Q-NORM INEQUALITIES FOR SEQUENCES OF HILBERT SPACE OPERATORS

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Abstract. We give some inequalities related to a large class of operator norm, the so-called Q-norms, for a (not necessary commutative) family of bounded linear operators acting on a Hilbert space that are related to the classical Schwarz inequality. Applications for vector inequalities are also provided.

1. Introduction and preliminaries

Let $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ be a real or complex Hilbert space, $\mathbb{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators acting on $\mathcal{H}$ and I denote the identity operator on $\mathcal{H}$.

In many estimates one needs to use upper bounds for the norm of a sum of products $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})$, where separate information for norms of operators are provided. The special case in which $B_i = \alpha_i I$ $(\alpha_i \in \mathbb{C}, 1 \leq i \leq n)$ and the norms in the question are the operator norm has already investigated in [4]. The celebrated Hölder's discrete inequality stating

$$\left| \sum_{i=1}^{n} \alpha_i \beta_i \right| \leq \left(\sum_{i=1}^{n} |\alpha_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |\beta_i|^q \right)^{1/q} \tag{1.1}$$

$$\left(1 < p, q\,; \frac{1}{p} + \frac{1}{q} = 1; \alpha_i, \beta_i \in \mathbb{C}\,, 1 \leq i \leq n \right)$$

and its variance in the special case $p = q = 2$ (the so-called Cauchy–Bunyakovsky–Schwarz discrete inequality) are of special interest and of larger utility.

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To each symmetric gauge functions defined on the sequences of real numbers there corresponds a unitarily invariant norm $|||\cdot|||$ defined on a two-sided ideal $C_{|||\cdot|||}$ of $\mathbb{B}(\mathcal{H})$ enjoying the invariant property $|||UAV||| = |||A|||$ for all $A \in C_{|||\cdot|||}$ and all unitary operators $U, V \in \mathbb{B}(\mathcal{H})$. It is known that $C_{|||\cdot|||}$ is a Banach space under the norm $|||\cdot|||$. If $A \in C_{|||\cdot|||}$, then $|||A^*||| = |||A|||$ and $|||BA||| \leq \|B\|\, |||A|||$ for all $B \in \mathbb{B}(\mathcal{H})$, from which we easily conclude that

$$|||BAC||| \leq \|B\|\, |||A|||\, \|C\|\,;\ (B, C \in \mathbb{B}(\mathcal{H})).$$

Some well-known examples of unitarily invariant norms are the Schatten p-norms $\|A\|_p := tr(|A|^p)^{1/p}$ for $1 \leq p < \infty$, operator norm $\|\cdot\|$, and the Ky-Fan norms; cf. [1, 6]. A unitarily invariant norm $\|\cdot\|_Q$ is called a Q-norm if there is a unitarily invariant norm $\|\cdot\|_{\widehat{Q}}$ such that $\|AA^*\|_{\widehat{Q}} = \|A\|_Q^2$.

The aim of the present paper is to establish some upper bounds of interest for the quantity $\|\sum_{i=1}^n A_iB_i\|_Q$ under certain assumptions on A_i's and B_i's. Our results are significant since we do not use any commutative assumption on the families of operators and as well we provide inequalities for a class of norms which are larger than operator norms (compare the results with those of [4]). Applications for vector inequalities are also given.

We need the following lemmas concerning real numbers (the second lemma is obvious):

Lemma 1.1. *(Young's inequality; see* [5, p. 30 or p. 49]*) If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then for all* $a, b > 0$ *one has*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 1.2. *For all* $a, b > 0$ *one has*

$$ab \leq \frac{1}{4}(a+b)^2 \leq \frac{1}{2}(a^2 + b^2)$$

Lemma 1.3. *(Daykin-Eliezer; see* [2]*) If* $a_k > 1$, $b_k > 1$ *and* $\dfrac{1}{p} + \dfrac{1}{q} < 1$*, then*

$$\left(\sum_{k=1}^n a_k b_k\right)^{\frac{1}{p}+\frac{1}{q}} \leq \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}.$$

