A COMPANION FOR THE OSTROWSKI AND THE GENERALISED TRAPEZOID INEQUALITIES

N.S. BARNETT, S.S. DRAGOMIR, AND I. GOMM

ABSTRACT. A companion for the Ostrowski and the generalised trapezoid inequalites for various classes of functions, including functions of bounded variation, Lipschitzian, convex and absolutely continuous functions is established. Applications for weighted means are also given.

1. INTRODUCTION

For a Lebesgue integrable function $f:[a,b]\to\mathbb{R}$ and for a given $t\in[a,b]$, it is natural to investigate the distances between the quantities

$$f(t), \ \frac{1}{b-a}\int_a^b f(s)\,ds \text{ and } \frac{(b-t)f(b)+(t-a)f(a)}{b-a}$$

respectively, and to seek sharp upper bounds for these distances in terms of different measures that can be associated with f, where f is restricted to particular classes of functions, such as the linear space of functions of bounded variation, the subspace of absolutely continuous functions on $[a,b]$, or the cone of all convex functions defined on the specified interval.

Such inequalities providing upper bounds for

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|, \qquad t\in[a,b], \tag{1.1}$$

are known in the literature as *Ostrowski type* inequalities. We note the original result obtained by Ostrowski in 1938, [13], that, if $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and such that $|f'(t)|\le M$ for $t\in(a,b)$, then

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\le\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M, \tag{1.2}$$

for each $t\in[a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

A similar result obtained by the second author in 1999 (see [7] or [6]) for functions of bounded variation is that

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\le\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f), \tag{1.3}$$

Date: February 15, 2008.

1991 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Ostrowski inequalities, Generalised trapezoid inequality, Functions of bounded variation, Lipschitzian functions, Convex functions, Absolutely continuous functions.

for each $t \in [a,b]$, where $\bigvee_a^b (f)$ is the total variation of f on $[a,b]$.

This same author, in [4], showed that if $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$, then

$$\begin{aligned} \frac{1}{2}\left[(b-t)^2 f'_+(t) - (t-a)^2 f'_-(t)\right] &\le \int_a^b f(s)\,ds - (b-a) f(t) \\ &\le \frac{1}{2}\left[(b-t)^2 f'_-(b) - (t-a)^2 f'_+(a)\right], \end{aligned} \tag{1.4}$$

for any $t \in (a,b)$, provided that the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite. The second inequality also holds for $t = a$ and $t = b$ and the constant $\frac{1}{2}$ is best possible in both inequalities.

Further, in [8, p. 2], it has been shown that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \tag{1.5}$$
$$\le \begin{cases} \left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1} + \left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1/q}\|f'\|_p & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2} + \left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1 \end{cases}$$

for any $t \in [a,b]$. The constants $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

For other recent results on Ostrowski type inequalities, see [1], [11], [14] and [15].

Inequalities providing upper bounds for the quantity

$$\left| \frac{(t-a) f(a) + (b-t) f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(s)\,ds \right|, \qquad t \in [a,b] \tag{1.6}$$

are known in the literature as *generalised trapezoid inequalities* and it has been shown in [3] that

$$\begin{aligned} &\left| \frac{(t-a) f(a) + (b-t) f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \\ &\qquad \le \left[\frac{1}{2} + \left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f) \end{aligned} \tag{1.7}$$

for any $t \in [a,b]$, provided that f is of bounded variation on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If f is absolutely continuous on $[a,b]$, then (see [2, p. 93])

$$\left|\frac{(t-a)f(a)+(b-t)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \tag{1.8}$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1/q}\|f'\|_p & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1 & \end{cases}$$

for any $t\in[a,b]$. The constants $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

Finally, for convex functions $f:[a,b]\to\mathbb{R}$, we have [5]

$$\frac{1}{2}\left[(b-t)^2 f'_+(t)-(t-a)^2 f'_-(t)\right] \tag{1.9}$$
$$\leq (b-t)f(f)+(t-a)f(a)-\int_a^b f(s)\,ds$$
$$\leq \frac{1}{2}\left[(b-t)^2 f'_-(b)-(t-a)^2 f'_-(a)\right]$$

for any $t\in(a,b)$, provided that $f'_-(b)$ and $f'_+(a)$ are finite. As above, the second inequality also holds for $t=a$ and $t=b$ and the constant $\frac{1}{2}$ is the best possible on both sides of (1.9).

