# Some Refinements of Discrete Jensen's Inequality and Some of Its Applications

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**Abstract**

In this paper, using some aspects of convex functions, we refine discrete Jensen's inequality via weight functions. Then, using these results, we give some applications in different abstract spaces and obtain some new interesting inequalities.

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## 1 Introduction

Jensen's inequality is sometimes called the king of inequalities [4] because it implies at once the main part of the other classical inequalities (e.g. those by Hölder, Minkowski, Young, and the AGM inequality, etc.). Therefore, it is worth studying it thoroughly and refine it from different points of view. There are numerous refinements of Jensen's inequality, see e.g. [3-5] and the references in them. In this paper, introducing suitable weight functions, first we give some refinements of discrete Jensen's inequality, and then using these refinements, we give several important applications in various abstract spaces, which extends the results obtained recently [5,6].
Throughout this paper, we suppose that $C$ is a convex subset of a real vector space, $x_1, \cdots, x_n \in C$, and $\varphi : C \to \mathbb{R}$ a convex mapping. Also, we suppose that $\mu = (\mu_1, \cdots, \mu_m)$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$ are two probability measures; i.e. $\mu_i,\ \lambda_j \geq 0$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ with

$$\sum_{i=1}^{m} \mu_i = 1 \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j = 1.$$

By a (discrete separately) weight function (with respect to $\mu$ and $\lambda$), we always mean a mapping

$$\omega : \{(i,j) : \ 1 \leq i \leq m,\ 1 \leq j \leq n\} \to [0, \infty),$$

such that

$$\sum_{i=1}^{m} \omega(i,j)\mu_i = 1 \qquad (j = 1, \cdots, n),$$

and

$$\sum_{j=1}^{n} \omega(i,j)\lambda_j = 1 \qquad (i = 1, \cdots, m).$$

For example, if $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_n)$ with $\|u\| = (\sum_{i=1}^{m} u_i^2)^{1/2} \leq 1$ and $\|v\| = (\sum_{j=1}^{n} v_j^2)^{1/2} \leq 1$ belong to $\mu^\perp$ and $\lambda^\perp$ respectively, then the function $\omega$ with $\omega(i,j) = 1 + u_i v_j$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is a weight function.
Also, we say that a quadratic matrix $A = [a_{ij}]_{n\times n}$ with nonnegative entries is a double stochastic matrix if the sum of each of its rows and columns is unit; that is

$$\sum_{i=1}^{n} a_{ij} = 1 \qquad (j = 1, \cdots, n),$$

and

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad (i = 1, \cdots, n).$$

If $\omega_1$ and $\omega_2$ are two weight functions, we denote by $\phi_{\omega_1,\omega_2}$ the real-valued function

$$\phi_{\omega_1,\omega_2}(t) = \sum_{i=1}^{m} \mu_i \varphi\left(\sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j x_j\right) \qquad (0 \leq t \leq 1), \tag{1}$$

and also, if $B = [b_{ij}]_{n\times n}$ and $C = [c_{ij}]_{n\times n}$ are two double stochastic matrices, we put

$$\phi_{B,C}(t) = \frac{1}{n}\sum_{i=1}^{n} \varphi\left(\sum_{j=1}^{n} [(1-t)b_{ij} + tc_{ij}]x_j\right) \qquad (0 \leq t \leq 1). \tag{2}$$

The aim of this paper is to refine discrete Jensen's inequality

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j \varphi(x_j)$$

via weight functions and give some applications in different abstract spaces which extend the results of [5] and [6].

## 2 Refinements

In this section, using the above notations, we refine discrete Jensen's inequality by one and two weight functions which extend the results of [5].

**Lemma 2.1.** *If $\omega$ is a weight function, then*

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{i=1}^{m} \mu_i \varphi\left(\sum_{j=1}^{n} \omega(i,j)\lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j \varphi(x_j). \tag{3}$$

*In particular, if $A = [a_{ij}]_{n\times n}$ is a double stochastic matrix, we have*

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{1}{n}\sum_{i=1}^{n}\varphi\left(\sum_{j=1}^{n}a_{ij}x_j\right) \le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n}. \tag{4}$$