2. Some General Results

The following result containing 9 different inequalities may be stated:

Theorem 2.1. *Let* $A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H})$ *and* $B_1, \ldots, B_n \in C_Q$. *Then*

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leq \begin{cases} A \\ B \\ C \end{cases} \tag{2.1}$$

where

$$A := \begin{cases} \max\limits_{1\le k\le n} \|A_k\|^2 \sum\limits_{i,j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}, \\[2ex] \max\limits_{1\le k\le n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|A_i\|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}\right)^s\right)^{\frac{1}{s}}, \\ \qquad\qquad\qquad\qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\[2ex] \max\limits_{1\le k\le n} \|A_k\| \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}\right), \end{cases}$$

$$B := \begin{cases} \max\limits_{1\le i\le n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} \\[2ex] \left(\sum\limits_{i=1}^{n} \|A_i\|^t\right)^{\frac{1}{t}} \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \\ \qquad\qquad\qquad\qquad \text{where } t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\[2ex] \sum\limits_{i=1}^{n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} \left\{\left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}\right\}, \end{cases} \tag{2.2}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$C := \begin{cases} \max\limits_{1\le i\le n} \|A_i\| \sum\limits_{k=1}^{n} \|A_k\| \sum\limits_{i=1}^{n} \max\limits_{1\le j\le n} \left\{ \left\|B_iB_j^*\right\|_{\widehat{Q}} \right\}, \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^m\right)^{\frac{1}{m}} \sum\limits_{k=1}^{n} \|A_k\| \left[\sum\limits_{i=1}^{n} \left(\max\limits_{1\le j\le n} \left\{ \left\|B_iB_j^*\right\|_{\widehat{Q}} \right\} \right)^l \right]^{\frac{1}{l}}, \\ \qquad\qquad\qquad \text{where m,l} > 1,\ \frac{1}{\text{m}} + \frac{1}{\text{l}} = 1; \\ \left(\sum\limits_{k=1}^{n} \|A_k\|\right)^2 \max\limits_{1\le i,j\le n} \left\{ \left\|B_iB_j^*\right\|_{\widehat{Q}} \right\}. \end{cases}$$

Proof. In the operator partial order of $\mathbb{B}(\mathcal{H})$, we have

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} B_iA_i\right)^* \left(\sum_{i=1}^{n} B_iA_i\right) \qquad (2.3) \\ &= \sum_{i=1}^{n} A_i^*B_i^* \sum_{j=1}^{n} B_jA_j = \sum_{i,j=1}^{n} A_i^*B_i^*B_jA_j. \end{aligned}$$

Taking the norm in (2.3) and noticing that $\|AA^*\|_{\widehat{Q}} = \|A\|_Q^2$ for any $A \in C_Q$, we have

$$\begin{aligned} \left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 &= \left\|\sum_{i=1}^{n}\sum_{j=1}^{n} A_i^*B_i^*B_jA_j\right\|_{\widehat{Q}} \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\| \|A_j\| \left\|B_iB_j^*\right\|_{\widehat{Q}} \\ &= \sum_{i=1}^{n} \|A_i\| \left(\sum_{j=1}^{n} \|A_j\| \left\|B_iB_j^*\right\|_{\widehat{Q}}\right) =: M. \end{aligned}$$

Utilizing Hölder's discrete inequality we have that

$$\sum_{j=1}^{n}\|A_j\|\left\|B_iB_j^*\right\|_{\widehat{Q}} \leq \begin{cases} \max\limits_{1\leq k\leq n}\|A_k\|\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}, \\ \\ \left(\sum\limits_{k=1}^{n}\|A_k\|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} \\ \qquad\qquad \text{where p} > 1,\ \frac{1}{\text{p}}+\frac{1}{\text{q}}=1; \\ \\ \sum\limits_{k=1}^{n}\|A_k\|\max\limits_{1\leq j\leq n}\left\|B_iB_j^*\right\|_{\widehat{Q}}, \end{cases}$$

for any $i\in\{1,\ldots,n\}$.

