REVERSES OF THE CBS INTEGRAL INEQUALITY IN HILBERT SPACES AND RELATED RESULTS

I. BRNETIĆ, S.S. DRAGOMIR, R. HOXHA, AND J. PEČARIĆ

Abstract. There are many known reverses of the Cauchy-Bunyakovsky-Schwarz (CBS) inequality in the literature. We obtain here a general integral inequality comprising some of those results and also provide other related inequalities. The discrete case, which is of interest in its own turn, is also analysed.

1. Introduction

Throughout the paper $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space over $\mathbb{C}$. Let us consider the Hilbert space $L^2(\Omega, H)$, of all strongly measurable functions f defined on the measurable set $\Omega \subset \mathbb{R}^n$ and with values in H for which the integral $\int_\Omega \|f(t)\|^2 d\mu(t)$ is finite. Here μ is a given positive measure on Ω. Then the following Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality may be stated:

$$\left(\int_\Omega \|f(t)\|^2 d\mu(t) \cdot \int_\Omega \|g(t)\|^2 d\mu(t) \right)^{\frac{1}{2}} \geq \left| \int_\Omega \langle f(t), g(t) \rangle d\mu(t) \right|. \tag{1.1}$$

This CBS inequality has several types of reverses. For a recent survey of these results see [6] and the references therein.

This article is divided in four parts.

In the introduction we recall some known reverses of the CBS inequalities from [1], [3] and [4]. In the second part, we state some known CBS inequalities from [1] and [3] in a more general form. Also, some easier proofs are given and new refinements are stated.

In the third part we obtain two new integral inequalities. One of them is of the Klamkin-McLenaghan type and the other can be compared with the result of S.S. Dragomir from [4].

In the fourth part we state analogous inequalities for the discrete case. These results are generalizations of those obtained in [5].

First, we recall some known reverses of the CBS inequalities that have been obtained previously when Ω was a compact interval in $\mathbb{R}$. Notice that the proofs in obtaining these inequalities do not depend on the domain Ω and therefore all the results can be translated for the more general case when Ω is a measurable set from $\mathbb{R}^n$.

Theorem A (S.S. Dragomir [3], see also [6])**.** *Let f, $g \in L^2_\rho([a,b]; H)$, where $L^2_\rho([a,b]; H)$ denotes the Hilbert space of all strongly measurable functions $h : [a,b] \to$*

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Inner product, Hilbert spaces, Lebesgue measure, Cauchy-Bunyakovski-Schwarz inequality.

H for which the integral $\int_a^b \rho(t)\,\|h(t)\|^2\,dt$ is finite (where $\rho:[a,b]\to[0,\infty)$ is measurable and given) and $c,C\in\mathbb{C}$ with $\operatorname{Re}(C\overline{c})>0$. If

$$\operatorname{Re}\langle Cg(t)-f(t),f(t)-cg(t)\rangle\geq 0 \tag{1.2}$$

for a.e. $t\in[a,b]$, then we have the inequalities

$$\begin{aligned}\left(\int_a^b \rho(t)\,\|f(t)\|^2\,dt\cdot\int_a^b \rho(t)\,\|g(t)\|^2\,dt\right)^{\frac12} &\leq \frac12\cdot\frac{\operatorname{Re}\left((\overline{C}+\overline{c})\cdot\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right)}{(\operatorname{Re}(C\overline{c}))^{\frac12}}\\ &\leq \frac12\cdot\frac{|C+c|}{(\operatorname{Re}(C\overline{c}))^{\frac12}}\cdot\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right|\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}0&\leq\int_a^b \rho(t)\,\|f(t)\|^2\,dt\cdot\int_a^b \rho(t)\,\|g(t)\|^2\,dt-\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right|^2\\ &\leq\frac14\cdot\frac{|C-c|^2}{\operatorname{Re}(C\overline{c})}\cdot\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right|^2.\end{aligned} \tag{1.4}$$

The constants $\frac12$ and $\frac14$ are best possible.