*Proof.* By the convexity of $\varphi$, we have

$$\sum_{i=1}^{m}\mu_i\varphi\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right) \le \sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\omega(i,j)\lambda_j\varphi(x_j)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\mu_i\omega(i,j)\right)\lambda_j\varphi(x_j) = \sum_{j=1}^{n}\lambda_j\varphi(x_j),$$

and

$$\sum_{i=1}^{m}\mu_i\varphi\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right) \ge \varphi\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\omega(i,j)\lambda_j x_j\right)$$

$$= \varphi\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\mu_i\omega(i,j)\right)\lambda_j x_j\right) = \varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right),$$

and (3) follows.
The inequalities in (4) follow from (3) by taking $m = n,\ \mu_i = \lambda_j = \frac{1}{n}$ and $\omega(i,j) = na_{ij}$ $(i,j = 1,\cdots,n)$.

Next, we give a refinement of discrete Jensen's inequality via two weights functions.

**Theorem 2.2.** *If $\omega_1$ and $\omega_2$ are two weight functions, then*

*(i)*

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le \phi_{\omega_1,\omega_2}(t) \le \sum_{j=1}^{n}\lambda_j\varphi(x_j) \qquad (0 \le t \le 1). \tag{5}$$

*(ii) For each $i$, the function*

$$t \longrightarrow \varphi\left(\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j x_j\right) \qquad (0 \le t \le 1),$$

*and so, $\phi_{\omega_1,\omega_2}$ is convex.*

*(iii)*

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le \int_0^1 \phi_{\omega_1,\omega_2}(t)dt \le \sum_{j=1}^{n}\lambda_j\varphi(x_j). \tag{6}$$

*In particular, if $C$ is an interval of $\mathbb{R}$,*

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \leq \sum_{i=1}^{m}\mu_i A\left(\varphi; \sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j\right) \leq \sum_{j=1}^{n}\lambda_j\varphi(x_j), \tag{7}$$

*where the arithmetic mean $A$ is defined for an integrable function $f$ over an interval with end points $a$ and $b$, by*

$$A(f;a,b) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{8}$$

*(We set $A(f;a,a) = f(a)$.)*

*(iv) Let $p_i \geq 0$ with $P_k = \sum_{i=1}^{k} p_i > 0$, and $t_i$ be in $[0,1]$ for all $i = 1, 2, \cdots, k$. Then*

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \leq \phi_{\omega_1,\omega_2}\left(\frac{1}{P_k}\sum_{i=1}^{k}p_i t_i\right) \leq \frac{1}{P_k}\sum_{i=1}^{k}p_i\phi_{\omega_1,\omega_2}(t_i) \leq \sum_{j=1}^{n}\lambda_j\varphi(x_j), \tag{9}$$

*which is a discrete version of Hadamard's result.*
*Proof*

*(i)* Since for each $t$ in $[0,1]$,

$$(i,j) \longrightarrow (1-t)\omega_1(i,j) + t\omega_2(i,j) \qquad (1 \leq i \leq m,\ 1 \leq j \leq n)$$

is a weight function, (5) follows from (3).

*(ii)* Let $\alpha,\ \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2$ be in $[0,1]$. For each $i$ with $1 \leq i \leq m$, we have

$$\begin{aligned}
&\varphi\left(\sum_{j=1}^{n}[(1-\alpha t_1-\beta t_2)\omega_1(i,j) + (\alpha t_1+\beta t_2)\omega_2(i,j)]\lambda_j x_j\right) \\
= \; &\varphi\left(\alpha\sum_{j=1}^{n}[(1-t_1)\omega_1(i,j) + t_1\omega_2(i,j)]\lambda_j x_j + \beta\sum_{j=1}^{n}[(1-t_2)\omega_1(i,j) + t_2\omega_2(i,j)]\lambda_j x_j\right) \\
\leq \; &\alpha\varphi\left(\sum_{j=1}^{n}[(1-t_1)\omega_1(i,j) + t_1\omega_2(i,j)]\lambda_j x_j\right) + \beta\varphi\left(\sum_{j=1}^{n}[(1-t_2)\omega_1(i,j) + t_2\omega_2(i,j)]\lambda_j x_j\right),
\end{aligned}$$

and (ii) follows.