For other recent results on the trapezoid inequality, see [9], [10], [12] and [16].

The main aim of this paper is to provide sharp upper bounds for the remaining difference

$$\Psi_f(t):=f(t)-\frac{f(a)(t-a)+(b-t)f(b)}{b-a},\qquad t\in[a,b]. \tag{1.10}$$

Obviously, if $O(t)$ is a bound for the Ostrowski difference (1.1) and $T(t)$ is a bound for the generalised trapezoid difference (1.6), then by the triangle inequality, $O(t)+T(t)$ is a bound for the absolute value of the difference $\Psi_f(t)$. However, using some integral representations for Ψ_f, we are able to obtain sharp upper bounds for $|\Psi_f(t)|$, which are better than the ones generated by the triangle inequality.

As applications, some bounds for the absolute value of the difference

$$\sum_{i=1}^n p_i f(x_i)-\frac{f(a)\left(\sum_{i=1}^n p_i x_i-a\right)+f(b)\left(b-\sum_{i=1}^n p_i x_i\right)}{b-a},$$

where $x_i\in[a,b]$, $p_i\geq 0$, $i\in\{1,\dots,n\}$ and $\sum_{i=1}^n p_i=1$, are also given.

2. The Case when f is of Bounded Variation

The following representation holds.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and let $T:[a,b]^2\to\mathbb{R}$ be given by*

$$T(t,s):=\begin{cases} t-a & \text{if } s\in[a,t], \\ t-b & \text{if } s\in(t,b]. \end{cases} \tag{2.1}$$

We then have the representation,

$$\Psi_f(t) = \frac{1}{b-a}\int_a^b T(t,s)\,df(s), \qquad t\in[a,b], \tag{2.2}$$

where the integral is considered in the Riemann-Stieltjes sense.

Proof. If f is bounded on $[a,b]$, then for any $t\in[a,b]$ the Riemann-Stieltjes integrals $\int_a^t df(s)$ and $\int_t^b df(s)$ exist and $\int_a^t df(s) = f(t)-f(a)$, $\int_t^b df(s) = f(b)-f(t)$. It follows that

$$\frac{1}{b-a}\int_a^b T(t,s)\,df(s) = (t-a)\int_a^t df(s) + (t-b)\int_t^b df(s) = (b-a)\,\Psi_f(t),$$

for any $t\in[a,b]$. □

The following provides a sharp bound for the absolute value of ψ_f where f is of bounded variation.

Theorem 1. *If $f:[a,b]\to\mathbb{R}$ is of bounded variation, then*

$$\begin{aligned}
|\Psi_f(t)| &\le \frac{1}{b-a}\left[(t-a)\bigvee_a^t(f) + (b-t)\bigvee_t^b(f)\right] \\
&\le \begin{cases}
\left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]\bigvee_a^b(f), \\
\left[\left(\frac{t-a}{b-a}\right)^q+\left(\frac{b-t}{b-a}\right)^q\right]^{\frac{1}{q}}\left[\left(\bigvee_a^t f\right)^p+\left(\bigvee_t^b f\right)^p\right]^{\frac{1}{p}} & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right|.
\end{cases}
\end{aligned} \tag{2.3}$$

and the first inequality is sharp. The constant $\frac{1}{2}$ is also the best possible in both branches of (2.3).

Proof. Utilising the represenation (2.2), we have

$$\begin{aligned}
|\Psi_f(t)| &= \frac{1}{b-a}\left|(t-a)\int_a^t df(s) + (t-b)\int_t^b df(s)\right| \\
&\le \frac{1}{b-a}\left[(t-a)\left|\int_a^t df(s)\right| + (t-b)\left|\int_t^b df(s)\right|\right] \\
&\le \frac{1}{b-a}\left[(t-a)\bigvee_a^t(f) + (b-t)\bigvee_t^b(f)\right],
\end{aligned} \tag{2.4}$$

which proves the first inequality in (2.3).

Further, on making use of the Hölder inequality, we also have

$$(t-a)\bigvee_a^t(f)+(b-t)\bigvee_t^b(f)$$
$$\leq\begin{cases}\max\{t-a,b-t\}\left[\bigvee_a^t(f)+\bigvee_t^b(f)\right],\\ \left[(t-a)^q+(b-t)^q\right]^{\frac{1}{q}}\left[\left(\bigvee_a^t f\right)^p+\left(\bigvee_t^b f\right)^p\right]^{\frac{1}{p}} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \max\left\{\bigvee_a^t(f),\bigvee_t^b(f)\right\}(t-a+b-t),\end{cases}$$

which together with (2.4) produces (2.3).

