Weighted Norm Inequalities for Commutators of One-sided Discrete Square Functions

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Abstract. The purpose of this paper is to prove the strong type inequalities with one-sided weights for commutators (with symbol $b \in Lip_\beta$) of one-sided discrete square functions. We also prove that $b \in Lip_\beta$ is a sufficient and necessary condition for the corresponding boundedness of commutators of one-sided maximal operators.

1 Introduction

A well known result of Coifman-Rochberg-Weiss[4] states that the commutator

$$T_b f = T(bf) - bT(f)$$

(where T is a Calderón-Zygmund singular integral operator) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$. There are other links between the boundedness properties of the operator T_b and the smoothness of b. A particular case of the result of Jason[6] states that T_b: $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded, $1 < p < q < \infty$, if and only if $b \in Lip_\beta$, $1/p - 1/q = \beta/n$. Here, Lip_β is the homogeneous Lipschitz space.

Many authors have studied strong and weak type inequalities for commutators with weights (see [2], [9], [10], [19]). Furthermore, many of the results have been generalized to commutators of other operators, not only Calderón-Zygmund operators (see [5], [21], [22]).

Very recently, Lorente-Riveros[11] proved the strong type inequalities with one-sided weights for commutators (with symbol $b \in \mathrm{BMO}$) of one-sided discrete square functions. Highly inspired by [6] and [11], we shall prove the strong type inequalities with one-sided weights for commutators (with symbol $b \in Lip_\beta$) of one-sided discrete square functions. We also prove that $b \in Lip_\beta$ is a sufficient and necessary condition for the corresponding boundedness of commutators of one-sided maximal operators.

Throughout this paper the letter C will be a positive constant, not necessarily the same at each occurrence. If $1 \le p \le \infty$, then its conjugate exponent will be denoted by p' and A_p will be the classical Muckenhoupt's class of weights (see [17]).

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2 Definitions and main results

Let f be a measurable function on $\mathbb{R}$. For each $n \in \mathbb{Z}$, let us define the operator A_n by $A_n = \frac{1}{2^n}\int_x^{x+2^n} f(y)dy$. It is a classical problem to study the different kinds of convergence of the $\{A_n f\}_n$ when the function f belongs to $L^p(\mathbb{R}, dx)$, being p in the range $1 \le p < \infty$. A method of measuring the speed of convergence of the sequence $\{A_n f\}_n$ is to analyze the boundedness of the square function

$$Sf(x) = \left(\sum_{n\in\mathbb{Z}} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{1/2}.$$

The square function S is of interest in ergodic theory and it has been extensively studied. In particular it has been proved in [7] that S^-(under the assumption of $n \in \mathbb{Z}^- \cup \{0\}$ in the definition of S, we denote S by S^-) is of weak type $(1,1)$, maps $L^p(\mathbb{R})$ into itself, $1 < p < \infty$. For the Ergodic theory and connections with Analysis and Probability, we choose to refer to [3] and [8].

It is not difficult to see that $Sf(x) = \|U^+ f(x)\|_{l^2}$, where U^+ is the sequence valued operator

$$U^+ f(x) = \int_{\mathbb{R}} H(x-y) f(y) dy,$$

where

$$H(x) = \left\{ \frac{1}{2^n}\chi_{(-2^n,0)} - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)} \right\}_{n\in\mathbb{Z}}.$$

(See [23].)

Definition 2.1 The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f|.$$

The good weights for these operators are the one-sided weights, A_p^+ and A_p^-:

$$\sup_{a<b<c} \frac{1}{(c-a)^p}\int_a^b \omega \left(\int_b^c \omega^{1-p'} \right)^{p-1} < \infty, \qquad 1 < p < \infty, \tag{A_p^+}$$

$$M^-\omega(x) < C\omega(x), \qquad a.e. \tag{A_1^+}$$

and

$$A_\infty^+ = \cup_{p\ge 1} A_p^+. \tag{A_∞^+}$$

The classes A_p^- are defined in a similar way. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneqq A_p^+$ and $A_p \subsetneqq A_p^-$. M^+ is bounded on $L^p(\omega)$ if and only if satisfies the A_p^+ condition. (See [13], [14], [20] for more definitions and results).