*(iii)* $\phi_{\omega_1,\omega_2}$ being bounded and convex on $[0,1]$ is Riemann integrable on $[0,1]$, and so by integrating, (6) follows from (5).
In particular, if $C$ is an interval of $\mathbb{R}$, then by the change of variables

$$u = \sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j x_j,$$

we have

$$\begin{aligned}\int_0^1 \phi_{\omega_1,\omega_2}(t)dt &= \sum_{i=1}^m \mu_i \int_0^1 \varphi\left(\sum_{j=1}^n [(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j x_j\right)dt \\ &= \sum_{i=1}^m \mu_i A\left(\varphi; \sum_{j=1}^n \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^n \omega_2(i,j)\lambda_j x_j\right),\end{aligned}$$

which by substituting it in (6), we get (7).

($iv$) The first and the third inequalities in (9) are obvious from (5), and the second inequality follows from Jensen's inequality applied for the convex function $\phi_{\omega_1,\omega_2}$.

**Corollary 2.3.** *If $B=[b_{ij}]_{n\times n}$ and $C=[c_{ij}]_{n\times n}$ are two double stochastic matrices, then* [5],

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) \le \phi_{B,C}(t) \le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n} \qquad (0\le t\le 1), \tag{10}$$

*and*

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) \le \int_0^1 \phi_{B,C}(t)dt \le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n}. \tag{11}$$

*In particular, if $C$ is an interval of $\mathbb{R}$, then*

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{1}{n}\sum_{i=1}^n A\left(\varphi; \sum_{j=1}^n b_{ij}x_j, \sum_{j=1}^n c_{ij}x_j\right) \le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n}, \tag{12}$$

*where $A$ is defined by* (8).
*Proof.* Take $m=n,\ \mu_i=\lambda_j=\frac{1}{n},\ \omega_1(i,j)=nb_{ij}$ and $\omega_2(i,j)=nc_{ij}$ $(i,j=1,\cdots,n)$ in (5), (6) and (7).

# 3 Applications

Throughout this section, we use the terminologies and results of the above sections, and as before, we suppose that $\omega_1$ and $\omega_2$ are two weight functions, and $B=[b_{ij}]_{n\times n}$ and $C=[c_{ij}]_{n\times n}$ are two double stochastic matrices.

**Theorem 3.1.** *Let $x_1,x_2,\cdots,x_n$ be $n$ positive numbers. Then, we have*

$$\prod_{j=1}^n x_j^{\lambda_j} \le \prod_{i=1}^m \left[I\left(\sum_{j=1}^n \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^n \omega_2(i,j)\lambda_j x_j\right)\right]^{\mu_i} \le \sum_{j=1}^n \lambda_j x_j, \tag{13}$$

*where the identric mean $I$ is defined for each $a,b>0$ by*

$$I(a,b)=\begin{cases} a & \text{if } a=b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\ne b. \end{cases} \tag{14}$$

*In particular* [5],

$$\sqrt[n]{x_1x_2\cdots x_n} \le \sqrt[n]{\prod_{i=1}^{n} I\left(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j\right)} \le \frac{x_1+x_2+\cdots+x_n}{n}. \tag{15}$$

*Proof.* The function $\varphi : (0,\infty) \to \mathbb{R},\ \varphi(x) = -\ln x$ is convex and $A(\varphi; a, b) = -\ln I(a, b)\ (a, b > 0)$. So, we have

$$\sum_{i=1}^{m} \mu_i A\left(\varphi; \sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right) = -\ln \prod_{i=1}^{m} \left[I\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right)\right]^{\mu_i},$$

which by substituting in (7) and taking into account that

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) = -\ln\left(\sum_{j=1}^{n} \lambda_j x_j\right) \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j \varphi(x_j) = -\ln\left(\prod_{j=1}^{n} x_j^{\lambda_j}\right),$$

we obtain (13).
The inequalities in (15) follow from (13) by taking $m = n,\ \mu_i = \lambda_j = \frac{1}{n},\ \omega_1(i,j) = nb_{ij}$ and $\omega_2(i,j) = nc_{ij}\ (i, j = 1, \cdots, n)$.

