Generalizations of Mitrinović's inequality and their applications

Mihály Bencze[1], Yan-Ping Zeng[2] and Shan-He Wu[2]
[1]Str. Harmanului 6, 505600 Sacele-Négyfalu, Jud. Brasov, Romania
E-mail: benczemihaly@yahoo.com
[2]Department of Mathematics and Computer Science, Longyan University, Longyan Fujian 364012, P.R. China
E-mail: wushanhe@yahoo.com.cn

Abstract: By introducing three parameters, a new generalization of Mitrinović's inequality is given, which contains several earlier results as special cases. The main result is then used to establish some parameterized triangle inequalities.
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1 Generalizations of Mitrinović's inequality

Let A, B, C be the angles of a triangle. Then

$$\cos A + \sqrt{2}\,(\cos B + \cos C) \le 2. \tag{1}$$

Inequality (1) is known in literature as Mitrinović's inequality [1, p.125]. We show here some generalizations of Mitrinović's inequality.

Theorem 1. *Let* $\lambda,\ \mu,\ A,\ B,\ C$ *be positive numbers with* $A+B+C=\theta,\ 0<\theta\le\pi$. *Then*

$$\cos A + \lambda\cos B + \mu\cos C \le \left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}+\lambda\mu\right)\cos\frac{\theta}{3}. \tag{2}$$

Proof. Note that the following result due to Wu and Debnath [2]:

$$yz\cos A + zx\cos B + xy\cos C \le (x^2+y^2+z^2)\cos\frac{\theta}{3}, \tag{3}$$

where $x,\ y,\ z,\ A,\ B,\ C\in\mathbb{R}^+$, $A+B+C=\theta,\ 0<\theta\le\pi$.
By using a substitution $x\to x,\ y\to\frac{x}{\lambda},\ z\to\frac{x}{\mu}$ $(x>0,\lambda>0,\mu>0)$ in (3), we obtain

$$\frac{x^2}{\lambda\mu}\cos A+\frac{x^2}{\mu}\cos B+\frac{x^2}{\lambda}\cos C\le\left(x^2+\frac{x^2}{\lambda^2}+\frac{x^2}{\mu^2}\right)\cos\frac{\theta}{3},$$

which leads to the desired inequality (2). The proof of Theorem 1 is complete.

Remark 1. Putting $\lambda=\mu=x$ in (2), we get the following result:

Corollary 1. *Let $x,\ A,\ B,\ C$ be positive numbers with $A+B+C=\theta,\ 0<\theta\le\pi$. Then*

$$\cos A + x(\cos B + \cos C) \le \left(x^2+2\right)\cos\frac{\theta}{3}. \tag{4}$$

In a special case when $\theta=\pi$, the inequality (4) reduce to the following generalization of Mitrinović's inequality.

Corollary 2. *Let x be a positive number, then for any triangle ABC the following inequality holds*

$$\cos A + x(\cos B + \cos C) \le \frac{x^2}{2}+1. \tag{5}$$

Remark 2. We can show that the inequality (5) holds for $x\in\mathbb{R}$ by the following fact.

$$\begin{aligned}
&\cos A + x(\cos B+\cos C) - \frac{2+x^2}{2}\\
&= -\cos(B+C) + x(\cos B+\cos C) - \frac{2+x^2}{2}\\
&= -\frac{1}{2}\left(2\cos B\cos C - 2\sin B\sin C - 2x\cos B - 2x\cos C + 2 + x^2\right)\\
&= -\frac{1}{2}\left[(\cos B+\cos C - x)^2 + (\sin B - \sin C)^2\right]\\
&\le 0.
\end{aligned}$$

Based on the above arguments, we have the following further extension of the Mitrinović's inequality.

Corollary 3. *Let x be a real number, then for any triangle ABC the following inequality holds*

$$\cos A + x(\cos B + \cos C) \le \frac{x^2}{2}+1. \tag{6}$$

Remark 3. If we put in the inequality (5) or (6) $x=\sqrt{2}$, the Mitrinović's inequality (1) is derived.