Theorem B (S.S. Dragomir [4], see also [6])**.** *Let $f,\ g\in L^2_\rho([a,b];H)$ and $c,C\in\mathbb{C}$ with $C\neq -c$. If the condition (1.2) is satisfied, then we have the inequalities*

$$\begin{aligned}0&\leq\left(\int_a^b \rho(t)\,\|f\,(t)\|^2\,dt\cdot\int_a^b \rho(t)\,\|g\,(t)\|^2\,dt\right)^{\frac12}\\ &\qquad-\left|\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right|\\ &\leq\left(\int_a^b \rho(t)\,\|f\,(t)\|^2\,dt\cdot\int_a^b \rho(t)\,\|g\,(t)\|^2\,dt\right)^{\frac12}\\ &\qquad-\left|\operatorname{Re}\left(\frac{\overline{C}+\overline{c}}{|C+c|}\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right)\right|\\ &\leq\left(\int_a^b \rho(t)\,\|f\,(t)\|^2\,dt\cdot\int_a^b \rho(t)\,\|g\,(t)\|^2\,dt\right)^{\frac12}\\ &\qquad-\operatorname{Re}\left(\frac{\overline{C}+\overline{c}}{|C+c|}\int_a^b\rho(t)\langle f(t),g(t)\rangle dt\right)\\ &\leq\frac14\cdot\frac{|C-c|^2}{|C+c|}\int_a^b\rho(t)\,\|g\,(t)\|^2\,dt.\end{aligned} \tag{1.5}$$

The constant $\frac14$ is best possible.

Theorem C (S.S. Dragomir [1], see also [6])**.** *Let f, $g \in L^2(\Omega, \mathbb{C})$, where $L^2(\Omega; \mathbb{C})$ denotes the Hilbert space of all strongly measurable functions $h : \Omega \to \mathbb{C}$ for which the integral $\int_\Omega |h(s)|^2 d\mu(s)$ is finite and $c, C \in \mathbb{C}$. If*

$$\int_\Omega \mathrm{Re}\left[(Cg(s) - f(s))(\overline{f(s)} - \overline{c} \cdot \overline{g}(s))\right] d\mu(s) > 0, \tag{1.6}$$

then the following inequality holds

$$\int_\Omega |f(s)|^2 d\mu(s) \cdot \int_\Omega |g(s)|^2 d\mu(s) - \left| \int_\Omega f(s)\overline{g(s)} d\mu(s) \right|^2 \leq \frac{1}{4}|C - c|^2 \cdot \left(\int_\Omega |g(s)|^2 d\mu(s) \right)^2. \tag{1.7}$$

In most of the theorems below we assume that the complex numbers c and C satisfy the following condition:

$$\mathrm{Re}(\overline{c}C) > 0. \tag{1.8}$$

2. Known Reverses of the CBS Inequality Revisited

The important result that we use in all our proofs is the following lemma:

Lemma 1. *Let f and g be vector functions from $L^2(\Omega, H)$ which satisfy the condition*

$$\mathrm{Re}\langle Cg(x) - f(x), f(x) - cg(x) \rangle \geq 0 \tag{2.1}$$

for $\mu - a.e.$ $x \in \Omega$. Then the following inequality holds

$$\begin{aligned} &\int_\Omega \|f(x)\|^2 d\mu(x) + \mathrm{Re}(\overline{c}C) \int_\Omega \|g(x)\|^2 d\mu(x) \\ &\leq \int_\Omega \mathrm{Re}[(\overline{c} + \overline{C})\langle f(x), g(x) \rangle] d\mu(x) \\ &\leq |c + C| \left| \int_\Omega \langle f(x), g(x) \rangle d\mu(x) \right|. \end{aligned} \tag{2.2}$$

Proof. Condition (2.1) is equivalent to the following one

$$\|f(x)\|^2 + \mathrm{Re}(\overline{c}C) \|g(x)\|^2 \leq \mathrm{Re}\left[(\overline{c} + \overline{C})\langle f(x), g(x) \rangle\right]$$

for $\mu - a.e.$ $x \in \Omega$.