This provides the following inequalities:

$$M\leq\begin{cases} \max\limits_{1\leq k\leq n}\|A_k\|\sum\limits_{i=1}^{n}\|A_i\|\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)=:M_1 \\ \\ \left(\sum\limits_{k=1}^{n}\|A_k\|^p\right)^{\frac{1}{p}}\sum\limits_{i=1}^{n}\|A_i\|\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}:=M_p \\ \qquad\qquad \text{where p} > 1,\ \frac{1}{\text{p}}+\frac{1}{\text{q}}=1; \\ \\ \sum\limits_{k=1}^{n}\|A_k\|\sum\limits_{i=1}^{n}\|A_i\|\left(\max\limits_{1\leq j\leq n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right):=M_\infty. \end{cases}$$

Utilizing Hölder's inequality for $r,s>1$, $\frac{1}{r}+\frac{1}{s}=1$, we have:

$$\sum_{i=1}^{n}\|A_i\|\left(\sum_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)\leq\begin{cases} \max\limits_{1\leq i\leq n}\|A_i\|\sum\limits_{i,j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}} \\ \\ \left(\sum\limits_{i=1}^{n}\|A_i\|^r\right)^{\frac{1}{r}}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right)^s\right]^{\frac{1}{s}} \\ \qquad\qquad \text{where r} > 1,\ \frac{1}{\text{r}}+\frac{1}{\text{s}}=1; \\ \\ \sum\limits_{i=1}^{n}\|A_i\|\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}\left\|B_iB_j^*\right\|_{\widehat{Q}}\right), \end{cases}$$

and thus we can state that

$$M_1 \leq \begin{cases} \max\limits_{1\leq k\leq n} \|A_k\|^2 \sum\limits_{i,j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}; \\ \max\limits_{1\leq k\leq n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|A_i\|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}\right)^s\right)^{\frac{1}{s}}, \\ \qquad\qquad\qquad\qquad \text{where r} > 1,\ \frac{1}{\text{r}} + \frac{1}{\text{s}} = 1; \\ \max\limits_{1\leq k\leq n} \|A_k\| \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}\right), \end{cases}$$

and the first part of the theorem is proved.

By Hölder's inequality we can also have that (for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$)

$$M_p \leq \left(\sum_{k=1}^{n} \|A_k\|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{1\leq i\leq n} \|A_i\| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \\ \qquad\qquad\qquad\qquad \text{where t} > 1,\ \frac{1}{\text{t}} + \frac{1}{\text{u}} = 1; \\ \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\leq i\leq n} \left\{\left(\sum\limits_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}}\right\}, \end{cases}$$

and the second part of (2.1) is proved.

Finally, we may state that

$$M_\infty \leq \sum_{k=1}^{n} \|A_k\| \times \begin{cases} \max\limits_{1\leq i\leq n} \|A_i\| \sum\limits_{i=1}^{n} \max\limits_{1\leq j\leq n} \left\{ \left\|B_i B_j^*\right\|_{\widehat{Q}} \right\} \\ \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^m\right)^{\frac{1}{m}} \left[\sum\limits_{i=1}^{n} \left(\max\limits_{1\leq j\leq n} \left\{ \left\|B_i B_j^*\right\|_{\widehat{Q}} \right\}\right)^l\right]^{\frac{1}{l}} \\ \qquad\qquad \text{where m, l} > 1,\ \frac{1}{\text{m}} + \frac{1}{\text{l}} = 1; \\ \\ \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1\leq i,j\leq n} \left\{ \left\|B_i B_j^*\right\|_{\widehat{Q}} \right\}, \end{cases}$$

giving the last part of (2.1). □

Remark 2.2. It is obvious that out of (2.1) one can obtain various particular inequalities. For instance, the choice $t = 2$, $p = 2$ (therefore $u = q = 2$) in the $B-$branch of (2.2) gives:

$$\left\|\sum_{i=1}^{n} A_i B_i\right\|_Q^2 \leq \sum_{i=1}^{n} \|A_i\|^2 \left(\sum_{i,j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}^2\right)^{\frac{1}{2}} \tag{2.4}$$