**Theorem 3.2.** *If $x_j \in (0, 1/2]$ $(j = 1, \cdots, n)$, and $A_n = \sum_{j=1}^{n} \lambda_j x_j$ and $G_n = \prod_{j=1}^{n} x_j^{\lambda_j}$ (also, $A'_n = \sum_{j=1}^{n} \lambda_j(1 - x_j)$ and $G'_n = \prod_{j=1}^{n}(1 - x_j)^{\lambda_j}$ ) are the arithmetic and geometric means of $x_1, \cdots, x_n$ (of $1 - x_1, \cdots, 1 - x_n$ ) respectively, then we have the following refinement of Ky Fan's inequality $\frac{A'_n}{A_n} \le \frac{G'_n}{G_n}$* [1]*:*

$$\frac{A'_n}{A_n} \le \prod_{i=1}^{m}\left(\frac{I\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j(1-x_j), \sum_{j=1}^{n} \omega_2(i,j)\lambda_j(1-x_j)\right)}{I\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right)}\right)^{\mu_i} \le \frac{G'_n}{G_n}, \tag{16}$$

*where the Identric mean $I$ is defined as in* (14).
*In particular* [6],

$$\frac{A'_n}{A_n} \le \sqrt[n]{\prod_{i=1}^{n}\left(\frac{I\left(\sum_{j=1}^{n} b_{ij}(1-x_j), \sum_{j=1}^{n} c_{ij}(1-x_j)\right)}{I\left(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j\right)}\right)} \le \frac{G'_n}{G_n}. \tag{17}$$

*Proof.* The function $\varphi(x) = \ln\frac{1-x}{x}$ is convex on $(0, 1/2]$, and $A(\varphi; a, b) = \ln\frac{I(1-a,1-b)}{I(a,b)}$ $(0 < a, b < 1)$. So, we have

$$\sum_{i=1}^{m} \mu_i A\left(\varphi; \sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right)$$
$$= \ln \prod_{i=1}^{m}\left(\frac{I\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j(1-x_j), \sum_{j=1}^{n} \omega_2(i,j)\lambda_j(1-x_j)\right)}{I\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right)}\right)^{\mu_i},$$

which by substituting in (7) and taking into account that

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) = \ln\frac{A'_n}{A_n} \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j \varphi(x_j) = \ln\frac{G'_n}{G_n},$$

we get (16).
In particular, (17) follows from (16) by taking $m = n,\ \mu_i = \lambda_j = 1/n,\ \omega_1(i,j) = nb_{ij}$ and $\omega_2(i,j) = nc_{ij}\ (i,j = 1,\cdots,n)$.

**Theorem 3.3.** *If $(X, \mathcal{A}, \mu)$ is a measure space, $p \geq 1$, and $f_1, f_2, \cdots, f_n$ belong to $L^p = L^p(\mu)$, then we have*

$$\left\|\sum_{j=1}^{n}\lambda_j f_j\right\|_p^p \leq \sum_{i=1}^{m}\mu_i\left\|L_p^p\left(\sum_{j=1}^{n}\omega_1(i,j)\lambda_j|f_j|,\ \sum_{j=1}^{n}\omega_2(i,j)\lambda_j|f_j|\right)\right\|_1 \leq \sum_{j=1}^{n}\lambda_j\|f_j\|_p^p, \tag{18}$$

*where the p-logarithmic mean is defined for $a, b \geq 0$, by*

$$L_p(a,b) = \begin{cases} a & \text{if } \ a = b, \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } \ a \neq b. \end{cases} \tag{19}$$

*In particular* [5],

$$\left\|\frac{f_1+\cdots+f_n}{n}\right\|_p^p \leq \frac{1}{n}\sum_{i=1}^{n}\left\|L_p^p\left(\sum_{j=1}^{n}b_{ij}|f_j|,\ \sum_{j=1}^{n}c_{ij}|f_j|\right)\right\|_1 \leq \frac{\|f_1\|_p^p+\cdots+\|f_n\|_p^p}{n}, \tag{20}$$

*and*

$$\left\|\frac{f_1+\cdots+f_n}{n}\right\|_p^p \leq \frac{1}{n}\sum_{i=1}^{n}\left\|L_p^p(|f_i|,\ |f_{n+1-i}|)\right\|_1 \leq \frac{\|f_1\|_p^p+\cdots+\|f_n\|_p^p}{n}. \tag{21}$$

