

BOUNDING THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL VIA A BEESACK INEQUALITY AND APPLICATIONS

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. Lower and upper bounds of the Čebyšev functional for the Riemann-Stieltjes integral are given. Applications for the three point quadrature rules of functions that are n -time differentiable are also provided.

1. INTRODUCTION

In 1975, P.R. Beesack [1] showed that, if y, v, w are real valued functions defined on a compact interval $[a, b]$, where w is of bounded variation with total variation $\bigvee_a^b(w)$, and such that the Riemann-Stieltjes integrals $\int_a^b y(t) dv(t)$ and $\int_a^b w(t) y(t) dv(t)$ both exist, then

$$\begin{aligned}
 (1.1) \quad & m \int_a^b y(t) dv(t) + \bigvee_a^b(w) \cdot \inf_{a \leq \alpha < \beta \leq b} \left[\int_\alpha^\beta y(t) dv(t) \right] \\
 & \leq \int_a^b w(t) y(t) dv(t) \\
 & \leq m \int_a^b y(t) dv(t) + \bigvee_a^b(w) \cdot \sup_{a \leq \alpha < \beta \leq b} \left[\int_\alpha^\beta y(t) dv(t) \right],
 \end{aligned}$$

where $m := \inf_{t \in [a, b]} \{w(t)\}$.

The second of the inequalities above extends a result of R. Darst and H. Pollard [5] who dealt with the case $y(t) = 1$, $t \in [a, b]$ and $v(t)$ continuous on $[a, b]$.

In [6], S.S. Dragomir has introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$\begin{aligned}
 (1.2) \quad T(f, g; u) := & \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) \\
 & - \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t),
 \end{aligned}$$

provided $u(b) \neq u(a)$ and the involved Riemann-Stieltjes integrals exist.

It has been shown in [6] that, if f, g are continuous, $m \leq f(t) \leq M$ for each $t \in [a, b]$ and u is of bounded variation, then the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of integrals, as described

Date: 30 April, 2007.

2000 Mathematics Subject Classification. Primary 26D15, 41A55.

Key words and phrases. Riemann-Stieltjes integral, Čebyšev functional, Integral inequalities, Quadrature rules.

in the definition of the Čebyšev functional (1.2), satisfies the inequality:

$$(1.3) \quad |T(f, g; u)| \leq \frac{1}{2} (M - m) \cdot \frac{1}{|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \bigvee_a^b(u),$$

where the constant $\frac{1}{2}$ is best possible and $\|\cdot\|_{\infty}$ is the sup-norm.

Moreover, if f, g are continuous, $m \leq f(t) \leq M$ for $t \in [a, b]$ and u is monotonic nondecreasing on $[a, b]$, then:

$$(1.4) \quad |T(f, g; u)| \leq \frac{1}{2} (M - m) \frac{1}{|u(b) - u(a)|} \cdot \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t)$$

and the constant $\frac{1}{2}$ here is also sharp.

Finally, if f, g are Riemann integrable and u is Lipschitzian with the constant $L > 0$ then also

$$(1.5) \quad |T(f, g; u)| \leq \frac{1}{2} (M - m) \frac{L}{|u(b) - u(a)|} \cdot \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt.$$

The constant $\frac{1}{2}$ is also best possible in (1.5) (see [7] and [8]).

The main aim of the present paper is to provide other bounds for the Čebyšev functional $T(f, g; u)$ by utilising the Beesack inequality (1.1). Applications for three point quadrature rules of functions that are $(n - 1)$ -differentiable ($n \geq 1$) with the derivative $f^{(n-1)}$ absolutely continuous are given as well.

2. THE RESULTS

The following result may be stated.

Theorem 1. *Let $f, g, u : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t) g(t) du(t)$, $\int_a^b f(t) du(t)$ and $\int_a^b g(t) du(t)$ exist. Then*

$$(2.1) \quad \begin{aligned} & \bigvee_a^b(f) \cdot \inf_{a \leq \alpha < \beta \leq b} \left[\int_{\alpha}^{\beta} g(t) du(t) - \frac{u(\beta) - u(\alpha)}{u(b) - u(a)} \cdot \int_a^b g(s) du(s) \right] \\ & \leq \int_a^b f(t) g(t) du(t) - \frac{1}{u(b) - u(a)} \cdot \int_a^b f(t) du(t) \cdot \int_a^b g(t) du(t) \\ & \leq \bigvee_a^b(f) \cdot \sup_{a \leq \alpha < \beta \leq b} \left[\int_{\alpha}^{\beta} g(t) du(t) - \frac{u(\beta) - u(\alpha)}{u(b) - u(a)} \cdot \int_a^b g(s) du(s) \right], \end{aligned}$$

provided $u(b) \neq u(a)$.