$$\leq \sum_{i=1}^{n} \|A_i\|^2 \left(\sum_{i,j=1}^{n} \|B_i^* B_i\|_{\widehat{Q}}^2 \left\|B_j^* B_j^*\right\|_{\widehat{Q}}^2\right)^{\frac{1}{2}}$$

$$= \sum_{i=1}^{n} \|A_i\|^2 \left(\sum_{i=1}^{n} \|B_i\|_Q^2 \sum_{j=1}^{n} \|B_j\|_Q^2\right)^{\frac{1}{2}}, \tag{2.5}$$

$$= \sum_{i=1}^{n} \|A_i\|^2 \sum_{i=1}^{n} \|B_i\|_Q^2 \; .$$

If we consider now the usual Cauchy–Bunyakovsky–Schwarz (CBS) inequality

$$\left\|\sum_{i=1}^{n} A_i B_i\right\|_Q^2 \leq \sum_{i=1}^{n} \|A_i\|_Q^2 \sum_{i=1}^{n} \|B_i\|_Q^2 \, , \tag{2.6}$$

then we can conclude that (2.4) is a refinement of the (CBS) inequality (2.6) whenever the operator norm $\|\cdot\|$ is majorized by $\|\cdot\|_Q$.

Corollary 2.3. *Let $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in C_Q$ so that $B_i B_j^* = 0$ with $i \neq j$. Then*

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leq \begin{cases} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{cases} \tag{2.7}$$

where

$$\tilde{A} := \begin{cases} \max\limits_{1 \leq k \leq n} \|A_k\|^2 \sum\limits_{i=1}^{n} \|B_i\|_Q^2 ; \\[2ex] \max\limits_{1 \leq k \leq n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|A_i\|^r \right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n} \|B_i\|_Q^{2s} \right)^{\frac{1}{s}}, \\ \qquad\qquad \text{where r} > 1,\ \frac{1}{\text{r}} + \frac{1}{\text{s}} = 1; \\[2ex] \max\limits_{1 \leq k \leq n} \|A_k\| \sum\limits_{i=1}^{n} \|A_i\| \max\limits_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}, \end{cases}$$

$$\tilde{B} := \begin{cases} \max\limits_{1 \leq i \leq n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p \right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} \|B_i\|_Q^2 ; \\[2ex] \left(\sum\limits_{i=1}^{n} \|A_i\|^t \right)^{\frac{1}{t}} \left(\sum\limits_{k=1}^{n} \|A_k\|^p \right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n} \|B_i\|_Q^{2u} \right]^{\frac{1}{u}}, \\ \qquad\qquad \text{where t} > 1,\ \frac{1}{\text{t}} + \frac{1}{\text{u}} = 1; \\[2ex] \sum\limits_{i=1}^{n} \|A_i\| \left(\sum\limits_{k=1}^{n} \|A_k\|^p \right)^{\frac{1}{p}} \max\limits_{1 \leq i \leq n} \left\{ \|B_i\|_Q^2 \right\}, \end{cases}$$

where $p > 1$ and

$$\tilde{C} := \begin{cases} \max\limits_{1 \leq i \leq n} \|A_i\| \sum\limits_{k=1}^{n} \|A_k\| \sum\limits_{i=1}^{n} \|B_i\|_Q^2 ; \\[2ex] \left(\sum\limits_{i=1}^{n} \|A_i\|^m \right)^{\frac{1}{m}} \sum\limits_{k=1}^{n} \|A_k\| \left(\sum\limits_{i=1}^{n} \|B_i\|_Q^{2l} \right)^{\frac{1}{l}}, \\ \qquad\qquad \text{where m,l} > 1,\ \frac{1}{\text{m}} + \frac{1}{\text{l}} = 1; \\[2ex] \left(\sum\limits_{k=1}^{n} \|A_k\| \right)^2 \max\limits_{1 \leq i,j \leq n} \left\{ \|B_i\|_Q^2 \right\}. \end{cases}$$