It is proved in [23] that $\omega \in A_p^+, 1 < p < \infty$, if and only if S is bounded from $L^p(\omega)$ to $L^p(\omega)$ and that $\omega \in A_1^+$ if and only if S is of weak-type $(1,1)$ with respect to ω.

Definition 2.2 The one-sided fractional maximal operator $M_\alpha^+, 0 < \alpha < 1$, is defined for locally integrable functions f by

$$M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f|.$$

It is proved in [1] that M_α^+ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ if and only if $\omega \in A^+(p,q)$, for $1 < p < q, 1/p - 1/q = \alpha$, where

$$\left(\frac{1}{h}\int_{x-h}^{x} \omega^q\right)^{1/q} \left(\frac{1}{h}\int_{x}^{x+h} \omega^{-p'}\right)^{1/p'} \leq C, \qquad (A^+(p,q))$$

$$\|\omega \chi_{[x-h,x]}\|_\infty \left(\frac{1}{h}\int_{x}^{x+h} \omega^{-p'}\right)^{1/p'} \leq C, \qquad (A^+(p,\infty))$$

for all $h > 0$ and $x \in \mathbb{R}$.

Definition 2.3[18] The Lipschitz space $Lip_\beta(\mathbb{R})$ is the space of functions f satisfying

$$\|f\|_{Lip_\beta(\mathrm{R}^n)} = \sup_{x,h\in\mathbb{R},h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

Now we shall state our results.

Theorem 2.4 *Let $b \in Lip_\beta(\mathbb{R})$, and $k \in \mathbb{N}$. The k-th order commutator of the one-sided discrete square function is defined by*

$$S_b^k f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x-y) f(y) dy \right\|_{l^2}.$$

Then for $\omega \in A^+(p,q)$, $1 < p < q < \infty, 1/p - 1/q = k\beta$, we have

$$\left(\int_{\mathbb{R}} |S_b^k f|^q \omega^q\right)^{1/q} \leq C \left(\int_{\mathbb{R}} |f|^p \omega^p\right)^{1/p},$$

for all bounded f with compact support.

Obviously, if $\beta > 1$, $Lip_\beta(\mathbb{R})$ contains only constants, $S_b^k \equiv 0$, so we will only concentrate our discussion on the cases $0 < \beta \leq 1$ in what follows. It should be pointed out that if $0 < \beta < 1$, $Lip_\beta(\mathbb{R}) = \dot{\Lambda}_\beta(\mathbb{R})$, where $\dot{\Lambda}_\beta(\mathbb{R})$ is the homogeneous Besov-Lipschitz space; but if $\beta = 1$, $Lip_\beta(\mathbb{R}) \subsetneqq \dot{\Lambda}_\beta(\mathbb{R})$.

Theorem 2.5 *Let k-th order commutator of the one-sided maximal operator be defined by*

$$M_b^{+,k}f(x)=\sup_{h>0}\frac{1}{h}\int_x^{x+h}|b(x)-b(y)|^k|f(y)|dy.$$

Then the following conditions are equivalent:

(i) $M_b^{+,k}$ *is bounded from* $L^p(\omega^p)$ *to* $L^q(\omega^q)$ *for pairs* (p,q), *such that* $1<p<q<\infty, 1/p-1/q=k\beta$ *and* $\omega\in A^+(p,q)$.

(ii) $M_b^{+,k}$ *is bounded from* $L^p(dx)$ *to* $L^q(dx)$ *for some pair* (p,q), *such that* $1<p<q<\infty, 1/p-1/q=k\beta$.

(iii) $b\in Lip_\beta(\mathbb{R})$.

Theorem 2.6 *Let k-th order commutator of the one-sided fractional maximal operator be defined by*

$$M_{\alpha,b}^{+,k}f(x)=\sup_{h>0}\frac{1}{h^{1-\alpha}}\int_x^{x+h}|b(x)-b(y)|^k|f(y)|dy.$$

Then the following conditions are equivalent:

(i) $M_{\alpha,b}^{+,k}$ *is bounded from* $L^p(\omega^p)$ *to* $L^q(\omega^q)$ *for pairs* (p,q), *such that* $1<p<q<\infty, 1/p-1/q=\alpha+k\beta$ *and* $\omega\in A^+(p,q)$.