*Proof.* We consider the convex function $\varphi : L^p \to \mathbb{R},\ \varphi(f) = \|f\|_p^p$. Clearly, the function $X\times[0,1] \to \mathbb{R}$ with

$$(x,t) \to \sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x),$$

is product-measurable. Since

$$\left|\sum_{j=1}^{n}\lambda_j f_j\right| \leq \sum_{j=1}^{n}\lambda_j|f_j|,$$

and the $L^p-$norms of $f_j$ and $|f_j|$ are equal $(j = 1,\cdots,n)$, it is sufficient to assume $f_j \geq 0$ $(j = 1,\cdots,n)$. Now, using Fubini's theorem and applying the change of variables

$$u = \sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x),$$

we have

$$\begin{aligned}\int_0^1 \phi_{\omega_1,\omega_2}(t)dt &= \sum_{i=1}^{m}\mu_i\int_0^1\left\|\sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j\right\|_p^p dt \\ &= \sum_{i=1}^{m}\mu_i\int_0^1\int_X\left(\sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)\right)^p d\mu(x)dt\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \mu_i \int_X \int_0^1 \left( \sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x) \right)^p dt d\mu(x) \\
&= \sum_{i=1}^{m} \mu_i \int_X L_p^p \left( \sum_{j=1}^{n} \omega_1(i,j)\lambda_j f_j,\ \sum_{j=1}^{n} \omega_2(i,j)\lambda_j f_j \right) d\mu \\
&= \sum_{i=1}^{m} \mu_i \left\| L_p^p \left( \sum_{j=1}^{n} \omega_1(i,j)\lambda_j f_j,\ \sum_{j=1}^{n} \omega_2(i,j)\lambda_j f_j \right) \right\|_1 ,
\end{aligned}
$$

which by substituting in (6) with $f_j$ instead of $x_j$, it yields (18).
In particular, (20) follows from (18) by taking $m = n,\ \mu_i = \lambda_j = \frac{1}{n},\ \omega_1(i,j) = nb_{ij},\ \omega_2(i,j) = nc_{ij}\ (i,j = 1, \cdots, n)$.
Finally, (21) follows from (20) by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}\ (i,j = 1, \cdots, n)$, where $\delta_{ij}$ is the Kronecker delta.

**Corollary 3.4.** *If $x_1, x_2, \cdots, x_n$ are nonnegative real numbers and $p \geq 1$, then*

$$
\sum_{j=1}^{n} \lambda_j^p x_j^p \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_i L_p^p \left( \omega_1(i,j)\lambda_j x_j,\ \omega_2(i,j)\lambda_j x_j \right) \leq \sum_{j=1}^{n} \lambda_j x_j^p. \tag{22}
$$

*As a consequence, if $p$ is a positive integer, then*

$$
n^{2-p} \leq \frac{1}{p+1} \sum_{i,j=1}^{n} \sum_{k=0}^{p} b_{ij}^k c_{ij}^{p-k} \leq n, \tag{23}
$$

*and*

$$
n^{2-p} \leq \frac{1}{p+1} \left( \sum_{i,j=1}^{n} b_{ij}^p + \sum_{i=1}^{n} \sum_{k=0}^{p-1} b_{ii}^k \right) \leq n. \tag{24}
$$

*Proof.* The inequalities in (22) follow from (18) by taking $X = \{1, 2, \cdots, n\}$ with counting measure and $f_j = x_j \chi_{\{j\}}$, where $\chi_{\{j\}}$ is the characteristic function of $\{j\}$ $(j = 1, 2, \cdots, n)$.
Now, (23) follows from (22) by taking $m = n,\ \mu_i = \lambda_j = \frac{1}{n},\ \omega_1(i,j) = nb_{ij},\ \omega_2(i,j) = nc_{ij},$ $x_j = 1\ (i,j = 1, \cdots, n)$ and expanding the $p$-logarithmic mean.
Finally, (24) follows from (23) by taking $c_{ij} = \delta_{ij}\ (i,j = 1, \cdots, n)$.

**Remark 3.5.** (i) Let $f_n$ be a sequence in $L^p(\mu)$ $(p \geq 1)$ converging with the $L^p$-norm and point-wise to an element $f$ of $L^p(\mu)$. Then, using Fatu's lemma and Cesaro's summability theorem, we have

$$
\|f\|_p^p \leq \liminf_{n\to\infty} \left\| \frac{f_1 + \cdots + f_n}{n} \right\|_p^p \leq \lim_{n\to\infty} \frac{\|f_1\|_p^p + \cdots + \|f_n\|_p^p}{n} = \|f\|_p^p,
$$

and so by (21),

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|L_p^p(|f_i|,\ |f_{n+1-i}|)\|_1 = \|f\|_p^p \qquad (p \geq 1). \tag{25}
$$

(ii) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $\mathcal{M}$ be the vector space of all measurable functions on $X$ with point-wise operations [2]. The set $C$, consisting of all nonnegative measurable functions on

$X$, is a convex subset of $\mathcal{M}$. Since the function $t \to \frac{t}{1+t}$ $(t \geq 0)$ is concave, the mapping $\varphi : C \to \mathbb{R}$ with

$$\varphi(f) = \int_X \frac{f}{1+f} d\mu \qquad (f \in C) \tag{26}$$

is concave.

