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NEW GENERALIZATIONS OF OSTROWSKI'S INEQUALITY ON TIME SCALES

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ABSTRACT. In this short paper, a time scales version of Ostrowski's inequality is further generalized via the ∇-integral and $\Diamond_\alpha$-dynamic integral, which is defined as a linear combination of the Δ- and ∇ integrals.

1. INTRODUCTION

The original renowned Ostrowski's inequality [5] reads as follows

Theorem A. *Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two non-proportional sequences of real numbers and*

$$|a| = \sum_{i=1}^{n} a_i^n \quad , \quad |b| = \sum_{i=1}^{n} b_i^n \quad \text{and} \quad a.b = \sum_{i=1}^{n} a_i b_i.$$

If $x = (x_1, x_2, ..., x_n)$ is any sequence of real numbers satisfying

$$\sum_{i=1}^{n} a_i x = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i x = 1$$

then

$$\sum_{i=1}^{n} x_i^2 \geqq \frac{|a|}{|a|\,|b| - |a.b|^2}.$$

Equality occurs if and only if

$$x_k = \frac{b_k\,|a| - a_k\,|b|}{|a|\,|b| - |a.b|^2} \quad \text{for all} \quad k = \overline{1, n}.$$

The development of the theory of time scales was initiated by Hilger [3] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales. For example, we refer the reader to [1, 2, 4]. In [9], Theorem A, by using Δ-integral, was generalized as follows.

Theorem B. *Let $f, g \in C_{rd}([a, b], \mathbb{R})$ be two linearly independent functions and*

$$A = \int_a^b f^2(t)\,\Delta t \quad , \quad B = \int_a^b g^2(t)\,\Delta t \quad \text{and} \quad C = \int_a^b f(t)\,g(t)\,\Delta t.$$

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(A) *If $x \in C_{rd}([a,b],\mathbb{R})$ is any function such that*

$$\int_a^b f(t)\,x(t)\Delta t = 0 \quad \textit{and} \quad \int_a^b g(t)\,x(t)\Delta t = 1 \tag{1}$$

then

$$\int_a^b x^2(t)\Delta t \geqq \frac{A}{AB - C^2} \tag{2}$$

with equality if and only if

$$x(t) = \frac{Ag(t) - Bf(t)}{AB - C^2} \quad \textit{a.e. on} \quad [a,b]\,.$$

(B) *If $y(t) = \frac{Ag(t)-Bf(t)}{AB-C^2}$ on $[a,b]$, then for any $\alpha \in [-1,1]$ and any arbitrary $x \in C_{rd}([a,b],\mathbb{R})$ satisfying condition (1), the function*

$$\alpha x(t) + (1-\alpha)\,y(t) \quad , \quad t \in [a,b]$$

satisfies condition (1) and

$$\int_a^b x^2(t)\Delta t \geqq \int_a^b \left(\alpha x(t) + (1-\alpha)\,y(t)\right)^2 \Delta t \geqq \frac{A}{AB - C^2}. \tag{3}$$

The first inequality in (3) becomes equality if and only if

$$|\alpha| = 1 \quad \textit{or} \quad \int_a^b x^2(t)\Delta t = \int_a^b y^2(t)\Delta t.$$

The second inequality in (3) becomes equality if and only if

$$\alpha = 0 \quad \textit{or} \quad x(t) = y(t) \quad \textit{a.e. on} \quad [a,b]\,.$$

The main aim of this paper is to further generalize Theorem B first using ∇-integral and later $\Diamond_\alpha$-integral on time scale. We refer the reader to [7, 8] for an account of the calculus corresponding to the ∇-derivative and $\Diamond_\alpha$-dynamic derivative respectively. Our main results are included in a couple of theorems below.