(ii) $M_{\alpha,b}^{+,k}$ *is bounded from* $L^p(dx)$ *to* $L^q(dx)$ *for some pair* (p,q), *such that* $1<p<q<\infty, 1/p-1/q=\alpha+k\beta$.

(iii) $b\in Lip_\beta(\mathbb{R})$.

Similarly, it is not difficult to prove strong type inequalities with pairs of related weights for commutators of one-sided singular integral (given by a Calderón-Zygmund kernel with support in $(-\infty,0)$, see [10]) and the weyl fractional integral.

3 Proof of main results

In order to prove our results, let us first introduce some lemmas and notations.

Lemma 3.1[18] *For any* $x,y\in\mathbb{R}$, *if* $f\in Lip_\beta(\mathbb{R})$, $0<\beta<1$, *then*

$$|f(x)-f(y)|\le|x-y|^\beta\|f\|_{Lip_\beta},$$

and given any interval I *in* $\mathbb{R}$, *there is*

$$\sup_{x\in I}|f(x)-f_I|\le C|I|^\beta\|f\|_{Lip_\beta},$$

if $I^*\subset I$, *then*

$$|f_{I^*}-f_I|\le C\|f\|_{Lip_\beta}|I|^\beta,$$

where

$$f_I=\frac{1}{|I|}\int_I f.$$

Lemma 3.2[18] *For $0 < \beta < 1, 1 \le q < \infty$, we have*

$$\|f\|_{Lip_\beta} \approx \sup_I \frac{1}{|I|^{1+\beta}} \int_I |f - f_I| \approx \sup_I \frac{1}{|I|^\beta} \left(\frac{1}{|I|} \int_I |f - f_I|^q \right)^{1/q},$$

for $q = \infty$ the formula should be interpreted appropriately.

The main tool for proving our results is a extrapolation theorem that appeared in [12], with slight modifications.

Lemma 3.3 *Let $1 < p_0 < \infty$ and T be a sublinear operator defined in C_c^∞. Assume that for all $\omega \in A^+(p_0, \infty)$ there exists $C = C(\omega)$ such that*

$$\|\omega T f\|_\infty \le C \|f\omega\|_{p_0}.$$

Then for all pairs (p, q) such that $1 < p < p_0, 1/p - 1/q = 1/p_0$ and all $\omega \in A^+(p, q)$, there exists $C = C(\omega)$ such that

$$\|\omega T f\|_q \le C \|f\omega\|_p,$$

provided the left hand side is finite.

We will also need the following result of Martín-Reyes and de la Torre (theorem 4 in [16]):

Lemma 3.4 *Let $1 < p < \infty$. If $\omega \in A_p^+$ and $M^+ f \in L^p(\omega)$, then there exits $C = C(\omega)$ such that*

$$\int_{\mathbb{R}} (M^+ f)^p \omega \le C \int_{\mathbb{R}} (f^{\sharp,+})^p \omega,$$

where

$$f^{\sharp,+}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy$$

and $z^+ = \max(z, 0)$.

It is proved in [16] that

$$f^{\sharp,+}(x) \le \sup_{h>0} \inf_{a \in \mathbb{R}} \frac{1}{h} \int_x^{x+h} (f(y) - a)^+ \, dy + \frac{1}{h} \int_{x+h}^{x+2h} (a - f(y))^+ \, dy \le C \|f\|_{\mathrm{BMO}}.$$

Lemma 3.5[15,20] *Let $\omega \in A_1^-$. Then there exits $s > 1$ such that $\omega^r \in A_1^-$, for all r such that $1 < r \le s$. Let $\omega \in A^+(p, q)$. Then $\omega^q \in A_q^+$ and $\omega^p \in A_p^+$, where $1 < p < q < \infty$.*

Applying Hölder's inequality in the definition of $A^+(p, q)$, we can get the following Lemma.

Lemma 3.6 *Let $\omega \in A^+(p,q)$. Then $\omega \in A^+(p_0,q)$ and $\omega \in A^+(p,p_0)$, where $1 < p < p_0 < q < \infty$.*

Proof of Theorem 2.4. Let $\omega \in A^+(p,q)$. Then $\omega^q \in A_q^+$. By Lemma 3.4, we have

$$\int_{\mathbb{R}} |S_b^k f|^q \omega^q \le C \int_{\mathbb{R}} |M^+(S_b^k f)|^q \omega^q \le C \int_{\mathbb{R}} |(S_b^k f)^{\sharp,+}|^q \omega^q.$$

To prove the theorem for any $b \in Lip_\beta$, we proceed in the same way as in [11]. We will control $(S_b^k f)^{\sharp,+}$ by some one-sided maximal operators. Using Lemma 3.3, we shall prove that they are bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$.