For $t=\frac{a+b}{2}$, we get, from (2.3),

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{2}\bigvee_a^b(f), \tag{2.5}$$

which will be shown to be sharp.

Assume that there exists a constant $A>0$ such that

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right|\leq A\bigvee_a^b(f). \tag{2.6}$$

Consider the function $f(t)=\left|t-\frac{a+b}{2}\right|$ which is of bounded variation on $[a,b]$, with $f(a)=f(b)=\frac{b-a}{2}$ and $\bigvee_a^b(f)=b-a$. For this function, the inequality (2.6) becomes $\frac{b-a}{2}\leq A(b-a)$ which implies that $A\geq\frac{1}{2}$. □

The following particular case is of interest for applications.

Corollary 1. *If $f:[a,b]\to\mathbb{R}$ is L_1-Lipschitzian on $[a,t]$ and L_2-Lipschitzian on $[t,b]$, $L_1,L_2>0$, $t\in[a,b]$, then*

$$|\Psi_f(t)|\leq\frac{1}{b-a}\left[L_1(t-a)^2+L_2(b-t)^2\right] \tag{2.7}$$
$$\leq\begin{cases}\max\{L_1,L_2\}\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a);\\ \left(L_1^q+L_2^q\right)^{\frac{1}{q}}\left[\left(\frac{t-a}{b-a}\right)^{2p}+\left(\frac{b-t}{b-a}\right)^{2p}\right]^{\frac{1}{p}}(b-a), \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ (L_1+L_2)\left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]^2(b-a).\end{cases}$$

In particular, if f is $L-$Lipschitzian, then

$$|\Psi_f(t)|\leq L\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) \tag{2.8}$$

for any $t\in[a,b]$, the constant $\frac{1}{4}$ being the best possible.

Proof. It is well known that if $g:[\alpha,\beta]\to\mathbb{R}$ is $L-$Lipschitzian, then g is of bounded variation and $\bigvee_\alpha^\beta(g)\leq L(\beta-\alpha)$. Therefore, by the first inequality in (2.3) we get the corresponding inequality in (2.7). The other inequalities follow by the Hölder inequality and the details are omitted.

If we now consider $t = \frac{a+b}{2}$ we obtain

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4} L (b - a), \tag{2.9}$$

for which we will show that $\frac{1}{4}$ is the best possible.

For this purpose, assume that there exists a $B > 0$ such that

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq BL (b - a). \tag{2.10}$$

Consider the function $f : [a, b] \to \mathbb{R}$, $f(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$ and so $f(a) = f(b) = \frac{(b-a)^2}{8}$, $f'(t) = t - \frac{a+b}{2}$ and $L = \sup_{t \in [a,b]} |f'(t)| = \frac{b-a}{2}$. If we replace these values in the above inequality (2.10), then we have $\frac{(b-a)^2}{8} \leq \frac{B(b-a)^2}{2}$, which implies that $B \geq \frac{1}{4}$. □

Corollary 2. *If $f : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then*

$$|\Psi_f(t)| \leq \frac{1}{b-a} \{(t - a)[f(t) - f(a)] + (b - t)[f(b) - f(t)]\} \tag{2.11}$$

$$\leq \begin{cases} \left[\frac{1}{2} + \left|\frac{t - \frac{a+b}{2}}{b-a}\right|\right][f(b) - f(a)], \\ \left[\left(\frac{t-a}{b-a}\right)^q + \left(\frac{b-t}{b-a}\right)^q\right]^{\frac{1}{q}} \left[[f(t) - f(a)]^p + [f(b) - f(t)]^p\right]^{\frac{1}{p}} \\ \qquad\qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2}[f(b) - f(a)] + \left|f(t) - \frac{f(a)+f(b)}{2}\right|. \end{cases}$$

The constant $\frac{1}{2}$ in the first inequality is the best possible.