2 Some applications

In what follows, we denote by A, B, C the angles of a triangle, s, R and r denote respectively the semi-perimeter, circumradius and inradius of a triangle. We will customarily use the notations of cyclic sum and cyclic product such as

$$\sum f(A) = f(A)+f(B)+f(C), \quad \prod f(A) = f(A)f(B)f(C).$$

Proposition 1. *Let x be a positive number, then for any triangle ABC the following inequality holds*

$$\frac{18R}{(3x^2+4)R-2r} \le \sum \frac{1}{1+\frac{x^2}{2}-\cos A} \le \frac{R(s^2+8Rr+5r^2)}{xr(s^2+2Rr+r^2)}. \tag{7}$$

Proof. It follows from Corollary 2 that

$$1+\frac{x^2}{2}-\cos A \ge x(\cos B+\cos C),$$

we hence have

$$\begin{aligned}\sum \frac{1}{1+\frac{x^2}{2}-\cos A} &\le \frac{1}{x}\sum \frac{1}{\cos B+\cos C}\\ &= \frac{\sum (\cos A+\cos B)(\cos A+\cos C)}{x\prod (\cos B+\cos C)}.\end{aligned}$$

Now, using the identities (see [3]):

$$\begin{aligned}\sum (\cos A+\cos B)(\cos A+\cos C) &= \frac{s^2+8Rr+5r^2}{4R^2},\\ \prod (\cos A+\cos B) &= \frac{r(s^2+2Rr+r^2)}{4R^3},\end{aligned}$$

we obtain that

$$\sum \frac{1}{1+\frac{x^2}{2}-\cos A} \le \frac{R(s^2+8Rr+5r^2)}{xr(s^2+2Rr+r^2)}.$$

In addition, by applying the Cauchy-Schwarz inequality [4] and the identity $\sum \cos A = (R+r)/R$, we have

$$\sum \frac{1}{1+\frac{x^2}{2}-\cos A} \ge \frac{9}{3+\frac{3}{2}x^2-\cos A-\cos B-\cos C} \ge \frac{18R}{(3x^2+4)R-2r}.$$

The inequality (7) is proved.

Putting $x=\sqrt{2}$ in (7), we get the following result:

Proposition 2. *For any triangle ABC we have the inequality*

$$\frac{9R}{5R-r} \le \sum \frac{1}{2-\cos A} \le \frac{R(s^2+8Rr+5r^2)}{\sqrt{2}r(s^2+2Rr+r^2)}. \tag{8}$$

Proposition 3. *In all triangle ABC, if $x>1$, then*

$$\min\{\cos A,\cos B,\cos C\} \ge \frac{2x(R+r)-(x^2+2)R}{2R(x-1)}. \tag{9}$$

If $x<1$, then

$$\max\{\cos A,\cos B,\cos C\} \le \frac{2x(R+r)-(x^2+2)R}{2R(x-1)}. \tag{10}$$

Proof. From Corollary 3 we have

$$\cos A + x(\cos B+\cos C) \le \frac{x^2}{2}+1$$

$$\Longleftrightarrow (1-x)\cos A + x(\cos A+\cos B+\cos C) \le \frac{x^2}{2}+1. \tag{11}$$

Applying the identity $\sum \cos A = (R+r)/R$ to inequality (11), we deduce the inequalities (9) and (10) immediately.

Proposition 4. *Let x be a real number, then for any triangle ABC the following inequality holds*

$$2(R+r)(x-1) \ge 6x(R+r)-3(x^2+2)R \tag{12}$$

Proof. From inequality (11) we have

$$(1-x)\sum\cos A+3x\sum\cos A\leq 3\left(\frac{x^2}{2}+1\right). \tag{13}$$

Applying the identity $\sum\cos A=(R+r)/R$ to inequality (13) leads to the inequality (12).
Proposition 5. *Let x be a real number, then for any triangle ABC the following inequality holds*

$$\sum\left(\frac{x^2}{2}+1-\cos A\right)\left(\frac{x^2}{2}+1-\cos B\right)\geq\frac{x^2(s^2+8Rr+5r^2)}{4R^2}. \tag{14}$$