Proof. We observe that the following identity holds true (see also [6])

$$(2.2) \quad [u(b) - u(a)] T(f, g; u) \\ = \int_a^b f(t) \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t).$$

Since f is of bounded variation, it follows that f is bounded below and if we denote by m the infimum of f on $[a, b]$, then on applying the Beesack inequality for the choices

$$w(t) = f(t), \quad y(t) = g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s)$$

and $v(t) = u(t)$, $t \in [a, b]$, we can write that:

$$(2.3) \quad m \int_a^b \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \\ + \bigvee_a^b(f) \cdot \inf_{a \leq \alpha < \beta \leq b} \left\{ \int_\alpha^\beta \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \right\} \\ \leq [u(b) - u(a)] T(f, g; u) \\ \leq m \int_a^b \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \\ + \bigvee_a^b(f) \cdot \sup_{a \leq \alpha < \beta \leq b} \left\{ \int_\alpha^\beta \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \right\}.$$

Since

$$\int_a^b \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) = 0$$

and

$$\int_\alpha^\beta \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \\ = \int_\alpha^\beta g(t) du(t) - \frac{u(\beta) - u(\alpha)}{u(b) - u(a)} \cdot \int_a^b g(s) du(s),$$

hence, by (2.3), we deduce the desired result (2.1). \square

The following corollary for weighted integrals may be stated:

Corollary 1. *Let $f, g, w : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b f(t) g(t) w(t) dt$, $\int_a^b f(t) w(t) dt$ and $\int_a^b g(t) w(t) dt$ exist.*

Then

$$\begin{aligned}
(2.4) \quad & \bigvee_a^b(f) \cdot \inf_{a \leq \alpha < \beta \leq b} \left[\int_{\alpha}^{\beta} g(t) w(t) dt - \frac{\int_{\alpha}^{\beta} w(s) ds}{\int_a^b w(s) ds} \cdot \int_a^b g(t) w(t) dt \right] \\
& \leq \int_a^b f(t) g(t) w(t) dt - \frac{1}{\int_a^b w(s) ds} \cdot \int_a^b f(t) w(t) dt \cdot \int_a^b g(t) w(t) dt \\
& \leq \bigvee_a^b(f) \cdot \sup_{a \leq \alpha < \beta \leq b} \left[\int_{\alpha}^{\beta} g(t) w(t) dt - \frac{\int_{\alpha}^{\beta} w(s) ds}{\int_a^b w(s) ds} \cdot \int_a^b g(t) w(t) dt \right],
\end{aligned}$$

provided $\int_a^b w(s) ds \neq 0$.

Remark 1. For the particular case when $w(t) = 1$, $t \in [a, b]$, then we get from (2.4) the following inequality:

$$\begin{aligned}
(2.5) \quad & \bigvee_a^b(f) \cdot \inf_{a \leq \alpha < \beta \leq b} \left[\int_{\alpha}^{\beta} g(t) dt - \frac{\beta - \alpha}{b - a} \cdot \int_a^b g(t) dt \right] \\
& \leq \int_a^b f(t) g(t) dt - \frac{1}{b - a} \int_a^b f(t) dt \cdot \int_a^b g(t) dt \\
& \leq \bigvee_a^b(f) \cdot \sup_{a \leq \alpha < \beta \leq b} \left[\int_{\alpha}^{\beta} g(t) dt - \frac{\beta - \alpha}{b - a} \cdot \int_a^b g(t) dt \right],
\end{aligned}$$

provided f is of bounded variation and the involved Riemann integrals exist.