Proof. The inequalities are obvious by Theorem 1. For $t = \frac{a+b}{2}$, we obtain

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2}[f(b) - f(a)]. \tag{2.12}$$

To show that $\frac{1}{2}$ is the best possible, assume that there is a constant $S > 0$ such that

$$\left| \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq S[f(b) - f(a)]. \tag{2.13}$$

Consider $f : [a, b] \to \mathbb{R}$,

$$f(t) = \begin{cases} 0 & \text{if } t \in \left[a, \frac{a+b}{2}\right] \\ k & \text{if } t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

with $k > 0$. Thus f is monotonic nondecreasing on $[a, b]$ and from (2.13) we get $\frac{k}{2} \leq Sk$, which implies that $S \geq \frac{1}{2}$. □

3. The Case when f is Absolutely Continuous

When f is absolutely continuous, the following representation of Φ can be determined.

Lemma 2. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$\Psi_f(t)=\frac{1}{b-a}\int_a^b T(t,s)\,f'(s)\,ds, \tag{3.1}$$

where the integral is considered in the Lebesgue sense and where the kernel $T:[a,b]^2\to\mathbb{R}$ has been defined in (2.1).

We can state the following result concerning estimates for the absolute value of $\psi_f(t)$ in terms of the Lebesgue norms $\|f'\|_{[a,b],p}$, $p\in[1,\infty]$, where

$$\|f'\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|f'(t)|,\qquad \|f'\|_{[a,b],p}:=\left(\int_a^b|f'(s)|^p\,ds\right)^{\frac{1}{p}},\quad p\geq 1.$$

Theorem 2. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$\begin{aligned}|\Psi_f(t)| &\leq \frac{1}{b-a}\left[(t-a)\|f'\|_{[a,t],1}+(b-t)\|f'\|_{[t,b],1}\right] \\ &\leq W(t),\qquad t\in[a,b],\end{aligned} \tag{3.2}$$

where

$$W(t):=\frac{1}{b-a}\times\begin{cases}(t-a)^2\|f'\|_{[a,t],\infty} & \text{if } f'\in L_\infty[a,b] \\ (t-a)^{1+\frac{1}{q}}\|f'\|_{[a,t],p} & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\end{cases}$$
$$+\frac{1}{b-a}\times\begin{cases}(b-t)^2\|f'\|_{[t,b],\infty} & \text{if } f'\in L_\infty[a,b] \\ (b-t)^{1+\frac{1}{\beta}}\|f'\|_{[t,b],\alpha} & \text{if } f'\in L_\alpha[a,b], \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\end{cases}$$

and W should be seen as all four possible combinations.

Proof. We have, by (3.1), that

$$\begin{aligned}|\Psi_f(t)| &= \frac{1}{b-a}\left|(t-a)[f(t)-f(a)]+(b-t)[f(b)-f(t)]\right| \\ &\leq \frac{1}{b-a}\left[(t-a)\left|\int_a^t f'(s)\,ds\right|+(b-t)\left|\int_t^b f'(s)\,ds\right|\right] \\ &\leq \frac{1}{b-a}\left[(t-a)\int_a^t|f'(s)|\,ds+(b-t)\int_t^b|f'(s)|\,ds\right],\end{aligned} \tag{3.3}$$

for $t\in[a,b]$, which proves the first inequality in (3.2).

Utilising the Hölder inequality, we have

$$\int_a^t |f'(s)|\,ds \le \begin{cases} (t-a)\, ess \sup_{s\in[a,t]} |f'(s)| & \text{if } f' \in L_\infty[a,b]; \\ (t-a)^{\frac{1}{q}} \left(\int_a^t |f'(s)|^p\,ds\right)^{\frac{1}{p}} & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \tag{3.4}$$

$$= \begin{cases} (t-a)\,\|f'\|_{[a,t],\infty} & \text{if } f' \in L_\infty[a,b] \\ (t-a)^{\frac{1}{q}}\,\|f'\|_{[a,t],p} & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

and, similarly,

$$\int_t^b |f'(s)|\,ds \le \begin{cases} (b-t)\,\|f'\|_{[t,b],\infty} & \text{if } f' \in L_\infty[a,b] \\ (b-t)^{\frac{1}{\beta}}\,\|f'\|_{[t,b],\alpha} & \text{if } f' \in L_\alpha[a,b], \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1. \end{cases} \tag{3.5}$$

Utilising (3.3) – (3.5) we deduce the desired inequality. □

Remark 1. *Inequality (3.2) has some particular instances of interest. The first is:*

$$|\Psi_f(t)| \le \frac{1}{b-a}\left[(t-a)\,\|f'\|_{[a,t],1} + (b-t)\,\|f'\|_{[t,b],1}\right] \tag{3.6}$$

$$\le \left[\frac{1}{2} + \left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_{[a,b],1},$$

for any $t \in [a,b]$, and the constant $\frac{1}{2}$ is the best possible.