Proof. From Corollary 3 we have

$$\cos A+x(\cos B+\cos C)\leq\frac{x^2}{2}+1$$

$$\iff\frac{x^2}{2}+1-\cos A\geq x(\cos B+\cos C). \tag{15}$$

When $x\geq 0$, we have

$$\sum\left(\frac{x^2}{2}+1-\cos A\right)\left(\frac{x^2}{2}+1-\cos B\right)\geq x^2\sum(\cos B+\cos C)(\cos C+\cos A)$$

$$=\frac{x^2(s^2+8Rr+5r^2)}{4R^2}. \tag{16}$$

When $x<0$, then $-x>0$, it follows from (16) that

$$\sum\left(\frac{x^2}{2}+1-\cos A\right)\left(\frac{x^2}{2}+1-\cos B\right)=\sum\left(\frac{(-x)^2}{2}+1-\cos A\right)\left(\frac{(-x)^2}{2}+1-\cos B\right)$$

$$\geq\frac{x^2(s^2+8Rr+5r^2)}{4R^2}. \tag{17}$$

Inequality (14) is proved.
Proposition 6. *Let x be a real number, then for any triangle ABC the following inequality holds*

$$\prod\left(\frac{x^2}{2}+1-\cos A\right)\geq\frac{x^3r(s^2+2Rr+r^2)}{4R^3}. \tag{18}$$

Proof. Obviously, when $x\leq 0$, inequality (18) is valid. When $x>0$, from inequality (15) we have

$$\prod(\frac{x^2}{2}+1-\cos A)\geq x^3\prod(\cos B+\cos C)=\frac{x^3r(s^2+2Rr+r^2)}{4R^3}.$$

Proposition 7. *Let x be a real number, then for any acute triangle ABC the following inequality holds*

$$\sum\frac{\frac{x^2}{2}+1-\cos A}{\cos A}\geq\frac{x((2R+r)^3+s^2r-2R(s^2+2Rr+r^2))}{R(s^2-(2R+r)^2)}. \tag{19}$$

Proof. From Corollary 3 we deduce that

$$1+\frac{x^2}{2}-\cos A \geq x(\cos B+\cos C)$$

$$\iff \frac{1+\frac{x^2}{2}-\cos A}{\cos A} \geq \frac{x(\cos B+\cos C)}{\cos A}.$$

By using the identities (see [3]):

$$\prod \cos A = \frac{s^2-(2R+r)^2}{4R^2},$$

$$\sum \cos A \cos B = \frac{s^2-4R^2+r^2}{4R^2},$$

we have

$$\begin{aligned}
\sum \frac{\frac{x^2}{2}+1-\cos A}{\cos A} &\geq x\sum \frac{\cos B+\cos C}{\cos A} \\
&= x\left[(\sum \cos A)(\sum \frac{1}{\cos A})-3)\right] \\
&= x\left[\frac{(\sum \cos A)(\sum \cos B\cos C)}{\prod \cos A}-3\right] \\
&= \frac{x((2R+r)^3+s^2r-2R(s^2+2Rr+r^2))}{R(s^2-(2R+r)^2)}.
\end{aligned}$$

Proposition 8. *Let x be a positive number, then for any triangle ABC the following inequality holds*

$$\frac{108R^2}{((3x^2+4)R-2r)^2} \leq \sum \frac{1}{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)} \leq \frac{8R^2(R+r)}{x^2r(s^2+2Rr+r^2)}. \tag{20}$$

Proof. From Corollary 3 we have

$$\cos A + x(\cos B+\cos C) \leq \frac{x^2}{2}+1$$

$$\iff \frac{x^2}{2}+1-\cos A \geq x(\cos B+\cos C). \tag{21}$$

Thus, it follows that

$$\begin{aligned}
\sum \frac{1}{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)} &\leq \frac{1}{x^2}\sum \frac{1}{(\cos B+\cos C)(\cos A+\cos C)} \\
&= \frac{2\sum \cos A}{x^2\prod(\cos A+\cos B)} \\
&= \frac{8R^2(R+r)}{x^2r(s^2+2Rr+r^2)}.
\end{aligned}$$

In addition, by applying the arithmetic-geometric means inequality and the identity $\sum \cos A = (R+r)/R$, we deduce that

$$\begin{aligned}\sum \frac{1}{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)} &= \frac{\sum(\frac{x^2}{2}+1-\cos A)}{\prod(\frac{x^2}{2}+1-\cos A)} \\ &\geq \frac{27}{(\sum(\frac{x^2}{2}+1-\cos A))^2} \\ &= \frac{108R^2}{((3x^2+4)R-2r)^2}.\end{aligned}$$