By integrating on Ω we obtain

$$\int_\Omega \|f(x)\|^2 d\mu(x) + \mathrm{Re}(\overline{c}C) \int_\Omega \|g(x)\|^2 d\mu(x) \leq \int_\Omega \mathrm{Re}[(\overline{c} + \overline{C})\langle f(x), g(x) \rangle] d\mu(x).$$

From this we obtain the desired inequality (2.2), since for the complex number w and the complex valued function $z(x)$, we have:

$$\begin{aligned} \int_\Omega \mathrm{Re}(w \cdot z(x)) d\mu(x) &= \mathrm{Re}\left(\int_\Omega w \cdot z(x) d\mu(x) \right) = \mathrm{Re}\left(w \int_\Omega z(x) d\mu(x) \right) \\ &\leq \left| w \int_\Omega z(x) d\mu(x) \right| = |w| \left| \int_\Omega z(x) d\mu(x) \right|. \end{aligned}$$

∎

Now we state the result of Theorem A in slightly generalized form. Also, the last inequality from Theorem A is given with the natural refinement.

Theorem 1. *If (1.8) and (2.1) are satisfied, then the following inequalities hold*

$$
\begin{aligned}
(2.3)\qquad &\int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x)\\
&\le \frac{\left(\int_\Omega \operatorname{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{4\operatorname{Re}(\overline{c}C)}\\
&\le \frac{|c+C|^2}{4\operatorname{Re}(\overline{c}C)}\cdot\left|\int_\Omega \langle f(x),g(x)\rangle d\mu(x)\right|^2
\end{aligned}
$$

and

$$
\begin{aligned}
(2.4)\qquad 0 &\le \int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x) - \left|\int_\Omega \langle f(x),g(x)\rangle d\mu(x)\right|^2\\
&\le \int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x)\\
&\qquad - \frac{\left(\int_\Omega \operatorname{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{|c+C|^2}\\
&\le \frac{|C-c|^2}{4\operatorname{Re}(\overline{c}C)}\left|\int_\Omega \langle f(x),g(x)\rangle d\mu(x)\right|^2.
\end{aligned}
$$

Proof. Inequalities (2.3) and (2.4) were essentially established by S.S. Dragomir in [2] and [3]. We give here a simpler proof as well.

Let

$$\alpha = \frac{\int_\Omega \operatorname{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)}{2(\operatorname{Re}(\overline{c}C))^{\frac{1}{2}}}.$$

By multiplying the inequality (2.2) with $\int_\Omega \|g(x)\|^2 d\mu(x)$ we obtain:

$$
\begin{aligned}
&\int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x)\\
&\qquad\le \int_\Omega \operatorname{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\\
&\qquad\qquad\times\int_\Omega \|g(x)\|^2 d\mu(x) - \operatorname{Re}(\overline{c}C)\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x)\\
&\qquad\le \alpha^2 - (\alpha - (\operatorname{Re}(\overline{c}C))^{\frac{1}{2}}\cdot\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2 \le \alpha^2,
\end{aligned}
$$

i.e., we obtained the first part of the inequality (2.3).

Using inequality (2.2), the second part of the inequality (2.3) follows immediately.

The second chain of inequalities is easy to prove. Of course, the first inequality in (2.4) is the CBS inequality, the second is equivalent to (2.2) and the last one, follows after trivial algebraic manipulations.

