AN OSTROWSKI-GRÜSS TYPE INEQUALITY ON TIME SCALES

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Abstract. In this paper we derive a new inequality of Ostrowski-Grüss type on time scales and thus unify corresponding continuous and discrete versions. We also apply our result to the quantum calculus case.

1. introduction

In 1997, Dragomir and Wang [7] proved the following Ostrowski-Grüss type integral inequality.

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I, a < b$. If $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \le f'(x) \le \Gamma$, $x \in [a, b]$. Then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma - \gamma), \tag{1}$$

for all $x \in [a, b]$.

This inequality is a connection between the Ostrowski inequality [12] and the Grüss inequality [13]. It can be applied to bound some special mean and some numerical quadrature rules. For other related results on the similar integral inequalities please see the papers [6, 10, 11, 14] and the references therein.

The aim of this paper is to extend a generalizations of Ostrowski-Grüss type integral inequality to an arbitrary time scale.

2. Time scales essentials

The development of the theory of time scales was initiated by Hilger [8] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales. For example, we refer the reader to [1, 4, 5, 15, 16].

Now we briefly introduce the time scales theory and refer the reader to Hilger [8] and the books [2, 3, 9] for further details.

Definition 1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers.*

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Definition 2. *For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(\mathrm{t}) = \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(\mathrm{t}) = \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(\mathrm{t}) > t$, then we say that t is right-scattered, while if $\rho(\mathrm{t}) < t$ then we say that t is left-scattered.*

Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(\mathrm{t}) = t$, the t is called *right-dense*, and if $\rho(\mathrm{t}) = t$ then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called dense.

Definition 3. *Let $t \in \mathbb{T}$, then two mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ satisfying*

$$\mu(t) := \sigma(\mathrm{t}) - t, \quad \nu(t) := t - \rho(\mathrm{t})$$

are called the graininess functions.

We now introduce the set $\mathbb{T}^\kappa$ which is derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t, then $\mathbb{T}^\kappa := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(\mathrm{t}) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(\mathrm{t})$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood U of t such that for all $s \in U$*

$$\left|f^\sigma(\mathrm{t}) - f(s) - f^\Delta(\mathrm{t})(\sigma(\mathrm{t}) - s)\right| \leqq \varepsilon |\sigma(\mathrm{t}) - s|.$$

We say that f is Δ-differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(\mathrm{t})$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 5. *A mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous (denoted by C_{rd}) provided if it satisfies*

(1) *f is continuous at each right-dense point or maximal element of $\mathbb{T}$.*
(2) *The left-sided limit $\lim\limits_{s \to t-} f(s) = f(t-)$ exists at each left-dense point t of $\mathbb{T}$.*

Remark 1. It follows from Theorem 1.74 of Bohner and Peterson [2] that every rd-continuous function has an anti-derivative.

Definition 6. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(\mathrm{t}) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the Δ-integral of f is defined by*

$$\int_a^b f(t)\,\Delta t = F(b) - F(a).$$

Proposition 1. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(4) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$,

(5) $\int_a^a f(t)\Delta t = 0.$

Definition 7. *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by*

$$h_0(t,s) = 1 \quad \text{for all} \quad s,t \in \mathbb{T}$$

and then resursively by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}\,.$$

The present paper is motivated by the following results: Grüss inequality on time scales and Ostrowski inequality on time scales due to Bohner and Matthews. More precisely, the following so-called Grüss inequality on time scales was established in [4].

Theorem 2 (See [4], Theorem 3.1)**.** *Let $a,b,s \in \mathbb{T}$, $f,g \in C_{rd}$ and $f,g : [a,b] \to \mathbb{R}$. Then for*

$$m_1 \le f(s) \le M_1, \qquad m_2 \le g(s) \le M_2, \tag{2}$$

we have

$$\left|\frac{1}{b-a}\int_a^b f^\sigma(s)g^\sigma(s)\Delta s - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\,\frac{1}{b-a}\int_a^b g^\sigma(s)\Delta s\right| \le \frac{1}{4}(M_1-m_1)(M_2-m_2). \tag{3}$$

The same authors also proved the following so-called Ostrowski inequality on time scales in [5].

Theorem 3 (See [5], Theorem 3.5)**.** *Let $a,b,s,t \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. Then*

$$\left|f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \le \frac{M}{b-a}(h_2(t,a) + h_2(t,b)), \tag{4}$$

where $M = \sup_{a<t<b}|f^\Delta(t)|$. This inequality is sharp in the sense that the right-hand side of (3) cannot be replaced by a smaller one.

In the present paper we shall first derive a new inequality of Ostrowski-Grüss type on time scales by using Theorem 2 and then unify corresponding continuous and discrete versions. We also apply our result to the quantum calculus case.

3. The Ostrowski-Grüss type inequality on time scales

Similarly as in [7], the Ostrowski-Grüss type inequality can be shown for general time scales.

Theorem 4. *Let $a,b,s,t \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. If f^Δ is rd-continuous and*

$$\gamma \le f^\Delta(t) \le \Gamma, \quad \forall\, t \in [a,b].$$

Then we have

$$\left|f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s - \frac{f(b)-f(a)}{(b-a)^2}\left(h_2(t,a)-h_2(t,b)\right)\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma), \quad (5)$$

for all $t \in [a,b]$.

To prove Theorem 4, we need the following Montgomery Identity.