As in the proof of the Theorem 2.1 one has

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leq \sum_{i=1}^{n} \|A_i\| \cdot \left(\sum_{j=1}^{n} \|A_j\| \cdot \|B_i B_j^*\|_{\widehat{Q}} \right) =: M$$

Using Hölder's generalized inequality from Lemma 1.3, and assuming that $\|A_j\| > 1$, $\|B_i B_j^*\|_{\widehat{Q}} > 1$, we get

$$\left(\sum_{j=1}^{n} \|A_j\| \cdot \|B_i B_j^*\|_{\widehat{Q}} \right) \leq \left(\sum_{k=1}^{n} \|A_k\|^p \right)^{\frac{q}{p+q}} \left(\sum_{j=1}^{n} \|B_i B_j^*\|_{\widehat{Q}}^q \right)^{\frac{P}{p+q}}.$$

Thus the following result is true:

Theorem 2.4. *Let $A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \ldots, B_n \in C_Q$. Assume that $\|A_j\| > 1$ and $\|B_i B_j^*\|_{\widehat{Q}} > 1$ for all $i, j \in \{1, 2, \ldots, n\}$. Then*

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leq \left(\sum_{k=1}^{n} \|A_k\|^p \right)^{\frac{q}{p+q}} \cdot \sum_{i=1}^{n} \|A_i\| \cdot \left(\sum_{j=1}^{n} \|B_i B_j^*\|_Q^q \right)^{\frac{p}{p+q}},$$

where $\frac{1}{p} + \frac{1}{q} < 1$.

3. Other Results

A different approach is embodied in the following theorem:

Theorem 3.1. *If $A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \ldots, B_n \in C_Q$, then*

$$\left\| \sum_{i=1}^{n} A_i B_i \right\|_Q^2 \leq \sum_{i=1}^{n} \|A_i\|^2 \sum_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}} \tag{3.1}$$

$$\leq \begin{cases} \sum_{i=1}^{n} \|A_i\|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}} \right]; \\ \left(\sum_{i=1}^{n} \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}} \right)^q \right]^{\frac{1}{q}} \\ \qquad \text{where p} > 1,\ \frac{1}{\text{p}} + \frac{1}{\text{q}} = 1; \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}}. \end{cases}$$

Proof. From the proof of Theorem 2.1 we have that

$$\left\|\sum_{i=1}^{n} A_i B_i\right\|_Q^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\| \, \|A_j\| \, \left\|B_i B_j^*\right\|_{\widehat{Q}} .$$

Using the simple observation that

$$\|A_i\| \, \|A_j\| \leq \frac{1}{2}\left(\|A_i\|^2 + \|A_j\|^2\right), \quad i, j \in \{1, \dots, n\},$$

we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \|A_i\| \, \|A_j\| \, \left\|B_i B_j^*\right\|_{\widehat{Q}} &\leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left[\|A_i\|^2 + \|A_j\|^2\right] \left\|B_i B_j^*\right\|_{\widehat{Q}} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left[\|A_i\|^2 \left\|B_i B_j^*\right\|_{\widehat{Q}} + \|A_j\|^2 \left\|B_j B_i^*\right\|_{\widehat{Q}}\right] \\ &= \sum_{i=1}^{n} \|A_i\|^2 \sum_{j=1}^{n} \left\|B_i B_j^*\right\|_{\widehat{Q}} ,\end{aligned}$$

which proves the first inequality in (3.1).

The second part follows by Hölder's inequality and the details are omitted.

□

Remark 3.2. If in (3.1) we choose $A_1 = \cdots = A_n = I$, then we get

$$\left\|\sum_{i=1}^{n} B_i\right\|_{\widehat{Q}} \leq \left(\sum_{i=1}^{n} \|B_i\|_Q^2 + \sum_{1 \leq i \neq j \leq n} \left\|B_i B_j^*\right\|_{\widehat{Q}}\right)^{1/2} \leq \sum_{i=1}^{n} \|B_i\|_{\widehat{Q}} ,$$

which is a refinement for the generalized triangle inequality for $\|\cdot\|_{\widehat{Q}}$, whenever $\|\cdot\|_Q$ is majorized by $\|\cdot\|_{\widehat{Q}}$.