**Theorem 3.6.** *With the notations of (ii) of Remark 3.5, if $f_1, \cdots, f_n$ belong to $C$ and $\varphi$ is as in* (26)*, then*

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j \varphi(f_j) \\ \leq \;& \mu(X) - \sum_{i=1}^{m} \mu_i \left\| L^{-1}\left(1 + \sum_{j=1}^{n} \omega_1(i,j)\lambda_j f_j,\ 1 + \sum_{j=1}^{n} \omega_2(i,j)\lambda_j f_j\right)\right\|_1 \\ \leq \;& \varphi\left(\sum_{j=1}^{n} \lambda_j f_j\right), \end{aligned} \tag{27}$$

*where the logarithmic mean $L$ is defined for each $a, b > 0$, by*

$$L(a,b) = \begin{cases} a & \text{if } \ a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if } \ a \neq b. \end{cases} \tag{28}$$

*In particular* [5],

$$\begin{aligned} &\frac{\varphi(f_1) + \cdots + \varphi(f_n)}{n} \\ \leq \;& \mu(X) - \frac{1}{n}\sum_{i=1}^{n} \left\| L^{-1}\left(1 + \sum_{j=1}^{n} b_{ij} f_j,\ 1 + \sum_{j=1}^{n} c_{ij} f_j\right)\right\|_1 \\ \leq \;& \varphi\left(\frac{f_1 + \cdots + f_n}{n}\right), \end{aligned} \tag{29}$$

*Proof.* Clearly, the mapping

$$(x,t) \to \sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)$$

on $X \times [0,1]$ is product-measurable. Since $\varphi$ is concave, $-\varphi$ is convex, and so by (6), we have

$$\sum_{j=1}^{n} \lambda_j \varphi(f_j) \leq \int_0^1 \phi_{\omega_1,\omega_2}(t)dt \leq \varphi\left(\sum_{j=1}^{n} \lambda_j f_j\right). \tag{30}$$

But

$$\begin{aligned} &\int_0^1 \phi_{\omega_1,\omega_2}(t)dt \\ = \;& \sum_{i=1}^{m} \mu_i \int_0^1 \int_X \frac{\sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)}{1 + \sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)} d\mu(x)dt \\ = \;& \sum_{i=1}^{m} \mu_i \int_X \int_0^1 \frac{\sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)}{1 + \sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)} dtd\mu(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\mu_i \int_X \frac{1}{\sum_{j=1}^{n}[\omega_2(i,j)-\omega_1(i,j)]\lambda_j f_j(x)} \int_{\sum_{j=1}^{n}\omega_1(i,j)\lambda_j f_j(x)}^{\sum_{j=1}^{n}\omega_2(i,j)\lambda_j f_j(x)} (1-\frac{1}{1+t})dt d\mu(x) \\
&= \mu(X) - \sum_{i=1}^{m}\mu_i \int_X \frac{1}{\sum_{j=1}^{n}[\omega_2(i,j)-\omega_1(i,j)]\lambda_j f_j} \ln \frac{1+\sum_{j=1}^{n}\omega_2(i,j)\lambda_j f_j}{1+\sum_{j=1}^{n}\omega_1(i,j)\lambda_j f_j} d\mu \\
&= \mu(X) - \sum_{i=1}^{m}\mu_i \left\| L^{-1}\left(1+\sum_{j=1}^{n}\omega_1(i,j)\lambda_j f_j,\ 1+\sum_{j=1}^{n}\omega_2(i,j)\lambda_j f_j\right)\right\|_1,
\end{aligned}
$$

which by substituting in (30), we obtain (27).
The inequalities in (29) follow from (27) by taking $m = n,\ \mu_i = \lambda_j = \frac{1}{n},\ \omega_1(i,j) = nb_{ij}$ and $\omega_2(i,j) = nc_{ij}\ (i,j = 1,\cdots,n)$.

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