3. APPLICATIONS FOR THREE POINT QUADRATURES

Recall that in [4] (see also [9, p. 223]) P. Cerone and S.S. Dragomir established the following identity concerning a three point quadrature rule for n -time differentiable functions $f : [a, b] \rightarrow \mathbb{R}$:

$$\begin{aligned}
(3.1) \quad \int_a^b f(t) dt &= \sum_{k=1}^n \frac{1}{k!} \left\{ (1 - \gamma)^k \left[(b - x)^k + (-1)^{k-1} (x - a)^k \right] f^{(k-1)}(x) \right. \\
&\quad \left. + \gamma^k \left[(x - a)^k f^{(k-1)}(a) + (-1)^{k-1} (b - x)^k f^{(k-1)}(b) \right] \right\} \\
&\quad + (-1)^n \int_a^b C_n(x, t) f^{(n)}(t) dt,
\end{aligned}$$

where the Peano kernel is given by:

$$(3.2) \quad C_n(x, t) := \begin{cases} \frac{[t - (\gamma x + (1 - \gamma)a)]^n}{n!} & \text{if } t \in [a, x], \\ \frac{[t - (\gamma x + (1 - \gamma)b)]^n}{n!} & \text{if } t \in (x, b], \end{cases}$$

and $\gamma \in [0, 1]$, $x \in [a, b]$.

We note that the above representation generalised the interior point quadrature rule obtained in 1999 by Cerone et al. in [2] for $\gamma = 0$ and the trapezoid type rule obtained in 2000 by Cerone et al. in [3] for $\gamma = 1$.

The function $C_n(x, \cdot)$ is of bounded variation for each fixed $x \in [a, b]$ and a simple calculation reveals that

$$\begin{aligned}
 (3.3) \quad & \bigvee_a^b ((-1)^n C_n(x, \cdot)) \\
 &= \int_a^x \left| \frac{dC_n(x, t)}{dt} \right| dt + \int_x^b \left| \frac{dC_n(x, t)}{dt} \right| dt \\
 &= \int_a^x \frac{|t - (\gamma x + (1 - \gamma)a)|^{n-1}}{(n-1)!} dt + \int_x^b \frac{|\gamma x + (1 - \gamma)b - t|^{n-1}}{(n-1)!} dt \\
 &= \frac{1}{n!} (x-a)^n [\gamma^n + (1-\gamma)^n] + \frac{1}{n!} (b-x)^n [\gamma^n + (1-\gamma)^n] \\
 &= \frac{1}{n!} [\gamma^n + (1-\gamma)^n] [(b-x)^n + (x-a)^n]
 \end{aligned}$$

for any $x \in [a, b]$.

Also,

$$\begin{aligned}
 (3.4) \quad & \int_a^b C_n(x, t) dt \\
 &= \frac{1}{n!} \int_a^x [t - (\gamma x + (1 - \gamma)a)]^n dt + \frac{1}{n!} \int_x^b [t - (\gamma x + (1 - \gamma)b)]^n dt \\
 &= \frac{1}{(n+1)!} \left\{ [x - (\gamma x + (1 - \gamma)a)]^{n+1} - [a - (\gamma x + (1 - \gamma)a)]^{n+1} \right. \\
 &\quad \left. + [b - (\gamma x + (1 - \gamma)b)]^{n+1} - [x - (\gamma x + (1 - \gamma)b)]^{n+1} \right\} \\
 &= \frac{1}{(n+1)!} \left\{ (1 - \gamma)^{n+1} (x-a)^{n+1} - (-1)^{n+1} \gamma^{n+1} (x-a)^{n+1} \right. \\
 &\quad \left. + \gamma^{n+1} (b-x)^{n+1} - (-1)^{n+1} (1 - \gamma)^{n+1} (b-x)^{n+1} \right\} \\
 &= \frac{1}{(n+1)!} \left\{ (b-x)^{n+1} [\gamma^{n+1} + (-1)^n (1 - \gamma)^{n+1}] \right. \\
 &\quad \left. + (-1)^n [\gamma^{n+1} + (-1)^n (1 - \gamma)^{n+1}] (x-a)^{n+1} \right\} \\
 &= \frac{1}{(n+1)!} [(b-x)^{n+1} + (-1)^n (x-a)^{n+1}] [\gamma^{n+1} + (-1)^n (1 - \gamma)^{n+1}]
 \end{aligned}$$

for any $x \in [a, b]$.