The following corollary may be stated:

Corollary 3.3. *If $B_1, \ldots, B_n \in C_Q$ are such that $B_i B_j^* = 0$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$, then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \|A_i\|^2 \|B_i\|_Q^2 \tag{3.2}$$

$$\leq \begin{cases} \sum_{i=1}^n \|A_i\|^2 \max_{1 \leq i \leq n} \|B_i\|_Q^2 ; \\ \\ \left(\sum_{i=1}^n \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum_{j=1}^n \|B_i\|_Q^{2q} \right]^{\frac{1}{q}} \\ \qquad \text{where p} > 1, \ \frac{1}{\text{p}} + \frac{1}{\text{q}} = 1; \\ \\ \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i=1}^n \|B_i\|_Q^2 \end{cases}$$

Theorem 3.4. *If $A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \ldots, B_n \in C_Q$, then*

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \begin{cases} \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \left\| B_i B_j^* \right\|_{\widehat{Q}} ; \\ \\ \left(\sum_{i=1}^n \|A_i\|^p \right)^{\frac{2}{p}} \left(\sum_{i,j=1}^n \left\| B_i B_j^* \right\|_{\widehat{Q}}^q \right)^{\frac{1}{q}} \\ \qquad \text{where p} > 1, \ \frac{1}{\text{p}} + \frac{1}{\text{q}} = 1; \\ \\ \left(\sum_{i=1}^n \|A_i\| \right)^2 \max_{1 \leq i,j \leq n} \left\{ \left\| B_i B_j^* \right\|_{\widehat{Q}} \right\}. \end{cases} \tag{3.3}$$

Proof. We know that

$$\left\| \sum_{i=1}^n A_i B_i \right\|_Q^2 \leq \sum_{i=1}^n \sum_{j=1}^n \|A_i\| \|A_j\| \left\| B_i B_j^* \right\|_{\widehat{Q}} =: P.$$

Firstly, we obviously have that

$$P \leq \max_{1 \leq i,j \leq n} \{ \|A_i\| \|A_j\| \} \sum_{i,j=1}^n \left\| B_i B_j^* \right\|_{\widehat{Q}} = \max_{1 \leq i \leq n} \|A_i\|^2 \sum_{i,j=1}^n \left\| B_i B_j^* \right\|_{\widehat{Q}} .$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned}
P &\le \left[\sum_{i,j=1}^{n} (\|A_i\|\,\|A_j\|)^p\right]^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{n} \|A_i\|^p \sum_{j=1}^{n} \|A_j\|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{n} \|A_i\|^p\right)^{\frac{2}{p}} \left(\sum_{i,j=1}^{n} \left\|B_iB_j^*\right\|_{\widehat{Q}}^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we have

$$\begin{aligned}
P &\le \max_{1\le i,j\le n} \left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\} \sum_{i,j=1}^{n} \|A_i\|\,\|A_j\| \\
&= \left(\sum_{i=1}^{n} \|A_i\|\right)^2 \max_{1\le i,j\le n} \left\{\left\|B_iB_j^*\right\|_{\widehat{Q}}\right\}
\end{aligned}$$

and the theorem is proved. □

Corollary 3.5. *If $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ and $B_1, \dots, B_n \in C_Q$ are such that $B_iB_j^* = 0$ for $i, j \in \{1, \dots, n\}$ with $i \ne j$, then*

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \le \begin{cases} \max\limits_{1\le i\le n} \|A_i\|^2 \sum\limits_{i=1}^{n} \|B_i\|_Q^2 ; \\[2ex] \left(\sum\limits_{i=1}^{n} \|A_i\|^p\right)^{\frac{2}{p}} \left(\sum\limits_{i=1}^{n} \|B_i\|_Q^{2q}\right)^{\frac{1}{q}}, \\ \qquad \text{where p} > 1,\ \frac{1}{\text{p}} + \frac{1}{\text{q}} = 1; \\[2ex] \left(\sum\limits_{i=1}^{n} \|A_i\|\right)^2 \max\limits_{1\le i\le n} \left\{\|B_i\|_Q^2\right\}. \end{cases} \tag{3.4}$$