Lemma 1 (Montgomery Identity, see [5])**.** *Let* $a, b, s, t \in \mathbb{T}$, $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then*

$$f(t) = \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s + \frac{1}{b-a}\int_a^b p(t,s) f^\Delta(s)\Delta s, \quad (6)$$

where

$$p(t,s) = \begin{cases} s-a, & a \le s < t, \\ s-b, & t \le s \le b. \end{cases}$$

Proof of Theorem 4. By applying Lemma 1, we get

$$f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s = \frac{1}{b-a}\int_a^b p(t,s) f^\Delta(s)\Delta s, \quad (7)$$

for all $t \in [a,b]$. It is clear that for all $t \in [a,b]$ and $s \in [a,b]$ we have

$$t-b \le p(t,s) \le t-a.$$

Applying Theorem 2 to the mapping $p(t,\cdot)$ and $f^\Delta(\cdot)$, we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b p(t,s) f^\Delta(s)\Delta s - \frac{1}{b-a}\int_a^b p(t,s)\Delta s \, \frac{1}{b-a}\int_a^b f^\Delta(s)\Delta s\right| \\ &\qquad \le \frac{1}{4}[(t-a)-(t-b)](\Gamma-\gamma) \\ &\qquad \le \frac{1}{4}(b-a)(\Gamma-\gamma). \quad (8) \end{aligned}$$

By a simple calculation we get

$$\begin{aligned} \int_a^b p(t,s)\Delta s &= \int_a^t (s-a)\Delta s + \int_t^b (s-b)\Delta s \\ &= \int_a^t (s-a)\Delta s - \int_b^t (s-b)\Delta s \\ &= h_2(t,a) - h_2(t,b) \end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f^\Delta(s)\Delta s = \frac{f(b)-f(a)}{b-a}.$$

By combining (7), (8) and the above two equalities, we obtain (5). □

If we apply the Ostrowski-Grüss type inequality to different time scales, we will get some well-known and some new results.

Corollary 1. (Continuous case). *Let $\mathbb{T} = \mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Hence,*

$$h_2(t,s) = \frac{(t-s)^2}{2}, \quad \text{for all} \quad t,s \in \mathbb{R}.$$

This leads us to state the following inequality

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(s)ds - \frac{f(b)-f(a)}{b-a}\left(t - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{9}$$

for all $t \in [a,b]$, where $\gamma \le f'(t) \le \Gamma$, which is exactly the Ostrowski-Grüss type inequality shown in Theorem 1.

Corollary 2. (Discrete case). *Let $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = n$, $s = j$, $t = i$ and $f(k) = x_k$. With these, it is known that*

$$h_k(t,s) = \binom{t-s}{k}, \quad \text{for all} \quad t,s \in \mathbb{Z}.$$

Therefore,

$$h_2(t,0) = \binom{t}{2} = \frac{t(t-1)}{2} \quad , \quad h_2(t,n) = \binom{t-n}{2} = \frac{(t-n)(t-n-1)}{2}.$$

Thus, we have

$$\left| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j - \frac{x_n}{n}\left(i - \frac{n+1}{2}\right)\right| \le \frac{1}{4}n(\Gamma-\gamma), \tag{10}$$

for all $i = \overline{1,n}$, where $\gamma \le \Delta x_i \le \Gamma$.

Corollary 3. (Quantum calculus case). *Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m, b = q^n$ with $m < n$. In this situation, one has*

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum\limits_{\mu=0}^{\nu} q^\mu}, \quad \text{for all} \quad t,s \in \mathbb{T}.$$

Therefore,

$$h_2(t,q^m) = \frac{(t-q^m)\left(t-q^{m+1}\right)}{1+q} \quad , \quad h_2(t,q^n) = \frac{(t-q^n)\left(t-q^{n+1}\right)}{1+q}.$$

Then

$$\begin{aligned} &\left| f(t) - \frac{1}{q^n - q^m}\int_{q^m}^{q^n} f^\sigma(s)\Delta s - \frac{f(q^n)-f(q^m)}{q^n-q^m}\left(t - \frac{q^{2n+1}-q^{2m+1}}{q+1}\right)\right| \\ &\qquad \le \frac{1}{4}(q^n-q^m)(\Gamma-\gamma), \end{aligned} \tag{11}$$

where

$$\gamma \le \frac{f(qt)-f(t)}{(q-1)(t)} \le \Gamma, \quad \forall\, t \in [a,b].$$

If f^{Δ} is bounded on $[a,b]$ then we have the following corollary.

Corollary 4. *With the assumptions in Theorem 4, if* $|f^{\Delta}(t)| \le M$ *for all* $t \in [a,b]$ *and some positive constant* M*, then we have*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s - \frac{f(b)-f(a)}{(b-a)^2}\left(h_2(t,a) - h_2(t,b)\right) \right| \le \frac{1}{2}(b-a)M, \quad (12)$$

for all $t \in [a,b]$.

Furthermore, choosing $t = (a+b)/2$ and $t = b$, respectively, in (5), we have the following corollary.

Corollary 5. *With the assumptions in Theorem 4, we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{f(b)-f(a)}{(b-a)^2}\left[h_2\left(\frac{a+b}{2},a\right) - h_2\left(\frac{a+b}{2},b\right)\right] - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s \right|$$
$$\le \frac{1}{4}(b-a)(\Gamma-\gamma) \qquad \left(\text{if } \frac{a+b}{2} \in \mathbb{T} \right) \quad (13)$$

and

$$\left| f(b) - \frac{f(b)-f(a)}{(b-a)^2}h_2(b,a) - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s \right| \le \frac{1}{4}(b-a)(\Gamma-\gamma). \quad (14)$$

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