Theorem 1. *Let $f, g \in C_{ld}([a,b],\mathbb{R})$ be two linearly independent functions and*

$$A = \int_a^b f^2(t)\,\nabla t \quad , \quad B = \int_a^b g^2(t)\,\nabla t \quad \textit{and} \quad C = \int_a^b f(t)\,g(t)\,\nabla t.$$

(A) *If $x \in C_{rd}([a,b],\mathbb{R})$ is any function such that*

$$\int_a^b f(t)\,x(t)\nabla t = 0 \quad \textit{and} \quad \int_a^b g(t)\,x(t)\nabla t = 1 \tag{4}$$

then

$$\int_a^b x^2(t)\nabla t \geqq \frac{A}{AB - C^2} \tag{5}$$

with equality if and only if

$$x(t) = \frac{Ag(t) - Bf(t)}{AB - C^2} \quad \textit{a.e. on} \quad [a,b]\,.$$

(B) *If* $y(t) = \frac{Ag(t)-Bf(t)}{AB-C^2}$ *on* $[a,b]$*, then for any* $\alpha \in [-1,1]$ *and any arbitrary* $x \in C_{rd}([a,b],\mathbb{R})$ *satisfying condition* (4)*, the function*

$$\alpha x(t) + (1-\alpha)\, y(t) \quad , \quad t \in [a,b]$$

satisfies condition (4) *and*

$$\int_a^b x^2(t)\nabla t \geqq \int_a^b \left(\alpha x(t) + (1-\alpha)\, y(t)\right)^2 \nabla t \geqq \frac{A}{AB-C^2}. \tag{6}$$

The first inequality in (6) *becomes equality if and only if*

$$|\alpha| = 1 \quad or \quad \int_a^b x^2(t)\nabla t = \int_a^b y^2(t)\nabla t.$$

The second inequality in (6) *becomes equality if and only if*

$$\alpha = 0 \quad or \quad x(t) = y(t) \quad a.e. \ on \quad [a,b]\,.$$

Theorem 2. *Let* $f, g : [a,b] \to \mathbb{R}$ *be two linearly independent* $\diamondsuit_\alpha$*-integrable functions and*

$$A = \int_a^b f^2\,(t)\,\diamondsuit_\alpha t \quad , \quad B = \int_a^b g^2\,(t)\,\diamondsuit_\alpha t \quad and \quad C = \int_a^b f\,(t)\,g\,(t)\,\diamondsuit_\alpha t.$$

(A) *If* $x : [a,b] \to \mathbb{R}$ *is any* $\diamondsuit_\alpha$*-integrable function such that*

$$\int_a^b f\,(t)\,x(t)\diamondsuit_\alpha t = 0 \quad and \quad \int_a^b g\,(t)\,x(t)\diamondsuit_\alpha t = 1 \tag{7}$$

then

$$\int_a^b x^2(t)\diamondsuit_\alpha t \geqq \frac{A}{AB-C^2} \tag{8}$$

with equality if and only if

$$x(t) = \frac{Ag\,(t) - Bf\,(t)}{AB-C^2} \quad a.e. \ on \quad [a,b]\,.$$

(B) *If* $y(t) = \frac{Ag(t)-Bf(t)}{AB-C^2}$ *on* $[a,b]$*, then for any* $\alpha \in [-1,1]$ *and any arbitrary* $\diamondsuit_\alpha$*-integrable function* $x : [a,b] \to \mathbb{R}$ *satisfying condition* (7)*, the function*

$$\alpha x(t) + (1-\alpha)\, y(t) \quad , \quad t \in [a,b]$$

satisfies condition (7) *and*

$$\int_a^b x^2(t)\diamondsuit_\alpha t \geqq \int_a^b \left(\alpha x(t) + (1-\alpha)\, y(t)\right)^2 \diamondsuit_\alpha t \geqq \frac{A}{AB-C^2}. \tag{9}$$

The first inequality in (9) *becomes equality if and only if*

$$|\alpha| = 1 \quad or \quad \int_a^b x^2(t)\diamondsuit_\alpha t = \int_a^b y^2(t)\diamondsuit_\alpha t.$$

The second inequality in (9) *becomes equality if and only if*

$$\alpha = 0 \quad or \quad x(t) = y(t) \quad a.e. \ on \quad [a,b]\,.$$

2. Preliminaries

Definition 1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers.*

The calculus of time scales was initiated by Stefan Hilger in his PhD thesis [3] in order to create a theory that can unify discrete and continuous analysis. Let $\mathbb{T}$ be a time scale. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

Definition 2. *Let $\sigma(\mathrm{t})$ and $\rho(\mathrm{t})$ be the forward and backward jump operators in $\mathbb{T}$, respectively. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(\mathrm{t}) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(\mathrm{t}) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(\mathrm{t}) > t$, then we say that t is right-scattered, while if $\rho(\mathrm{t}) < t$ then we say that t is left-scattered.