Let λ be an arbitrary constant. Then $b(x) - b(y) = (b(x) - \lambda) - (b(y) - \lambda)$ and

$$\begin{aligned}
S_b^k f(x) &= \Big\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x-y) f(y) dy \Big\|_{l^2} \\
&= \Big\| \sum_{j=0}^{k} C_{j,k} (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x-y) f(y) dy \Big\|_{l^2} \\
&\le \Big\| \int_{\mathbb{R}} (b(y) - \lambda)^k H(x-y) f(y) dy \Big\|_{l^2} \\
&\quad + \Big\| \sum_{j=1}^{k} C_{j,k} (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x-y) f(y) dy \Big\|_{l^2} \\
&\le S((b-\lambda)^k f)(x) \\
&\quad + \Big\| \sum_{j=1}^{k} \sum_{s=0}^{k-j} C_{j,k,s} (b(x) - \lambda)^{s+j} \int_{\mathbb{R}} (b(x) - b(y))^{k-j-s} H(x-y) f(y) dy \Big\|_{l^2} \\
&\le S((b-\lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m} |b(x) - \lambda|^{k-m} S_b^m f(x),
\end{aligned}$$

where $C_{j,k}$ (respectively $C_{j,k,s}$) are absolute constants depending only on j and k (respectively j, k and s). Let $x \in \mathbb{R}, h > 0$. Let $i \in \mathbb{Z}$ be such that $2^i \le h < 2^{i+1}$ and set $J = [x, x + 2^{i+3}]$. Then, write $f = f_1 + f_2$, where $f_1 = f\chi_J$ and set $\lambda = b_J$. Then

$$\begin{aligned}
&\frac{1}{h} \int_x^{x+h} |S_b^k f(y) - S((b-b_J)^k f_2)(x)| dy \\
&\le \frac{1}{h} \int_x^{x+h} |S((b-b_J)^k f_1)(y)| dy \\
&\quad + \frac{1}{h} \int_x^{x+h} |S((b-b_J)^k f_2)(y) - S((b-b_J)^k f_2)(x)| dy \\
&\quad + \sum_{m=0}^{k-1} C_{k,m} \frac{1}{h} \int_x^{x+h} |b(y) - b_J|^{k-m} |S_b^m f(y)| dy \\
&= I(x) + II(x) + III(x).
\end{aligned}$$

For $II(x)$, we have

$$II(x) \leq \frac{1}{h} \int_x^{x+2^{i+3}} ||U^+((b-b_J)^k f_2)(y) - U^+((b-b_J)^k f_2)(x)||_{l^2} dy,$$

and

$$||U^+((b-b_J)^k f_2)(y) - U^+((b-b_J)^k f_2)(x)||_{l^2} \leq \int_{x+2^{i+3}}^{\infty} |b(t)-b_J|^k |f(t)| ||H(y-t)-H(x-t)||_{l^2} dt.$$

Consider the following sublinear operators defined in C_c^∞:

$$M_1^+ f(x) = \sup_{i\in\mathbb{Z}} \frac{1}{2^i} \int_x^{x+2^{i+2}} |S((b-b_J)^k f\chi_J)(y)| dy;$$

$$M_2^+ f(x) = \sup_{i\in\mathbb{Z}} \frac{1}{2^i} \int_x^{x+2^{i+3}} \int_{x+2^{i+3}}^{\infty} |b(t)-b_J|^k |f(t)| ||H(y-t)-H(x-t)||_{l^2} dt dy;$$

and

$$M_{3,m}^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+2h} |b(y) - b_{[x,x+8h]}|^{k-m} |f(y)| dy, \qquad 0 \leq m \leq k-1, k \geq 1.$$

The above definitions give that

$$(S_b^k f)^{\sharp,+} \leq C \left(M_1^+ f(x) + M_2^+ f(x) + \sum_{m=0}^{k-1} M_{3,m}^+ (S_b^m f)(x) \right).$$

We shall prove, using Lemma 3.3, that these operators are bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$, $\omega \in A^+(p,q)$, $1 < p < q < \infty, 1/p - 1/q = k\beta$.