We can state the following result:

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be an $(n - 1)$ -differentiable function ($n \geq 1$) with the derivative $f^{(n-1)}$ absolutely continuous on $[a, b]$. Then we have

$$(3.5) \quad \int_a^b f(t) dt = \sum_{k=1}^n \frac{1}{k!} \left\{ (1-\gamma)^k \left[(b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \right. \\ \left. + \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \right\} \\ + \frac{1}{(n+1)!} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right] \\ \times \left[(-1)^n \gamma^{n+1} + (1-\gamma)^{n+1} \right] + E_n(f, x, \gamma; a, b),$$

where the remainder $E_n(f, x, \gamma; a, b)$ (which is defined implicitly by (3.5)) satisfies the bounds:

$$(3.6) \quad \frac{1}{n!} [\gamma^n + (1-\gamma)^n] [(b-x)^n + (x-a)^n] \inf_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)] \\ \leq E_n(f, x, \gamma; a, b) \\ \leq \frac{1}{n!} [\gamma^n + (1-\gamma)^n] [(b-x)^n + (x-a)^n] \sup_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)]$$

and

$$(3.7) \quad \delta_n(f; \alpha, \beta) = f^{(n-1)}(\beta) - f^{(n-1)}(\alpha) - \frac{\beta - \alpha}{b - a} \left[f^{(n-1)}(b) - f^{(n-1)}(a) \right],$$

where $\gamma \in [0, 1]$ and $x \in [a, b]$.

Proof. Apply the inequality (2.5) for the functions $f = (-1)^n C_n(x, \cdot)$ and $g = f^{(n)}$ to get

$$(3.8) \quad \frac{1}{n!} [\gamma^n + (1-\gamma)^n] [(b-x)^n + (x-a)^n] \inf_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)] \\ \leq (-1)^n \int_a^b C_n(x, t) f^{(n)}(t) dt - \frac{1}{b-a} (-1)^n \int_a^b C_n(x, t) dt \cdot \int_a^b f^{(n)}(t) dt \\ \leq \frac{1}{n!} [\gamma^n + (1-\gamma)^n] [(b-x)^n + (x-a)^n] \sup_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)].$$

Since, by (3.3)

$$\int_a^b ((-1)^n C_n(x, \cdot)) = \frac{1}{n!} [\gamma^n + (1-\gamma)^n] [(b-x)^n + (x-a)^n]$$

and by (3.4)

$$(-1)^n \int_a^b C_n(x, t) dt \\ = \frac{1}{(n+1)!} \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right] \left[(-1)^n \gamma^{n+1} + (1-\gamma)^{n+1} \right],$$

then, on utilising the inequality (3.8), we have

$$\begin{aligned}
 (3.9) \quad & \frac{1}{n!} [\gamma^n + (1 - \gamma)^n] [(b - x)^n + (x - a)^n] \inf_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)] \\
 & \leq (-1)^n \int_a^b C_n(x, t) f^{(n)}(t) dt \\
 & - \frac{1}{(n+1)!} [(b - x)^{n+1} + (-1)^n (x - a)^{n+1}] [(-1)^n \gamma^{n+1} + (1 - \gamma)^{n+1}] \\
 & \quad \times \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a} \right] \\
 & \leq \frac{1}{n!} [\gamma^n + (1 - \gamma)^n] [(b - x)^n + (x - a)^n] \sup_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)].
 \end{aligned}$$

Now, due to the fact that, by the representation (3.1) we have

$$\begin{aligned}
 (3.10) \quad & (-1)^n \int_a^b C_n(x, t) f^{(n)}(t) dt \\
 & = \int_a^b f(t) dt - \sum_{k=1}^n \frac{1}{k!} \left\{ (1 - \gamma)^k [(b - x)^k + (-1)^{k-1} (x - a)^k] f^{(k-1)}(x) \right. \\
 & \quad \left. + \gamma^k [(x - a)^k f^{(k-1)}(a) + (-1)^{k-1} (b - x)^k f^{(k-1)}(b)] \right\}
 \end{aligned}$$

then, on making use of remainder's representation $E_n(f, x, \gamma; a, b)$ (which is defined implicitly by (3.5)), we deduce from (3.9) the desired result (3.6). \square

Remark 2. For $\gamma = 0$, we get from Theorem 2:

$$\begin{aligned}
 (3.11) \quad & \int_a^b f(t) dt = \sum_{k=1}^n \frac{1}{k!} [(b - x)^k + (-1)^{k-1} (x - a)^k] f^{(k-1)}(x) \\
 & + \frac{1}{(n+1)!} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a} \right] [(b - x)^{n+1} + (-1)^n (x - a)^{n+1}] \\
 & \quad + F_n(f, x; a, b),
 \end{aligned}$$

where the remainder satisfies the bounds

$$\begin{aligned}
 (3.12) \quad & \frac{1}{n!} [(b - x)^n + (x - a)^n] \inf_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)] \\
 & \leq F_n(f, x; a, b) \\
 & \leq \frac{1}{n!} [(b - x)^n + (x - a)^n] \sup_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)]
 \end{aligned}$$

for $x \in [a, b]$.