Point that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(\mathrm{t}) = t$, the t is called *right-dense*, and if $\rho(\mathrm{t}) = t$ then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called dense.

Definition 3. *Let $t \in \mathbb{T}$, then two mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ satisfying*

$$\mu(t) := \sigma(\mathrm{t}) - t \quad , \quad \nu(t) := t - \rho(\mathrm{t})$$

are called the graininess functions.

We now introduce the sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $\mathbb{T}^\kappa_\kappa$ which are derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t, then $\mathbb{T}^\kappa := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. If $\mathbb{T}$ has a right-scattered maximum t, then $\mathbb{T}_\kappa := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. Finally, $\mathbb{T}^\kappa_\kappa := \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(\mathrm{t}) = f(\sigma(t))$ for all $t \in \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\rho : \mathbb{T} \to \mathbb{R}$ by $f^\rho(\mathrm{t}) = f(\rho(t))$ for all $t \in \mathbb{T}$.

Definition 4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(\mathrm{t})$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood U of t such that for all $s \in U$*

$$\left| f^\sigma(\mathrm{t}) - f(s) - f^\Delta(\mathrm{t})\left(\sigma(\mathrm{t}) - s\right) \right| \leqq \varepsilon \left| \sigma(\mathrm{t}) - s \right|.$$

We say that f is Δ-differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(\mathrm{t})$ exists for all $t \in \mathbb{T}^\kappa$. Similarly, for $t \in \mathbb{T}_\kappa$, we define $f^\nabla(\mathrm{t})$ to be the number value, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood V of t such that for all $s \in V$

$$\left| f^\rho(\mathrm{t}) - f(s) - f^\nabla(\mathrm{t})\left(\rho(\mathrm{t}) - s\right) \right| \leqq \varepsilon \left| \rho(\mathrm{t}) - s \right|.$$

We say that f is ∇-differentiable on $\mathbb{T}_\kappa$ provided $f^\nabla(\mathrm{t})$ exists for all $t \in \mathbb{T}_\kappa$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$ $(t \neq \min \mathbb{T})$. Then we have the following

(i) If f is Δ-differentiable at t, then f is continuous at t.

(ii) If f is left continuous at t and t is right-scattered, then f is Δ-differentiable at t with

$$f^{\Delta}(\mathrm{t})=\frac{f^{\sigma}(\mathrm{t})-f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is Δ-differentiable at t if and only if

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists a finite number. In this case

$$f^{\Delta}(\mathrm{t})=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

(iv) If f is Δ-differentiable at t, then

$$f^{\sigma}(\mathrm{t})=f(t)+\mu(t)f^{\Delta}(\mathrm{t}).$$

Assume that $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}_{\kappa}$ ($t\neq\max\mathbb{T}$). Then we have the following

(i) If f is ∇-differentiable at t, then f is continuous at t.

(ii) If f is right continuous at t and t is left-scattered, then f is ∇-differentiable at t with

$$f^{\nabla}(\mathrm{t})=\frac{f(t)-f^{\rho}(\mathrm{t})}{\nu(t)}.$$

(iii) If t is left-dense, then f is ∇-differentiable at t if and only if

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists a finite number. In this case

$$f^{\nabla}(\mathrm{t})=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

(iv) If f is ∇-differentiable at t, then

$$f^{\rho}(\mathrm{t})=f(t)+\nu(t)f^{\nabla}(\mathrm{t}).$$

Remark 1 (See [7])**.** *Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}$. The existence of the Δ-derivative of f at t does not imply the existence of the ∇-derivative at t, and vice versa.*

Definition 5.