Boundedness of M_1^+: Let $\omega \in A^+(1/k\beta, \infty)$, then $\omega^{-1/(1-k\beta)} \in A_1^-$. Therefore, there exists $t > 1$ such that $\omega^{-t/(1-k\beta)} \in A_1^-$. Let $s > 1, r > 1$ be such that $s = t/(1-k\beta)$ and $1/r - 1/s = k\beta$. Then, using Hölder's inequality and the fact that S maps $L^r(\mathbb{R})$ into itself, we get

$$\begin{aligned}
&\frac{1}{2^i} \int_x^{x+2^{i+2}} |S((b-b_J)^k f\chi_J)(y)| dy \\
&\leq \left(\frac{1}{2^i} \int_x^{x+2^{i+2}} |S((b-b_J)^k f\chi_J)(y)|^r dy \right)^{1/r} \\
&\leq \left(\frac{1}{2^i} \int_x^{x+2^{i+3}} |(b-b_J)^k f(y)|^r dy \right)^{1/r} \\
&\leq C \sup_{y\in J} |b(y) - b_J|^k \left(\frac{1}{2^i} \int_x^{x+2^{i+3}} |f(y)|^r dy \right)^{1/r}
\end{aligned}$$

$$
\begin{aligned}
&= C2^{ik\beta}\|b\|^k_{Lip_\beta}\left(\frac{1}{2^i}\int_x^{x+2^{i+3}}|f(y)|^r\omega^r\omega^{-r}dy\right)^{1/r}\\
&\le C2^{ik\beta}\|b\|^k_{Lip_\beta}\left(\frac{1}{2^i}\int_x^{x+2^{i+3}}|f(y)\omega|^{1/k\beta}dy\right)^{k\beta}\left(\frac{1}{2^i}\int_x^{x+2^{i+3}}\omega^{-s}dy\right)^{1/s}\\
&\le C\|b\|^k_{Lip_\beta}\|f\omega\|_{1/k\beta}\omega^{-1}(x)
\end{aligned}
$$

The last inequality is deduced by the fact $\omega^{-s}\in A_1^-$.

As a consequence,

$$\|\omega M_1^+ f\|_\infty \le C\|b\|^k_{Lip_\beta}\|f\omega\|_{1/k\beta}.$$

Then, by Lemma 3.3, for all $\omega\in A^+(p,q)$, $1/p-1/q=k\beta$,

$$\|M_1^+ f\|_{\omega^q,q}\le C\|b\|^k_{Lip_\beta}\|f\|_{\omega^p,p}.$$

Boundedness of M_2^+: Set $I_j=[x,x+2^{j+1}]$, we have that

$$
\begin{aligned}
&\int_{x+2^{i+3}}^\infty |b(t)-b_J|^k|f(t)|||H(y-t)-H(x-t)||_{l^2}dt\\
&\le C\sum_{j=i+3}^\infty\int_{x+2^j}^{x+2^{j+1}}|b(t)-b_{I_j}|^k|f(t)|||H(y-t)-H(x-t)||_{l^2}dt\\
&\quad +C\sum_{j=i+3}^\infty|b_{I_j}-b_J|^k\int_{x+2^j}^{x+2^{j+1}}|f(t)|||H(y-t)-H(x-t)||_{l^2}dt\\
&= II_1(x)+II_2(x).
\end{aligned}
$$

We proceed in the same way as in the estimates of M_1^+, choose r' such that $1/r+1/r'=1$, by Hölder's inequality, we get

$$
\begin{aligned}
II_1(x) &\le C\sum_{j=i+3}^\infty\left(\int_{I_j}|(b-b_{I_j})^k f|^r\right)^{1/r}\left(\int_{x+2^j}^{x+2^{j+1}}||H(y-t)-H(x-t)||^{r'}_{l^2}dt\right)^{1/r'}\\
&\le C\sum_{j=i+3}^\infty\sup_{I_j}|b-b_{I_j}|^k\left(\int_{I_j}|f|^r\right)^{1/r}\left(\int_{x+2^j}^{x+2^{j+1}}||H(y-t)-H(x-t)||^{r'}_{l^2}dt\right)^{1/r'}\\
&\le C\|b\|^k_{Lip_\beta}\|f\omega\|_{1/k\beta}\omega^{-1}(x)\sum_{j=i+3}^\infty 2^{j/r}\left(\int_{x+2^j}^{x+2^{j+1}}||H(y-t)-H(x-t)||^{r'}_{l^2}dt\right)^{1/r'}.
\end{aligned}
$$