Actually, from (2.3) we have

$$\int_\Omega \|f(x)\|^2 d\mu(x) \cdot \int_\Omega \|g(x)\|^2 d\mu(x) - \frac{\left(\int_\Omega \operatorname{Re}[(\overline{c}+\overline{C})\langle f(x), g(x)\rangle] d\mu(x)\right)^2}{|c+C|^2} \tag{2.5}$$
$$\leq \left(\frac{1}{4\operatorname{Re}(\overline{c}C)} - \frac{1}{|c+C|^2}\right) \cdot \left(\int_\Omega \operatorname{Re}[(\overline{c}+\overline{C})\langle f(x), g(x)\rangle] d\mu(x)\right)^2 .$$

Also, we have

$$\begin{aligned} \frac{1}{4\operatorname{Re}(\overline{c}C)} - \frac{1}{|c+C|^2} &= \frac{|c+C|^2 - 4\operatorname{Re}(\overline{c}C)}{4\operatorname{Re}(\overline{c}C) \cdot |c+C|^2} \\ &= \frac{(C+c)(\overline{C}+\overline{c}) - 4\operatorname{Re}(\overline{c}C)}{4\operatorname{Re}(\overline{c}C) \cdot |c+C|^2} \\ &= \frac{|C|^2 + |c|^2 - 2\operatorname{Re}(\overline{c}C)}{4\operatorname{Re}(\overline{c}C) \cdot |c+C|^2} \\ &= \frac{|C-c|^2}{4\operatorname{Re}(\overline{c}C) \cdot |c+C|^2}. \end{aligned} \tag{2.6}$$

Now, the last part of the inequality (2.4) follows from (2.5), (2.6) and (2.2). ∎

Remark 1. *For $d\mu(x) = \rho(x)dx$, we obtain the results of Theorem A.*

From Theorem 1 we can easily obtain the following chain of inequalities:

Theorem 2. *If (1.8) and (2.1) with $g(x) = e, ||e|| = 1$ are satisfied, then the following inequalities hold*

$$\begin{aligned} \int_\Omega \|f(x)\| d\mu(x) &\leq \left(\mu(\Omega) \cdot \int_\Omega \|f(x)\|^2 d\mu(x)\right)^{\frac{1}{2}} \\ &\leq \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}} \cdot \left|\int_\Omega \langle f(x), e\rangle d\mu(x)\right| \\ &\leq \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}} \cdot \left\|\int_\Omega f(x) d\mu(x)\right\|. \end{aligned} \tag{2.7}$$

Proof. The first inequality follows directly from the quadratic-arithmetic mean inequality for integrals:

$$\frac{1}{\mu(\Omega)} \int_\Omega \|f(x)\| d\mu(x) \leq \left(\frac{1}{\mu(\Omega)} \int_\Omega \|f(x)\|^2 d\mu(x)\right)^{\frac{1}{2}} .$$

The second one is an inequality (2.2) with $g(x) = e$, $||e|| = 1$:

$$\left(\mu(\Omega) \cdot \int_\Omega \|f(x)\|^2 d\mu(x)\right)^{\frac{1}{2}} \leq \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}} \cdot \left|\int_\Omega \langle f(x), e\rangle d\mu(x)\right| .$$

The last inequality is a consequence of the CBS inequality:

$$\begin{aligned} \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}} \cdot \left|\int_\Omega \langle f(x), e\rangle d\mu(x)\right| &= \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}} \cdot \left|\left\langle \int_\Omega f(x) d\mu(x), e\right\rangle\right| \\ &\leq \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}} \cdot \left\|\int_\Omega f(x) d\mu(x)\right\| . \end{aligned}$$

∎

Now, we generalize the reverses of the CBS inequality stated in Theorem C with a refinement. Also, we offer a simpler proof.

Theorem 3. *If (2.1) is valid, then the following inequalities hold*

$$
\begin{aligned}
&\int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x) \\
&\qquad -\left|\int_\Omega \langle f(x),g(x)\rangle d\mu(x)\right|^2 \\
&\le \int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x) \\
&\qquad -\frac{\left(\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{|c+C|^2} \\
&\le \frac{1}{4}|C-c|^2\cdot\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2.
\end{aligned}
\tag{2.8}
$$

Proof. The first inequality in (2.8) is equivalent to (2.2). Let us prove the second inequality.