For $\gamma = \frac{1}{2}$, we get from Theorem 2 that:

$$(3.13) \quad \int_a^b f(t) dt = \sum_{k=1}^n \frac{1}{2^k k!} \left\{ \left[(b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \right. \\ \left. + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \right\} + \frac{[1 + (-1)^n]}{2^{n+1} (n+1)!} \\ \times \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right] \\ + G_n(f, x; a, b),$$

where the remainder satisfies the inequality:

$$(3.14) \quad \frac{1}{2^{n-1} n!} [(b-x)^n + (x-a)^n] \inf_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)] \\ \leq G_n(f, x; a, b) \\ \leq \frac{1}{2^{n-1} n!} [(b-x)^n + (x-a)^n] \sup_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)],$$

for $x \in [a, b]$.

Finally, for $\gamma = 1$, we obtain from Theorem 2 that:

$$(3.15) \quad \int_a^b f(t) dt = \sum_{k=1}^n \frac{1}{k!} \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \\ + \frac{(-1)^n}{(n+1)!} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right] \\ + H_n(f, x; a, b)$$

where the remainder $H_n(f, x; a, b)$ satisfies the bounds:

$$(3.16) \quad \frac{1}{n!} [(b-x)^n + (x-a)^n] \inf_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)] \\ \leq H_n(f, x; a, b) \\ \leq \frac{1}{n!} [(b-x)^n + (x-a)^n] \sup_{a \leq \alpha < \beta \leq b} [\delta_n(f; \alpha, \beta)]$$

for $x \in [a, b]$.

The following particular case may be useful in applications:

If $n = 1$ and $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous function on $[a, b]$ then we have the representation:

$$(3.17) \quad \int_a^b f(t) dt = (1-\gamma)(b-a)f(x) + \gamma[(x-a)f(a) + (b-x)f(b)] \\ + [f(b) - f(a)] \left(\frac{a+b}{2} - x \right) (1-2\gamma) + E(f, x, \gamma; a, b)$$

and the remainder $E(f, x, \gamma; a, b)$ satisfies the bounds

$$(3.18) \quad (b-a) \inf_{a \leq \alpha < \beta \leq b} [\delta(f; \alpha, \beta)] \leq E(f, x, \gamma; a, b) \leq (b-a) \sup_{a \leq \alpha < \beta \leq b} [\delta(f; \alpha, \beta)]$$

where

$$\delta(f; \alpha, \beta) := f(\beta) - f(\alpha) - \frac{\beta - \alpha}{b - a} [f(b) - f(a)],$$

and $x \in [a, b]$ while $\gamma \in [0, 1]$.

One must observe that for $n = 1$ the bounds for the error are independent of x and γ . However, this quality is not inherited for the quadrature rules with $n \geq 2$.

REFERENCES

- [1] P.R. BEESACK, Bounds for Riemann-Stieltjes integrals, *Rocky Mountain J. Math.*, **5**(1) (1975), 75-78.
- [2] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Some Ostrowski type inequalities for n -time differentiable mappings and applications, *Demonstratio Math.*, **32**(2) (1999), 697-712.
- [3] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS and J. SÜNDE, A new generalisation of the trapezoid formula for n -time differentiable mappings and applications, *Demonstratio Math.*, **33**(4) (2000), 719-736.
- [4] P. CERONE and S.S. DRAGOMIR, Three point identities and inequalities for n -time differentiable functions, *SUT J. of Math. (Japan)*, **36**(2) (2000), 351-383.
- [5] R. DARST and H. POLLARD, An inequality for the Riemann-Stieltjes integral, *Proc. Amer. Math. Soc.*, **25** (1970), 912-913.
- [6] S.S. DRAGOMIR, Sharp bounds of Čebyšev functional for Stieltjes integrals and applications, *Bull. Austral. Math. Soc.*, **67** (2003), 257-266.
- [7] S.S. DRAGOMIR, New estimates of the Čebyšev functional for Stieltjes integrals and applications, *J. Korean Math. Soc.*, **41**(2) (2004), 249-264.
- [8] S.S. DRAGOMIR, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.*, **26** (2004), 89-122.
- [9] S.S. DRAGOMIR and Th.M. RASSIAS, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, VIC 8001, AUSTRALIA.

E-mail address: `pietro.cerone@vu.edu.au`

URL: `http://rgmia.vu.edu.au/cerone`

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`