]'$ is increasing on $\mathbb{R}$. Since

$$\frac{y^t\ln y - x^t \ln x}{y^t - x^t} = \frac{(y/x)^t \ln y - \ln x}{(y/x)^t - 1} = \frac{\ln y - (x/y)^t \ln x}{1 - (x/y)^t},$$

then it follows easily that

$$\lim_{t\to-\infty}[\ln g_{x,y}(t)]' = \ln x \quad \text{and} \quad \lim_{t\to\infty}[\ln g_{x,y}(t)]' = \ln y.$$

L'Hôspital's rule reveals that

$$\begin{aligned}\lim_{t\to0}\{[\ln g_{x,y}(t)]'\} &= \lim_{t\to0}\frac{t(y^t\ln y - x^t\ln x) - (y^t - x^t)}{t(y^t - x^t)} \\ &= \lim_{t\to0}\frac{y^t(\ln y)^2 - x^t(\ln x)^2}{(y^t - x^t)/t + (y^t\ln y - x^t \ln x)} \\ &= \frac{\ln y + \ln x}{2}.\end{aligned}$$

The proof of Theorem 2 is complete. □

Remark 6. In the proof of Theorem 2, Hermite-Hadamard's integral inequality (16) for convex functions are not used.

Remark 7. Now it is natural to ponder what about the monotonicity or convexity of the function $[g_{x,y}(t)]'''$. By the noted software MATHEMATICS 5.2, the curves of three functions

$$[\ln g_{2,9}(t)]''', \quad [\ln g_{0.2,9}(t)]''' \quad \text{and} \quad [\ln g_{0.2,0.9}(t)]'''$$

are plotted. Their shapes of these three curves look like Figure 1. This shows that the function $g_{x,y}(t)$ is not 4-log-convex or 4-log-concave on any one of three intervals $(-\infty, 0)$, $(0, \infty)$ and $(-\infty, \infty)$, although it is completely monotonic on at least one of three intervals $(-\infty, 0)$, $(0, \infty)$ and $(-\infty, \infty)$ under some special cases on x and y, as shown in [47].

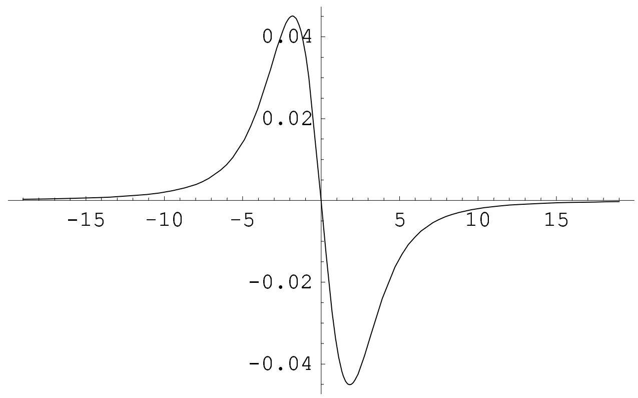

FIGURE 1. The curve of the function $[\ln g_{2,9}(t)]'''$

Proof of Theorem 3. By Theorem 1, the function $F_{a,b}(t)$ is logarithmically concave in $(-\infty,\infty)$, which means that

$$
\begin{gathered}
F_{a,b}(s)F_{a,b}(t) \le F_{a,b}^2\left(\frac{s+t}{2}\right),\\
\left(\frac{s}{e^{bs}-e^{as}}\right)\left(\frac{t}{e^{bt}-e^{at}}\right) \le \left[\frac{(s+t)/2}{e^{b(s+t)/2}-e^{a(s+t)/2}}\right]^2,\\
\left[\sum_{n=0}^{\infty}\frac{B_n(e^a,e^b)}{n!}s^n\right]\left[\sum_{n=0}^{\infty}\frac{B_n(e^a,e^b)}{n!}t^n\right] \le \left[\sum_{n=0}^{\infty}\frac{B_n(e^a,e^b)}{n!}\left(\frac{s+t}{2}\right)^n\right]^2\\
= \frac{1}{4}\left[\sum_{n=0}^{\infty}\frac{B_n(e^{a/2},e^{b/2})}{n!}(s+t)^n\right]^2,\\
\sum_{n=0}^{\infty}\sum_{i+j=n}\frac{B_i(e^a,e^b)B_j(e^a,e^b)}{i!j!}s^it^j \le \sum_{n=0}^{\infty}\sum_{i+j=n}\frac{B_i(e^a,e^b)B_j(e^a,e^b)}{i!j!}\left(\frac{s+t}{2}\right)^n\\
= \frac{1}{4}\sum_{n=0}^{\infty}\sum_{i+j=n}\frac{B_i(e^{a/2},e^{b/2})B_j(e^{a/2},e^{b/2})}{i!j!}(s+t)^n,\\
\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{B_i(e^a,e^b)B_{n-i}(e^a,e^b)}{i!(n-i)!}s^it^{n-i} \le \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{i=0}^{n}\frac{B_i(e^a,e^b)B_{n-i}(e^a,e^b)}{i!(n-i)!}(s+t)^n\\
= \frac{1}{4}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{B_i(e^{a/2},e^{b/2})B_{n-i}(e^{a/2},e^{b/2})}{i!(n-i)!}\sum_{k=0}^{n}\binom{n}{k}s^kt^{n-k}\\
= \frac{1}{4}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{k=0}^{n}\frac{B_k(e^{a/2},e^{b/2})B_{n-k}(e^{a/2},e^{b/2})}{k!(n-k)!}\binom{n}{i}s^it^{n-i}.
\end{gathered}
$$

Equating on both sides of above inequalities and equations yields equality (13) and inequality (14). The proof of Theorem 3 is complete. □

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