Another inequality of interest is

$$|\Psi_f(t)| \le \left[\left(\frac{t-a}{b-a}\right)^2 \|f'\|_{[a,t],\infty} + \left(\frac{b-t}{b-a}\right)^2 \|f'\|_{[t,b],\infty}\right](b-a) \tag{3.7}$$

$$\le \left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,\|f'\|_{[a,b],\infty}$$

for any $t \in [a,b]$, provided that $f' \in L_\infty[a,b]$. The constant $\frac{1}{4}$ is the best possible.

If we choose $\alpha = p$ and $\beta = q$ with $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, then we also have

$$|\Psi_f(t)| \le \left[\left(\frac{t-a}{b-a}\right)^{1+\frac{1}{q}} \|f'\|_{[a,t],p} + \left(\frac{b-t}{b-a}\right)^{1+\frac{1}{q}} \|f'\|_{[t,b],p}\right](b-a)^{\frac{1}{q}} \tag{3.8}$$

$$\le \left[\left(\frac{t-a}{b-a}\right)^{q+1} + \left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \|f'\|_{[a,b],p}\,(b-a)^{\frac{1}{q}}$$

for any $t \in [a,b]$, provided that $f' \in L_p[a,b]$.

4. The Case when f is Convex

The following result for convex functions can be stated as well.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $f'_{-}(b)$ and $f'_{+}(a)$ finite, then, for any $t\in(a,b)$, we have*

$$\frac{1}{b-a}\left[(t-a)^2 f'_{+}(a)-(b-t)^2 f'_{-}(b)\right] \leq \Psi_f(t) \leq \frac{1}{b-a}\left[(t-a)^2 f'_{-}(t)-(b-t)^2 f'_{+}(t)\right]. \tag{4.1}$$

The first inequality also holds for $t=a$ and $t=b$. The constant 1 is the best possible on both sides of (4.1).

Proof. From Lemma 1,

$$(b-a)\Psi_f(t)=(t-a)[f(t)-f(a)]-(b-t)[f(b)-f(t)], \qquad t\in[a,b]. \tag{4.2}$$

Let $t\in(a,b)$, then, by the convexity of f we have

$$(t-a)f'_{-}(t)\geq f(t)-f(a)\geq f'_{+}(a)(t-a) \tag{4.3}$$

and

$$(b-t)f'_{-}(b)\geq f(b)-f(t)\geq (b-t)f'_{+}(t). \tag{4.4}$$

If we multiply (4.3) by $t-a>0$ and (4.4) by $b-t>0$, we can write

$$(t-a)^2 f'_{-}(t)\geq (t-a)[f(t)-f(a)]\geq f'_{+}(a)(t-a)^2 \tag{4.5}$$

and

$$-(b-t)^2 f'_{+}(t)\geq -(b-t)[f(b)-f(t)]\geq -(b-t)^2 f'_{-}(b). \tag{4.6}$$

Finally, on adding (4.5) to (4.6) we deduce the desired result (4.1).

When $t=\frac{a+b}{2}$ in (4.1), we obtain

$$\frac{b-a}{4}\left[f'_{+}(a)-f'_{-}(b)\right]\leq f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2} \leq \frac{b-a}{4}\left[f'_{-}\left(\frac{a+b}{2}\right)-f'_{+}\left(\frac{a+b}{2}\right)\right]. \tag{4.7}$$

For $f(t)=\left|t-\frac{a+b}{2}\right|$, we have $f'_{+}(a)=-1$, $f'_{-}(b)=1$, $f'_{+}\left(\frac{a+b}{2}\right)=1$, $f'_{-}\left(\frac{a+b}{2}\right)=-1$, $f(a)=f(b)=\frac{b-a}{2}$ and in (4.7) all terms are $-\frac{b-a}{2}$. □

Remark 2. *If f is differentiable on (a,b), then the second inequality can be written in the following simpler form*

$$\Psi_f(t)\leq 2\left(t-\frac{a+b}{2}\right)f'(t), \tag{4.8}$$

for any $t\in(a,b)$.