(1) *A mapping $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided if it satisfies*
   (a) *f is continuous at each right-dense point or maximal element of $\mathbb{T}$.*
   (b) *The left-sided limit $\lim_{s\to t-}f(s)=f(t-)$ exists at each left-dense point t of $\mathbb{T}$.*

(2) *A mapping $f:\mathbb{T}\to\mathbb{R}$ is called ld-continuous provided if it satisfies*
   (a) *f is continuous at each left-dense point or minimal element of $\mathbb{T}$.*
   (b) *The right-sided limit $\lim_{s\to t+}f(s)=f(t+)$ exists at each right-dense point t of $\mathbb{T}$.*

Remark 2.

(1) *It follows from Theorem 1.74 of Bohner and Peterson* [1] *that every rd-continuous function has an anti-derivative.*

(2) *It follows from Theorem 8.45 of Bohner and Peterson* [1] *that every ld-continuous function has an anti-derivative.*

Definition 6.

(1) *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a* Δ*-antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided* $F^{\Delta}(\mathrm{t}) = f(t)$ *holds for all* $t \in \mathbb{T}^{\kappa}$*. Then the* Δ*-integral of* f *is defined by*

$$\int_a^b f(t)\,\Delta t = F(b) - F(a).$$

(2) *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a* ∇*-antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided* $F^{\nabla}(\mathrm{t}) = f(t)$ *holds for all* $t \in \mathbb{T}_{\kappa}$*. Then the* ∇*-integral of* f *is defined by*

$$\int_a^b f(t)\,\nabla t = F(b) - F(a).$$

Now we briefly introduce the $\diamondsuit_\alpha$-dynamic derivative and the $\diamondsuit_\alpha$-dynamic integration and we refer the reader to [7, 8] for a comprehensive development of the calculus of the $\diamondsuit_\alpha$-dynamic derivative and the $\diamondsuit_\alpha$-dynamic integration.

Definition 7. *Let* $\mathbb{T}$ *be a time scale. We define* $f^{\diamondsuit_\alpha}(t)$ *to be the value, if one exists, such that for all* $\varepsilon > 0$ *there is a neighborhood* U *of* t *such that for all* $s \in U$

$$\left|\alpha\left(f^{\sigma}(t) - f(s)\right)\eta_{ts} + (1-\alpha)\left(f^{\rho}(t) - f(s)\right)\mu_{ts} - f^{\diamondsuit_\alpha}(t)\,\mu_{ts}\eta_{ts}\right| < \varepsilon\left|\mu_{ts}\eta_{ts}\right|.$$

We say that f *is* $\diamondsuit_\alpha$*-differentiable on* $\mathbb{T}^{\kappa}_{\kappa}$ *provided* $f^{\diamondsuit_\alpha}(t)$ *exists for all* $t \in \mathbb{T}^{\kappa}$*.*

Remark 3. *It is clear that* $f^{\diamondsuit_\alpha}(t)$ *reduces to* $f^{\Delta}(\mathrm{t})$ *for* $\alpha = 1$ *and* $f^{\nabla}(\mathrm{t})$ *for* $\alpha = 0$*. Also, the above definition is well-defined, see* [7]*.*

Theorem 3 (See [7], Theorem 3.2)**.** *Let* $0 \leqq \alpha \leqq 1$*. If* f *is both* Δ*- and* ∇*-differentiable at* $t \in \mathbb{T}$*, then* f *is* $\diamondsuit_\alpha$*-differentiable at* t *and*

$$f^{\diamondsuit_\alpha}(t) = \alpha f^{\Delta}(\mathrm{t}) + (1-\alpha) f^{\nabla}(\mathrm{t}).$$

Theorem 4 (See [7], Theorem 3.9)**.** *Let* $\mathbb{T}$ *be a time scale and* $0 < \alpha < 1$*. If* f *is* $\diamondsuit_\alpha$*-differentiable at* t *then* f *is both* Δ*- and* ∇*-differentiable at* t*.*

Remark 4. *Note that the strict inequalities in* $0 < \alpha < 1$ *are necessary for the results above. In the case* $\alpha = 1$*, the* $\diamondsuit_\alpha$*-derivative reduces to the* Δ*-derivative, which does not imply the existence of the* ∇*-derivative. Similarly for* $\alpha = 0$*.*