From Lemma 1 it follows that

$$
\begin{aligned}
&\int_\Omega \|f(x)\|^2 d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x)-\frac{\left(\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{|c+C|^2} \\
&\le \int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x) \\
&\qquad -\mathrm{Re}(\overline{c}C)\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2-\frac{\left(\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{|c+C|^2} \\
&= \frac{1}{4}|C-c|^2\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2 \\
&\qquad -\left[\left(\mathrm{Re}(\overline{c}C)+\frac{1}{4}|C-c|^2\right)\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2\right. \\
&\qquad -\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x) \\
&\qquad \left.+\frac{\left(\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{|c+C|^2}\right] \\
&= \frac{1}{4}|C-c|^2\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2-\left[\frac{1}{4}|C+c|^2\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^2\right. \\
&\qquad -\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\cdot\int_\Omega \|g(x)\|^2 d\mu(x) \\
&\qquad \left.+\frac{\left(\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x),g(x)\rangle]d\mu(x)\right)^2}{|c+C|^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}|C-c|^2 \Big(\int_\Omega \|g(x)\|^2 \, d\mu(x)\Big)^2 \\
&\qquad - \left(\frac{|c+C|}{2} \cdot \int_\Omega \|g(x)\|^2 \, d\mu(x) - \frac{\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x), g(x)\rangle] d\mu(x)}{|c+C|} \right)^2 \\
&\le \frac{1}{4}|C-c|^2 \cdot \left(\int_\Omega \|g(x)\|^2 \, d\mu(x) \right)^2 .
\end{aligned}
$$

And the proof is complete. ∎

3. Some New Inequalities

Now we are going to prove some new similar inequalities:

Theorem 4. *If (1.8) and (2.1) are satisfied, with $C \neq -c$, then the following inequalities hold*

$$
\begin{aligned}
(3.1) \quad & \int_\Omega \|f(x)\| \, d\mu(x) \cdot \int_\Omega \|g(x)\| \, d\mu(x) - \left| \int_\Omega \langle f(x), g(x)\rangle d\mu(x) \right| \\
&\le \int_\Omega \|f(x)\| \, d\mu(x) \cdot \int_\Omega \|g(x)\| \, d\mu(x) - \frac{\int_\Omega \mathrm{Re}\left[(\overline{c}+\overline{C})\langle f(x), g(x)\rangle\right] d\mu(x)}{|c+C|} \\
&\le \frac{|C-c|^2}{4|c+C|} \cdot \int_\Omega \|g(x)\|^2 \, d\mu(x).
\end{aligned}
$$

Proof. The first inequality in (3.1) is equivalent to inequality (2.2). Let us prove the second inequality.

By using the result of Lemma 1, we have

$$
\begin{aligned}
& \int_\Omega \|f(x)\| \, d\mu(x) \cdot \int_\Omega \|g(x)\| \, d\mu(x) - \frac{\int_\Omega \mathrm{Re}[(\overline{c}+\overline{C})\langle f(x), g(x)\rangle] d\mu(x)}{|c+C|} \\
&\le \int_\Omega \|f(x)\| \, d\mu(x) \cdot \int_\Omega \|g(x)\| \, d\mu(x) \\
&\qquad - \frac{1}{|c+C|} \cdot \int_\Omega \|f(x)\|^2 \, d\mu(x) - \frac{\mathrm{Re}(\overline{c}C)}{|c+C|} \cdot \int_\Omega \|g(x)\|^2 \, d\mu(x) \\
&= \left(\frac{|c+C|}{4} - \frac{\mathrm{Re}(\overline{c}C)}{|c+C|} \right) \cdot \int_\Omega \|g(x)\|^2 \, d\mu(x) \\
&\qquad - \frac{1}{|c+C|} \left(\int_\Omega \|f(x)\| \, d\mu(x) - \frac{|c+C|}{2} \cdot \int_\Omega \|g(x)\| \, d\mu(x) \right)^2 \\
&\le \frac{|c+C|^2 - 4\,\mathrm{Re}(\overline{c}C)}{4|c+C|} \cdot \int_\Omega \|g(x)\|^2 \, d\mu(x) \\
&= \frac{|C-c|^2}{4|c+C|} \cdot \int_\Omega \|g(x)\|^2 \, d\mu(x).
\end{aligned}
$$

This concludes the proof. ∎

Remark 2. *It is interesting to compare the inequalities given in Theorem 4 with the inequalities given in Theorem B. The details are left to the interested reader.*

Remark 3. *Notice that the discrete version of (3.1) was obtained in* [5, Theorem 4].