Proposition 9. *Let x be a positive number, then for any acute triangle ABC the following inequality holds*

$$\sum \frac{\cos B \cos C}{\frac{x^2}{2}+1-\cos A} \leq \frac{(s^2+r^2-4R^2)^2-4R(R+r)(s^2-(2R+r)^2)}{4xRr(s^2+2Rr+r^2)}. \tag{22}$$

Proof. From Corollary 3 we have

$$\cos A + x(\cos B + \cos C) \leq \frac{x^2}{2}+1$$

$$\iff \frac{x^2}{2}+1-\cos A \geq x(\cos B+\cos C).$$

Further, we have

$$\begin{aligned}\sum \frac{\cos B \cos C}{\frac{x^2}{2}+1-\cos A} &\leq \frac{1}{x}\sum \frac{\cos B \cos C}{\cos B+\cos C} \\ &= \frac{\prod \cos A \sum \cos A + \sum \cos^2 B \cos^2 C}{x\prod(\cos B+\cos C)} \\ &= \frac{(\sum \cos B \cos C)^2 - \prod \cos A \sum \cos A}{x\prod(\cos B+\cos C)} \\ &= \frac{(s^2+r^2-4R^2)^2-4R(R+r)(s^2-(2R+r)^2)}{4xRr(s^2+r^2+2Rr)}.\end{aligned}$$

Proposition 10. *Let x be a real number, then for any triangle ABC the following inequality holds*

$$\sum \left(\frac{x^2}{2}+1-\cos A\right)^2 \geq \frac{x^2(2(2R+r)^2+r^2-s^2)}{2R^2}. \tag{23}$$

Proof. In order to prove Proposition 10, it is enough to prove that the inequality (23) holds for $x \geq 0$. We deduce from Corollary 3 that

$$\begin{aligned}\sum \left(\frac{x^2}{2}+1-\cos A\right)^2 &\geq x^2 \sum (\cos B+\cos C)^2 \\ &= x^2 \sum (\cos^2 B + \cos^2 C + 2\cos B \cos C) \\ &= x^2(2\sum \cos^2 A + 2\sum \cos B \cos C) \\ &= \frac{x^2(2(2R+r)^2+r^2-s^2)}{2R^2}.\end{aligned}$$

Proposition 11. *Let x be a real number, then for any acute triangle ABC the following inequality holds*

$$\sum \frac{\frac{x^2}{2}+1-\cos A}{\cos B\cos C} \geq \frac{2x(s^2-4R^2+r^2)}{s^2-(2R+r)^2}. \tag{24}$$

Proof. From Corollary 3 we have

$$\begin{aligned}\sum \frac{\frac{x^2}{2}+1-\cos A}{\cos B\cos C} &\geq x\sum \frac{\cos B+\cos C}{\cos B\cos C} \\ &= \frac{2x\sum\cos A\cos B}{\prod\cos A} \\ &= \frac{2x(s^2-4R^2+r^2)}{s^2-(2R+r)^2}.\end{aligned}$$

Proposition 12. *Let x be a real number, then for any acute triangle ABC the following inequality holds*

$$\sum \frac{(\frac{x^2}{2}+1-\cos A)^2}{\cos B\cos C} \geq x^2\left(3+\frac{(R+r)(s^2-4R^2+r^2)}{R(s^2-(2R+r)^2)}\right). \tag{25}$$

Proof. In order to prove Proposition 12, it is enough to prove that the inequality (25) holds for $x \geq 0$. We deduce from Corollary 3 that

$$\begin{aligned}\sum \frac{(\frac{x^2}{2}+1-\cos A)^2}{\cos B\cos C} &\geq x^2\sum \frac{(\cos B+\cos C)^2}{\cos B\cos C} \\ &= x^2\sum\left(\frac{\cos B}{\cos C}+\frac{\cos C}{\cos B}+2\right) \\ &= x^2\left(\frac{\sum\cos A}{\cos C}+\frac{\sum\cos A}{\cos B}+\frac{\sum\cos A}{\cos A}+3\right) \\ &= x^2\left(\frac{(\sum\cos A)(\sum\cos A\cos B)}{\prod\cos A}+3\right) \\ &= x^2\left(3+\frac{(R+r)(s^2-4R^2+r^2)}{R(s^2-(2R+r)^2)}\right).\end{aligned}$$