Definition 8. *Let* $a, t \in \mathbb{T}$ *and* $h : \mathbb{T} \to \mathbb{R}$*. Then the* $\diamondsuit_\alpha$*-integral from* a *to* t *of* h *is defined by*

$$\int_a^t h(\tau)\,\diamondsuit_\alpha\tau = \alpha\int_a^t h(\tau)\,\Delta\tau + (1-\alpha)\int_a^t h(\tau)\,\nabla\tau \quad , \quad 0 \leqq \alpha \leqq 1.$$

You may notice that since $\diamondsuit_\alpha$-integral is a combined Δ- and ∇-integral, we in general do not have

$$\left(\int_a^t h(\tau)\,\diamondsuit_\alpha\tau\right)^{\diamondsuit_\alpha} = h(t) \quad , \quad t \in \mathbb{T}.$$

Throughout this paper, we suppose that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval means the intersection of real interval with the given time scale.

3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1.
Proof of (A). Since $f,g\in C_{ld}([a,b],\mathbb{R})$ then A, B and C are well-defined. Clearly, $AB-C^2>0$. Let

$$y(t):=\frac{Ag\left(t\right)-Cf\left(t\right)}{AB-C^2}\quad,\quad a\leqq t\leqq b.$$

We see that

$$\begin{aligned}\int_a^b y^2(t)\nabla t&=\frac{1}{\left(AB-C^2\right)^2}\int_a^b\left(Ag\left(t\right)-Cf\left(t\right)\right)^2\nabla t\\&=\frac{1}{\left(AB-C^2\right)^2}\int_a^b\left(A^2g^2\left(t\right)-2ACf\left(t\right)g\left(t\right)+C^2f^2\left(t\right)\right)\nabla t\\&=\frac{A^2B-2AC^2+C^2A}{\left(AB-C^2\right)^2}\\&=\frac{A}{AB-C^2},\end{aligned}$$

and

$$\begin{aligned}\int_a^b f\left(t\right)y(t)\nabla t&=\frac{1}{AB-C^2}\int_a^b\left(Af\left(t\right)g\left(t\right)-Cf^2\left(t\right)\right)\nabla t\\&=\frac{1}{AB-C^2}\int_a^b Af\left(t\right)g\left(t\right)\nabla t-\frac{1}{AB-C^2}\int_a^b Cf^2\left(t\right)\nabla t\\&=0,\end{aligned}$$

and

$$\begin{aligned}\int_a^b g\left(t\right)y(t)\nabla t&=\frac{1}{AB-C^2}\int_a^b\left(Ag^2\left(t\right)-Cf\left(t\right)g\left(t\right)\right)\nabla t\\&=\frac{1}{AB-C^2}\int_a^b Ag^2\left(t\right)\nabla t-\frac{1}{AB-C^2}\int_a^b Cf\left(t\right)f\left(t\right)\nabla t\\&=1.\end{aligned}$$

Now for $x(t)\in C_{ld}([a,b],\mathbb{R})$ satisfying (4), we have

$$\begin{aligned}\int_a^b x(t)y(t)\nabla t&=\frac{1}{AB-C^2}\int_a^b\left(Ag\left(t\right)x(t)-Cf\left(t\right)x(t)\right)\nabla t\\&=\frac{A}{AB-C^2}\int_a^b g\left(t\right)x(t)\nabla t\\&=\frac{A}{AB-C^2}\\&=\int_a^b y^2(t)\nabla t.\end{aligned}$$

On the other hand,

$$\begin{aligned}0\leqq\int_a^b\left(x(t)-y(t)\right)^2\nabla t&=\int_a^b x^2(t)\nabla t-2\int_a^b x(t)y(t)\nabla t+\int_a^b y^2(t)\nabla t\\&=\int_a^b x^2(t)\nabla t-\int_a^b y^2(t)\nabla t.\end{aligned}$$