5. Applications for Weighted Means

In this section we show that the above result can be useful in providing various bounds for the weighted mean:

$$M_f(p;x) := \sum_{i=1}^{n} p_i f(x_i), \quad x_i \in [a,b], \ p_i \geq 0, \ i \in \{1,\dots,n\}, \quad \sum_{i=1}^{n} p_i = 1. \tag{5.1}$$

For $f(t) = t$, we denote by $A(p;x)$ the *weighted arithmetic mean.*

The following result can be stated.

Proposition 1. *If the function $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned} &\left| M_f(p;x) - \frac{f(a)[A(p;x)-a] + [b-A(p;x)]f(b)}{b-a} \right| \\ &\qquad \leq \left[\frac{1}{2} + \frac{1}{b-a} \sum_{i=1}^{n} p_i \left| x_i - \frac{a+b}{2} \right| \right] \bigvee_a^b (f), \end{aligned} \tag{5.2}$$

the constant $\frac{1}{2}$ being the best possible.

Proof. We use the first branch of the second inequality in (2.3) to state:

$$\left| f(x_i) - \frac{f(a)(x_i-a) + f(b)(b-x_i)}{b-a} \right| \leq \left[\frac{1}{2} + \frac{1}{b-a} \left| x_i - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{5.3}$$

for each $i \in \{1,\dots,n\}$.

Now, if we multiply (5.3) by $p_i \geq 0$, sum over i from $\{1,\dots,n\}$ and use the generalised triangle inequality, we deduce the desired result (5.1).

The fact that $\frac{1}{2}$ is the best possible follows from the fact that it is the best possible for $n = 1$. □

In a similar manner, on utilising the inequality (2.8), we can state the following result.

Proposition 2. *If the function $f : [a,b] \to \mathbb{R}$ is $L-$Lipschitzian, then*

$$\begin{aligned} &\left| M_f(p;x) - \frac{f(a)[A(p;x)-a] + [b-A(p;x)]f(b)}{b-a} \right| \\ &\qquad \leq L(b-a) \left[\frac{1}{4} + \frac{1}{(b-a)^2} \sum_{i=1}^{n} p_i \left(x_i - \frac{a+b}{2} \right)^2 \right], \end{aligned} \tag{5.4}$$

the constant $\frac{1}{4}$ being the best possible.

Finally, on utilising the first inequality in (3.2) we can also state that:

Proposition 3. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$\begin{aligned} &\left| M_f(p;x) - \frac{f(a)[A(p;x)-a] + [b-A(p;x)]f(b)}{b-a} \right| \\ &\qquad \leq \frac{1}{b-a} \left[\sum_{i=1}^{n} p_i (x_i - a) \|f'\|_{[a,x_i],1} + \sum_{i=1}^{n} p_i (b-x_i) \|f'\|_{[t,x_i],1} \right]. \end{aligned} \tag{5.5}$$

The above results can be useful in providing various inequalities between the weighted arithmetic mean $A(p;x)$ and the *weighted geometric mean* $G(p;x) := \prod_{i=1}^{n} x_i^{p_i}$, for which the following well-known inequality holds:

$$A(p;x) \geq G(p;x) . \tag{5.6}$$

If we consider the function $f(t) = \ln t$, $f : [a,b] \to \mathbb{R} \subset (0,\infty)$, then $M_{\ln(\cdot)}(p;x) = \ln G(p;x)$ and

$$\frac{f(a)[A(p;x)-a]+[b-A(p;x)]f(b)}{b-a} = \ln\left[a^{\frac{A(p;x)-a}{b-a}} \cdot b^{\frac{b-A(p;x)}{b-a}}\right].$$

Also,

$$\sum_{i=1}^{n} p_i (x_i - a) \|f'\|_{[a,x_i],1} = \sum_{i=1}^{n} p_i (x_i - a)[\ln(x_i) - \ln a]$$

and

$$\sum_{i=1}^{n} p_i (b - x_i) \|f'\|_{[t,x_i],1} = \sum_{i=1}^{n} p_i (b - x_i)[\ln b - \ln(x_i)].$$