Theorem 3.6. *If $\dfrac{1}{p}+\dfrac{1}{q} < 1$ and $\|A_i\| > 1$ and $\|B_iB_j^*\| > 1$ for all $i, j = \overline{1, n}$, then*

$$\left\|\sum_{i=1}^{n} A_iB_i\right\|_Q^2 \le \left(\sum_{i=1}^{n} \|A_i\|^p\right)^{\frac{2q}{p+q}} \left(\sum_{i,j=1}^{n} \|B_iB_j^*\|_{\widehat{Q}}^q\right)^{\frac{p}{p+q}}.$$

Proof. Using Lemma 1.3 for double sums, we have, as in the proof of Theorem 3.4, that

$$p = \sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|\cdot\|A_j\|\cdot\|B_iB_j^*\|_{\widehat{Q}} \leq \left(\sum_{i,j=1}^{n}(\|A_i\|\cdot\|A_j\|)^p\right)^{\frac{q}{p+q}}\cdot\left(\sum_{i,j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}}^q\right)^{\frac{p}{p+q}},$$

if we assume that $\|A_i\| > 1$ and $\|B_iB_j^*\| > 1$ for all $i, j = \overline{1,n}$.

Thus, we get

$$p \leq \left(\sum_{i=1}^{n}\|A_i\|^p\right)^{\frac{2q}{p+q}}\cdot\left(\sum_{i,j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}}^q\right)^{\frac{p}{p+q}},$$

if $\dfrac{1}{p}+\dfrac{1}{q}<1$. □

Finally, the following result may be stated as well:

Theorem 3.7. *If* $A_1,\dots,A_n \in \mathbb{B}(\mathcal{H})$ *and* $B_1,\dots,B_n \in C_Q$*, then*

$$\left\|\sum_{i=1}^{n}A_iB_i\right\|_Q^2 \leq \left(\frac{1}{p}\sum_{i=1}^{n}\|A_i\|^p + \frac{1}{q}\sum_{i=1}^{n}\|A_i\|^q\right)\max_{1\leq i\leq n}\left[\sum_{j=1}^{n}\|B_iB_j^*\|_Q\right].$$

Proof. As in the proof of Theorem 3.1 we have

$$\left\|\sum_{i=}^{n}A_iB_i\right\|_Q^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|\cdot\|A_j\|\cdot\|B_iB_j^*\|_{\widehat{Q}}.$$

Now, using Lemma 1.1, we can write

$$\|A_i\|\cdot\|A_j\| \leq \frac{1}{p}\|A_i\|^p + \frac{1}{q}\|A_j\|^q.$$

Thus, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|\cdot\|A_j\|\cdot\|B_iB_j^*\|_{\widehat{Q}} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{1}{p}\|A_i\|^p+\frac{1}{q}\|A_j\|^q\right]\|B_iB_j^*\|_{\widehat{Q}} =$$
$$= \frac{1}{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|^p\cdot\|B_iB_j^*\|_{\widehat{Q}}+\frac{1}{q}\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_j\|^q\cdot\|B_iB_j^*\|_{\widehat{Q}} =$$
$$= \frac{1}{p}\sum_{i=1}^{n}\|A_i\|^p\cdot\sum_{j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}}+\frac{1}{q}\sum_{j=1}^{n}\|A_j\|^q\cdot\sum_{i=1}^{n}\|B_iB_j^*\|_{\widehat{Q}} \le$$
$$\le \left(\frac{1}{p}\sum_{i=1}^{n}\|A_i\|^p+\frac{1}{q}\sum_{j=1}^{n}\|A_j\|^q\right)\max_{1\le i\le n}\left[\sum_{j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}}\right].$$