It is proved in theorem 1.6 of [23] that for all $y\in[x,x+2^{i+3}]$ the kernel H satisfies

$$\left(\int_{x+2^j}^{x+2^{j+1}}||H(y-t)-H(x-t)||^{r'}_{l^2}dt\right)^{1/r'}\le C\frac{2^{i/r'}}{2^j}.$$

Then we get

$$\begin{aligned} II_1(x) &\le C\|b\|_{Lip_\beta}^k \|f\omega\|_{1/k\beta}\omega^{-1}(x)\sum_{j=i+3}^{\infty}\left(\frac{2^i}{2^j}\right)^{1/r'} \\ &\le C\|b\|_{Lip_\beta}^k \|f\omega\|_{1/k\beta}\omega^{-1}(x). \end{aligned}$$

Observe that $|b_{I_j} - b_J| \le C2^{j\beta}\|b\|_{Lip_\beta}$, similar to the estimates of $II_1(x)$, we can get

$$II_2(x) \le C\|b\|_{Lip_\beta}^k \|f\omega\|_{1/k\beta}\omega^{-1}(x).$$

As a consequence,

$$\|\omega M_2^+ f\|_\infty \le C\|b\|_{Lip_\beta}^k \|f\omega\|_{1/k\beta}.$$

Then, by Lemma 3.3, for all $\omega \in A^+(p,q)$, $1/p - 1/q = k\beta$,

$$\|M_2^+ f\|_{\omega^q,q} \le C\|b\|_{Lip_\beta}^k \|f\|_{\omega^p,p}.$$

Boundedness of $M_{3,m}^+$: We shall prove that $M_{3,m}^+$ are bounded from $L^{p_0}(\omega^{p_0})$ to $L^q(\omega^q)$, $\omega \in A^+(p_0,q)$, $1 < p < p_0 < q < \infty, 1/p_0 - 1/q = (k-m)\beta$.

Let $\omega \in A^+(\frac{1}{(k-m)\beta}, \infty)$, then $\omega^{-1/(1-(k-m)\beta)} \in A_1^-$. Therefore, there exists $t_0 > 1$ such that $\omega^{-t_0/(1-(k-m)\beta)} \in A_1^-$. Let $s_0 > 1, r_0 > 1$ be such that $s_0 = t_0/(1-(k-m)\beta)$ and $1/r_0 - 1/s_0 = (k-m)\beta$. Then, using Hölder's inequality, we get

$$\begin{aligned} &\frac{1}{h}\int_x^{x+2h} |b(y) - b_{[x,x+8h]}|^{k-m}|f(y)|dy \\ &\le \left(\frac{1}{h}\int_x^{x+2h} |f(y)|^{r_0}\omega^{r_0}\omega^{-r_0}dy\right)^{1/r_0}\left(\frac{1}{h}\int_x^{x+2h} |b(y) - b_{[x,x+8h]}|^{(k-m)r_0'}dy\right)^{1/r_0'} \\ &\le Ch^{(k-m)\beta}\|b\|_{Lip_\beta}^{k-m}\left(\frac{1}{h}\int_x^{x+2h} |f(y)\omega|^{1/(k-m)\beta}dy\right)^{(k-m)\beta}\left(\frac{1}{h}\int_x^{x+2h} \omega^{-s_0}dy\right)^{1/s_0} \\ &\le C\|b\|_{Lip_\beta}^{k-m}\|f\omega\|_{1/(k-m)\beta}\omega^{-1}(x). \end{aligned}$$

The last inequality is deduced by the fact $\omega^{-s_0} \in A_1^-$.