Proposition 13. *Let x be a real number, then for any acute triangle ABC the following inequality holds*

$$\sum \frac{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)}{\cos A\cos B} \geq x^2\left(3+\frac{2(R+r)(6R^2+r^2+4Rr-s^2)}{R(s^2-(2R+r)^2)}\right). \tag{26}$$

Proof. In order to prove Proposition 13, it is enough to prove that the inequality (26) holds for $x \geq 0$. We deduce from Corollary 3 that

$$
\begin{aligned}
\sum \frac{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)}{\cos A\cos B} &\geq x^2 \sum \frac{(\cos B+\cos C)(\cos A+\cos C)}{\cos A\cos B}\\
&= x^2 \sum \frac{\cos A\cos B+\cos B\cos C+\cos C\cos A+\cos^2 C}{\cos A\cos B}\\
&= x^2\left(3+\frac{(\sum\cos^2 A)\sum\cos A}{\prod\cos A}\right)\\
&= x^2\left(3+\frac{((\sum\cos A)^2-2\sum\cos A\cos B)\sum\cos A}{\prod\cos A}\right)\\
&= x^2\left(3+\frac{2(R+r)(6R^2+r^2+4Rr-s^2)}{R(s^2-(2R+r)^2)}\right).
\end{aligned}
$$

Proposition 14. *Let x be a real number, then for any acute triangle ABC the following inequality holds*

$$\sum \frac{\cos A\cos B}{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)} \leq \frac{1}{x^2}\left(1-\frac{2R(s^2-(2R+r)^2)}{r(s^2+2Rr+r^2)}\right). \tag{27}$$

Proof. In order to prove Proposition 14, it is enough to prove that the inequality (27) holds for $x \geq 0$. We deduce from Corollary 3 that

$$
\begin{aligned}
\sum \frac{\cos A\cos B}{(\frac{x^2}{2}+1-\cos A)(\frac{x^2}{2}+1-\cos B)} &\leq \frac{1}{x^2}\sum\frac{\cos A\cos B}{(\cos B+\cos C)(\cos A+\cos C)}\\
&= \frac{\sum\cos A\cos B(\cos A+\cos B)}{x^2\prod(\cos B+\cos C)}\\
&= \frac{(\sum\cos A\cos B)(\sum\cos A)-3\cos A\cos B\cos C)}{x^2\prod(\cos B+\cos C)}\\
&= \frac{\prod(\cos B+\cos C)-2\cos A\cos B\cos C)}{x^2\prod(\cos B+\cos C)}\\
&= \frac{1}{x^2}\left(1-\frac{2R(s^2-(2R+r)^2)}{r(s^2+2Rr+r^2)}\right).
\end{aligned}
$$

Proposition 15. *Let x be a positive number, then for any acute triangle ABC the following inequality holds*

$$\sum\frac{\cos A}{\frac{x^2}{2}+1-\cos A} \leq \frac{R(s^2+8Rr+5r^2)-2r(s^2-Rr-r^2)}{xr(s^2+2Rr+r^2)}. \tag{28}$$

Proof. From Corollary 3 we deduce that

$$1+\frac{x^2}{2}-\cos A \geq x(\cos B+\cos C)$$

$$\iff \frac{\cos A}{1+\frac{x^2}{2}-\cos A} \leq \frac{\cos A}{x(\cos B+\cos C)}.$$

Thus, we have

$$
\begin{aligned}
\sum \frac{\cos A}{\frac{x^2}{2}+1-\cos A} &\leq \frac{1}{x}\sum \frac{\cos A}{\cos B+\cos C} \\
&= \frac{1}{x}\left((\sum \cos A)(\sum \frac{1}{\cos B+\cos C}) - 3\right) \\
&= \frac{1}{x}\left(\frac{(\sum \cos A)\sum (\cos C+\cos A)(\cos A+\cos B)}{\prod (\cos B+\cos C)} - 3\right) \\
&= \frac{R(s^2+8Rr+5r^2)-2r(s^2-Rr-r^2)}{xr(s^2+2Rr+r^2)}.
\end{aligned}
$$

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