□

Remark 3.8. Another variant may be obtained via Lemma 1.2:

$$\|A_i\|\cdot\|A_j\| \le \frac{1}{4}\left(\|A_i\|+\|A_j\|\right)^2, \text{ so}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\|A_i\|\cdot\|A_j\|\cdot\|B_iB_j^*\|_{\widehat{Q}} \le \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|A_i\|+\|A_j\|\right)^2\cdot\|B_iB_j^*\|_{\widehat{Q}} \le$$
$$\le \sup_{1\le i\le n}\sum_{j=1}^{n}\|B_iB_j^*\|_{\widehat{Q}}\left(\sum_{i,j=1}^{n}\left(\|A_i\|+\|A_j\|\right)^2\right).$$

4. Applications

Throughout this section we assume $\|\cdot\|_Q$ (and hence $\|\cdot\|_{\widehat{Q}}$) to be the operator norm. If by $M(\mathbf{B},\mathbf{A})$ we denote any of the bounds provided by (2.1), (2.4), (3.1) or (3.3) for the quantity $\left\|\sum_{i=1}^{n}A_iB_i\right\|^2$, then we may state the following general fact:

Under the assumptions of Theorem 2.1, we have:

$$\left\|\sum_{i=1}^{n}A_iB_ix\right\|^2 \le \|x\|^2 M(\mathbf{B},\mathbf{A}). \tag{4.1}$$

for any $x \in \mathcal{H}$ *and*

$$\left| \sum_{i=1}^{n} \langle A_i B_i x, y \rangle \right|^2 \le \|x\|^2 \|y\|^2 M(\mathbf{B}, \mathbf{A}). \tag{4.2}$$

for any $x, y \in \mathcal{H}$, *respectively.*

The proof follows by the Schwarz inequality in the Hilbert space $(\mathcal{H}, \langle .,. \rangle)$.

Now, we consider the non zero vectors $y_1, \ldots, y_n \in \mathcal{H}$. Define $B_i : \mathcal{H} \to \mathcal{H}$ by $B_i = \frac{y_i \otimes y_i}{\|y_i\|}$ $(1 \le i \le n)$. Then

$$\left\| B_i B_j^* \right\| = \frac{\| \langle y_j, y_i \rangle y_i \otimes y_j \|}{\|y_i\| \|y_j\|} = |\langle y_i, y_j \rangle| ; \quad i, j \in \{1, \ldots, n\}.$$

If $(y_i)_{i=\overline{1,n}}$ is an orthogonal family on $\mathcal{H}$, then $\|B_i\| = 1$ and $B_i B_j = 0$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$.

Now, utilising, for instance, the inequalities in Theorem 3.1 we may state that:

$$\left\| \sum_{i=1}^{n} A_i \frac{\langle x, y_i \rangle}{\|y_i\|} y_i \right\|^2 \le \|x\|^2 \sum_{i=1}^{n} \|A_i\|^2 \sum_{j=1}^{n} |\langle y_i, y_j \rangle| \tag{4.3}$$

$$\le \|x\|^2 \times \begin{cases} \sum\limits_{i=1}^{n} \|A_i\|^2 \max\limits_{1 \le i \le n} \left[\sum\limits_{j=1}^{n} |\langle y_i, y_j \rangle| \right]; \\ \\ \left(\sum\limits_{i=1}^{n} \|A_i\|^{2p} \right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |\langle y_i, y_j \rangle| \right)^q \right]^{\frac{1}{q}} \\ \qquad \text{where p} > 1, \ \frac{1}{\text{p}} + \frac{1}{\text{q}} = 1; \\ \\ \max\limits_{1 \le i \le n} \|A_i\|^2 \sum\limits_{i,j=1}^{n} |\langle y_i, y_j \rangle|. \end{cases}$$

for any $x, y_1, \ldots, y_n \in \mathcal{H}$ and $A_1, \ldots, A_n \in \mathbb{B}(\mathcal{H})$.

The choice $A_i = \|y_i\| I$ $(i = 1, \cdots, n)$ will produce some interesting bounds for the norm of the Fourier series $\|\sum_{i=1}^{n} \langle x, y_i \rangle\|$. Notice the vectors y_i $(i = 1, \cdots, n)$ are not necessarily orthonormal.

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