Also, by using Lemma 1, we prove an inequality of the Klamkin-McLenaghan type:

Theorem 5. *If (1.8) and (2.1) are satisfied, then the following inequality holds*

$$\frac{\int_\Omega \|f(x)\|^2 d\mu(x)}{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|} - \frac{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|}{\int_\Omega \|g(x)\|^2 d\mu(x)} \le |C+c| - 2(\mathrm{Re}(\overline{c}C))^{\frac{1}{2}}. \tag{3.2}$$

Proof. Again we start with the inequality (2.2):

$$\frac{\int_\Omega \|f(x)\|^2 d\mu(x)}{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|} \le |c+C| - \mathrm{Re}(\overline{c}C)\frac{\int_\Omega \|g(x)\|^2 d\mu(x)}{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|}.$$

Then we have

$$\begin{aligned}
&\frac{\int_\Omega \|f(x)\|^2 d\mu(x)}{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|} - \frac{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|}{\int_\Omega \|g(x)\|^2 d\mu(x)}\\
&\le |c+C| - \mathrm{Re}(\overline{c}C)\frac{\int_\Omega \|g(x)\|^2 d\mu(x)}{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|} - \frac{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|}{\int_\Omega \|g(x)\|^2 d\mu(x)}\\
&= |C+c| - 2(\mathrm{Re}(\overline{c}C))^{\frac{1}{2}}\\
&\quad - \left[(\mathrm{Re}(\overline{c}C))^{\frac{1}{2}} \cdot \frac{\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^{\frac{1}{2}}}{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|^{\frac{1}{2}}} - \frac{\left|\int_\Omega \langle f(x), g(x)\rangle d\mu(x)\right|^{\frac{1}{2}}}{\left(\int_\Omega \|g(x)\|^2 d\mu(x)\right)^{\frac{1}{2}}}\right]^2\\
&\le |C+c| - 2(\mathrm{Re}(\overline{c}C))^{\frac{1}{2}}.
\end{aligned}$$

The proof is completed. ∎

Remark 4. *Notice that the discrete version of the above inequality (3.2) has been obtained in* [5, Theorem 2].

4. Discrete Inequalities

Now we state analogous inequalities for sums instead of integrals which will provide generalization for the results from [5]. Again, we suppose that (1.8) is satisfied for the complex numbers c and C.

First, the analogue version of Lemma 1 for finite sequences is as follows:

Lemma 2. *Let $x_1, ..., x_n$ and $y_1, ...y_n$ be vectors from inner product space $(H, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$ or $\mathbb{R}$ which satisfy the following condition:*

$$\mathrm{Re}\langle Cy_k - x_k, x_k - cy_k\rangle \ge 0, \quad \forall k = 1, ..., n. \tag{4.1}$$

Then the following inequality holds

$$\begin{aligned}
\sum_{k=1}^{n} ||x_k||^2 + \mathrm{Re}(\overline{c}C)\sum_{k=1}^{n} ||y_k||^2 &\le \sum_{k=1}^{n} \mathrm{Re}[(\overline{c}+\overline{C})\langle x_k, y_k\rangle]\\
&\le |c+C|\left|\sum_{k=1}^{n}\langle x_k, y_k\rangle\right|.
\end{aligned} \tag{4.2}$$

Now we state all the theorems for discrete case:

Theorem 6. *If (1.8) and (4.1) are satisfied, then the following inequalities hold*

$$\sum_{k=1}^{n}||x_k||^2\cdot\sum_{k=1}^{n}||y_k||^2 \le \frac{\left(\sum_{k=1}^{n}\operatorname{Re}[(\overline{c}+\overline{C})\langle x_k,y_k\rangle]\right)^2}{4\operatorname{Re}(\overline{c}C)} \le \frac{|c+C|^2}{4\operatorname{Re}(\overline{c}C)}\cdot\left|\sum_{k=1}^{n}\langle x_k,y_k\rangle\right|^2 \tag{4.3}$$

and

$$\begin{aligned} 0 &\le \sum_{k=1}^{n}||x_k||^2\cdot\sum_{k=1}^{n}||y_k||^2-\left|\sum_{k=1}^{n}\langle x_k,y_k\rangle\right|^2 \\ &\le \sum_{k=1}^{n}||x_k||^2\cdot\sum_{k=1}^{n}||y_k||^2-\frac{\left(\sum_{k=1}^{n}\operatorname{Re}[(\overline{c}+\overline{C})\langle x_k,y_k\rangle]\right)^2}{|c+C|^2} \\ &\le \frac{|C-c|^2}{4\operatorname{Re}(\overline{c}C)}\left|\sum_{k=1}^{n}\langle x_k,y_k\rangle\right|^2 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \sum_{k=1}^{n}||x_k|| &\le \left(n\cdot\sum_{k=1}^{n}||x_k||^2\right)^{\frac{1}{2}} \le \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}}\cdot\left|\sum_{k=1}^{n}\langle x_k,e\rangle\right| \\ &\le \frac{|c+C|}{2\sqrt{\operatorname{Re}(\overline{c}C)}}\cdot\left\|\sum_{k=1}^{n}x_k\right\|, \end{aligned} \tag{4.5}$$

respectively.

Theorem 7. *If (4.1) is valid, then the following inequalities hold*

$$\begin{aligned} &\sum_{k=1}^{n}||x_k||^2\cdot\sum_{k=1}^{n}||y_k||^2-\left|\sum_{k=1}^{n}\langle x_k,y_k\rangle\right|^2 \\ &\le \sum_{k=1}^{n}||x_k||^2\cdot\sum_{k=1}^{n}||y_k||^2-\frac{\left(\sum_{k=1}^{n}\operatorname{Re}[(\overline{c}+\overline{C})\langle x_k,y_k\rangle]\right)^2}{|c+C|^2} \\ &\le \frac{1}{4}|C-c|^2\cdot\left(\sum_{k=1}^{n}||y_k||^2\right)^2. \end{aligned} \tag{4.6}$$

Finally, we can state that:

Theorem 8. *If (1.8) and (4.1) are satisfied, then the following inequality is valid*

$$\frac{\sum_{k=1}^{n}||x_k||^2}{|\sum_{k=1}^{n}\langle x_k,y_k\rangle|}-\frac{|\sum_{k=1}^{n}\langle x_k,y_k\rangle|}{\sum_{k=1}^{n}||y_k||^2} \le |C+c|-2(\operatorname{Re}(\overline{c}C))^{\frac{1}{2}}. \tag{4.7}$$

Moreover, if $C \neq -c$, then we also have

$$\begin{aligned} &\sum_{k=1}^{n} ||x_k|| \cdot \sum_{k=1}^{n} ||y_k|| - \left| \sum_{k=1}^{n} \langle x_k, y_k \rangle \right| \\ &\leq \sum_{k=1}^{n} ||x_k|| \cdot \sum_{k=1}^{n} ||y_k|| - \frac{\sum_{k=1}^{n} \operatorname{Re}[(\overline{c} + \overline{C}) \langle x_k, y_k \rangle]}{|c + C|} \\ &\leq \frac{|C - c|^2}{4|c + C|} \cdot \sum_{k=1}^{n} ||y_k||^2. \end{aligned} \tag{4.8}$$

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Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10 000 Zagreb, Croatia

E-mail address: `ilko.brnetic@fer.hr`

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`

Faculty of Applied Technical Sciences, University of Prishtina, Mother Theresa 5, 38 000 Prishtina, Kosova.

E-mail address: `razimhoxha@yahoo.com`

Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.

E-mail address: `pecaric@element.hr`