It follows that

$$\int_a^b x^2(t)\nabla t \geqq \int_a^b y^2(t)\nabla t \geqq \frac{A}{AB - C^2}$$

which gives the desires result.
Proof of (B). For $-1 \leqq \alpha \leqq 1$,

$$\begin{aligned}
\int_a^b x^2(t)\nabla t &= \int_a^b \alpha^2 x^2(t)\nabla t + \int_a^b \left(1-\alpha^2\right) x^2(t)\nabla t \\
&\geqq \int_a^b \alpha^2 x^2(t)\nabla t + \int_a^b \left(1-\alpha^2\right) y^2(t)\nabla t \\
&= \int_a^b \alpha^2 x^2(t)\nabla t + \int_a^b \left[(1-\alpha)^2 + 2\alpha(1-\alpha)\right] y^2(t)\nabla t \\
&= \int_a^b \alpha^2 x^2(t)\nabla t + \int_a^b 2\alpha(1-\alpha) y^2(t)\nabla t + \int_a^b (1-\alpha)^2 y^2(t)\nabla t \\
&= \int_a^b \alpha^2 x^2(t)\nabla t + \int_a^b 2\alpha(1-\alpha) x(t)y(t)\nabla t + \int_a^b (1-\alpha)^2 y^2(t)\nabla t \\
&= \int_a^b \left(\alpha x(t) + (1-\alpha) y(t)\right)^2 \nabla t
\end{aligned}$$

which completes the proof of the first part inequality in (6). Equality occurs if and only if either $|\alpha| = 1$ or

$$\int_a^b x^2(t)\nabla t = \int_a^b y^2(t)\nabla t.$$

Next,

$$\begin{aligned}
&\int_a^b (\alpha x(t) + (1-\alpha)y(t)^2\nabla t = \int_a^b \left(\alpha^2 x^2(t) + 2\alpha(1-\alpha)x(t)y(t) + (1-\alpha)^2 y^2(t)\right)\nabla t \\
&\quad = \alpha^2 \int_a^b \left(x^2(t) - 2x(t)y(t) + y^2(t)\right)\nabla t + (1-2\alpha)\int_a^b y^2(t)\nabla t + 2\alpha\int_a^b x(t)y(t)\nabla t \\
&\quad = \alpha^2 \int_a^b (x(t) - y(t))^2 \nabla t + \frac{A}{AB - C^2} \\
&\quad \geqq \frac{A}{AB - C^2}.
\end{aligned}$$

Equality occurs if and only if either $\alpha = 0$ or $x(t) = y(t)$ a.e. on $[a, b]$. □

Proof of Theorem 2. This Theorem is a direct extension of the Theorem 1. So we omit its proof. □

4. Discussion

Integral inequalities play an important role in the development of a time scales calculus. Ostrowski's inequality on the time scales was obtained by Yeh [9] using Δ-integral, we offered inequalities first using ∇-integral and later $\Diamond_\alpha$-integral which is the linear combination of the Δ- and ∇-integrals.

References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Series*, Birkhäuser, Boston, 2001.

[2] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Series*, Birkhäuser, Boston, 2003.

[3] S. Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrmsmannigfaltingkeiten*, PhD thesis, Univarsi. Würzburg, 1988.

[4] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, 1996.

[5] A.M. Ostrowski, *Vorlesungen über Differential-und Integralrechnung II*, Basel, 1951.

[6] U. M. Özkan, M.Z. Sarikaya, and H.Yildirim, Extensions of certain integral inequalities on time scales, *Applied Mathematics Letters* (2007), doi:10.1016/j.aml.2007.06.008.

[7] James W. Rogers and Jr. Qin Sheng, Notes on the diamond-α dynamic derivative on time scales, *J. Math. Anal. Appl.* **326** (2007), 228-241.

[8] Q. Sheng, M. Fadag, J. Henderson, and J.M. Davi, An exploration of combined dynamic derivatives on time scales and their applications, *Nonlinear Anal (RWA)*, **7**(2006), 395-413.

[9] C.-C. Yeh, Ostrowski's inequality on time scales, *Applied Mathematics Letters* (2007), doi:10.1016/j.aml.2007.03.024.

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