As a consequence,

$$\|\omega M_{3,m}^+ f\|_\infty \le C\|b\|_{Lip_\beta}^{k-m}\|f\omega\|_{1/(k-m)\beta}.$$

Then, by Lemma 3.3, for all $\omega \in A^+(p_0,q)$, $1/p_0 - 1/q = (k-m)\beta$,

$$\|M_{3,m}^+ f\|_{\omega^q,q} \le C\|b\|_{Lip_\beta}^{k-m}\|f\|_{\omega^{p_0},p_0}.$$

Specially, when $k = 1$, by all the above estimates, we can deduce

$$\|S_b f\|_{\omega^q,q} \le C\|b\|_{Lip_\beta}\|f\|_{\omega^{p_1},p_1},$$

where $\omega \in A^+(p_1, q)$, $1/p_1 - 1/q = \beta$,

Using the induction principle, let $\omega \in A^+(p, p_0)$, $1 < p < p_0 < \infty, 1/p - 1/p_0 = m\beta$. Then $1/p - 1/q = k\beta$, $1 < p < q < \infty$, by Lemma 3.6, we get that, for all $\omega \in A^+(p, q)$,

$$\|M^+_{3,m}(S^m_b f)\|_{\omega^q,q} \leq C\|b\|^{k-m}_{Lip_\beta}\|S^m_b f\|_{\omega^{p_0},p_0} \leq C\|b\|^k_{Lip_\beta}\|f\|_{\omega^p,p}.$$

Proof of Theorem 2.5.

(iii)⇒(i) By Lemma 3.1, we have

$$\frac{1}{h}\int_x^{x+h} |b(x) - b(y)|^k |f(y)|dy \leq C\frac{\|b\|^k_{Lip_\beta}}{h^{1-k\beta}}\int_x^{x+h} |f(y)|dy.$$

Using the fact that $M^+_{k\beta}$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ if and only if $\omega \in A^+(p, q)$, for $1 < p < q, 1/p - 1/q = k\beta$, we can get the desired result.

(i)⇒(ii) Given an appropriate pair (p, q), set $\omega \equiv 1$.

(ii)⇒(iii) Set $I = (a, b)$, $I^+ = (b, c)$, and $|I| = |I^+|$. Then

$$\begin{aligned}
\frac{1}{|I|^{1+\beta}}\int_I |b(y) - b_I|dy &\leq \frac{C}{|I|^{1+\beta}}\int_I |b(y) - b_{I^+}|dy \\
&\leq \frac{C}{|I|^\beta}\left(\frac{1}{|I|}\int_I |b(y) - b_{I^+}|^k dy\right)^{1/k} \\
&\leq \frac{C}{|I|^\beta}\left(\frac{1}{|I|}\int_I \left|\frac{1}{|I^+|}\int_{I^+}(b(y) - b(x))dx\right|^k dy\right)^{1/k} \\
&\leq \frac{C}{|I|^\beta}\left(\frac{1}{|I|}\int_I \left(\frac{1}{|I^+|}\int_{I^+}|b(y) - b(x)|^k dx\right) dy\right)^{1/k}.
\end{aligned}$$

Observe that, for $y \in I$,

$$\frac{1}{|I^+|}\int_{I^+}|b(y) - b(x)|^k dx = \frac{1}{|I^+|}\int_y^c |b(y) - b(x)|^k \chi_{I^+}(x)dx \leq CM^{+,k}_b \chi_{I^+}(y).$$

Then by Hölder's inequality and (ii),

$$\begin{aligned}
\frac{1}{|I|^{1+\beta}}\int_I |b(y) - b_I|dy &\leq \frac{C}{|I|^\beta}\left(\frac{1}{|I|}\int_I M^{+,k}_b \chi_{I^+}(y)dy\right)^{1/k} \\
&\leq \frac{C}{|I|^\beta}\left(\frac{1}{|I|}\int_I |M^{+,k}_b \chi_{I^+}(y)|^q dy\right)^{1/qk} \\
&\leq \frac{C}{|I|^\beta}\frac{1}{|I|^{1/qk}}\left(\int_{\mathbb{R}} |\chi_{I^+}(y)|^p dy\right)^{1/pk} \\
&\leq C\frac{|I^+|^{1/pk}}{|I|^{\beta+1/qk}} = C.
\end{aligned}$$

So, by Lemma 3.2, we get $b \in Lip_\beta$.

Similar to Theorem 2.5, we can finish the proof of Theorem 2.6 easily. We omit the details here.

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