Utilising the inequality

$$\frac{\ln\beta - \ln\alpha}{\beta - \alpha} \leq \frac{1}{\sqrt{\alpha\beta}}, \qquad 0 < \alpha, \beta$$

and taking into account that $0 < a \leq x_i \leq b$, $i \in \{1, \ldots, n\}$, then we have

$$(x_i - a)[\ln(x_i) - \ln a] \leq \frac{(x_i - a)^2}{\sqrt{x_i a}} \leq \frac{(x_i - a)^2}{a}$$

and

$$(b - x_i)[\ln b - \ln(x_i)] \leq \frac{(b - x_i)^2}{\sqrt{x_i b}} \leq \frac{(b - x_i)^2}{\sqrt{ab}} \leq \frac{(b - x_i)^2}{a},$$

which implies that

$$\sum_{i=1}^{n} p_i (x_i - a)[\ln(x_i) - \ln a] \leq \frac{1}{a}\sum_{i=1}^{n} p_i (x_i - a)^2$$

and

$$\sum_{i=1}^{n} p_i (b - x_i)[\ln b - \ln(x_i)] \leq \frac{1}{a}\sum_{i=1}^{n} p_i (b - x_i)^2 .$$

Now, by (5.5),

$$\left|\ln\left[\frac{G(p;x)}{a^{\frac{A(p;x)-a}{b-a}} \cdot b^{\frac{b-A(p;x)}{b-a}}}\right]\right| \leq \frac{2}{a(b-a)}\left[\frac{1}{4}(b-a)^2 + \sum_{i=1}^{n} p_i \left(x_i - \frac{a+b}{2}\right)^2\right]. \tag{5.7}$$

Finally, on utilising (5.2) we also have

$$\left|\ln\left[\frac{G(p;x)}{a^{\frac{A(p;x)-a}{b-a}} \cdot b^{\frac{b-A(p;x)}{b-a}}}\right]\right| \leq \left[\frac{1}{2} + \frac{1}{b-a}\sum_{i=1}^{n} p_i \left|x_i - \frac{a+b}{2}\right|\right] \ln\left(\frac{b}{a}\right). \tag{5.8}$$

References

[1] P. CERONE and S.S. DRAGOMIR, Midpoint-type rules from an inequalities point of view, in *Handbook of Analytic-Computational Methods in Applied Mathematics*, G. Anastassiou (Ed.), CRC Press, NY, 2000, 135-200.

[2] P. CERONE and S.S. DRAGOMIR, Trapezoidal-type rules from an inequalities point of view, in *Handbook of Analytic-Computational Methods in Applied Mathematics*, G. Anastassiou (Ed.), CRC Press, NY, 2000, 65-134.

[3] P. CERONE, S.S. DRAGOMIR and C.E.M. PEARCE, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2) (2000), 147-163.

[4] S.S. DRAGOMIR, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Ineq. Pure & Appl. Math.,* **3**(2) (2002), Art. 31. [ONLINE: http://jipam.vu.edu.au/article.php?sid=183].

[5] S.S. DRAGOMIR, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Ineq. Pure & Appl. Math.,* **3**(3) (2002), Art. 35. [ONLINE: http://jipam.vu.edu.au/article.php?sid=187].

[6] S.S. DRAGOMIR, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.*, **4**(1) (2001), 59–66.

[7] S.S. DRAGOMIR, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, **60** (1999), 495–508.

[8] S.S. DRAGOMIR and Th.M. RASSIAS (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[9] A. I. KECHRINIOTIS and N.D. ASSIMAKIS, Generalizations of the trapezoid inequalities based on a new mean value theorem for the remainder in Taylor's formula. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 3, Article 90, 13 pp. (electronic).

[10] Z. LIU, Some inequalities of perturbed trapezoid type. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 47, 9 pp. (electronic).

[11] Z. LIU, Some Ostrowski-Grüss type inequalities and applications. *Comput. Math. Appl.* **53** (2007), no. 1, 73–79.

[12] McD. A. MERCER, On perturbed trapezoid inequalities. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 118, 7 pp. (electronic).

[13] A. OSTROWSKI, Über die absolutabweichung einer differentiierbaren funktion von ihrem integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226–227.

[14] B.G. PACHPATTE, New inequalities of Ostrowski-Grüss type. *Fasc. Math.* No. **38** (2007), 97–104.

[15] E.C. POPA, An inequality of Ostrowski type via a mean value theorem. *Gen. Math.* **15** (2007), no. 1, 93–100.

[16] N. UJEVIĆ, Error inequalities for a generalized trapezoid rule. *Appl. Math. Lett.* **19** (2006), no. 1, 32–37.

E-mail address: neil.barnett@vu.edu.au

E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir

E-mail address: ian.gomm@vu.edu.au

School of Computer Science and Mathematics, Victoria University, PO Box 14428,, Melbourne City MC, Victoria 8001, Australia.