REFINEMENTS, GENERALIZATIONS AND APPLICATIONS OF JORDAN'S INEQUALITY AND RELATED PROBLEMS

FENG QI AND DA-WEI NIU

Abstract. This is a survey and expository article. Some new developments on refinements, generalizations, applications of Jordan's inequality and related problems, including some results about Wilker-Anglesio's inequality, some estimates for three kinds of complete elliptic integrals and several inequalities for the remainder of power series expansion of e^x, are summarized.

Contents

2000 *Mathematics Subject Classification.* 26D05, 26D15, 33B10, 33B15, 33C10, 33C15, 33C75.

Key words and phrases. Jordan's inequality, Kober's inequality, Yang's inequality, refinement, generalization, application, L'Hôspital's rule, auxiliary function, Wilker-Anglesio's inequality, sine function, cosine function, Bessel function, complete elliptic integral, hyperbolically trigonometric function, exponential function.

The first author was partially supported by the China Scholarship Council and the NSF of Henan University, China.

1. Jordan's and related inequalities

1.1. **Jordan's inequality.** The well-known Jordan's inequality (see [20, p. 143], [27], [48, p. 269] and [57, p. 33]) reads that

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \tag{1.1}$$

for $0 < |x| \le \frac{\pi}{2}$. The equality in (1.1) is valid if and only if $x = \frac{\pi}{2}$.

The inequality (1.1) is an immediate consequence of the concavity of $x \longmapsto \sin x$ on the interval $\left[0, \frac{\pi}{2}\right]$. The straight line $y = \frac{2}{\pi}x$ is a chord of $y = \sin x$, which joints the points $(0,0)$ and $\left(\frac{\pi}{2}, 1\right)$. The straight line $y = x$ is a tangent to $y = \sin x$ at the origin. Hence, the graph of $y = \sin x, x \in \left[0, \frac{\pi}{2}\right]$ lies between these straight lines. See [57, p. 33, Remark 1].

The very origin of Jordan's inequality (1.1) is not found in the references listed in this paper, therefore, it is unknown that why the inequality (1.1) is named after Jordan and to which Jordan, to the best of our knowledge. Although the Name Index on [57, p. 391] hints us that the inequality (1.1) is due to C. Jordan (1838–1922), but no references related to C. Jordan was listed.

1.2. **Kober's inequality.** The following inequality is due to H. Kober [45, p. 22]:

$$1 - \frac{2}{\pi}x \le \cos x \le 1 - \frac{x^2}{\pi}, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{1.2}$$

See also [48, pp. 274–275].

In [46] and [47, p. 313], it was listed that for $x \in [0, \pi]$,

$$\cos x \le 1 - \frac{2}{\pi^2}x^2. \tag{1.3}$$

The left-hand side inequalities in (1.1) and (1.2) are equivalent to each other, since they can be deduced from each other via the transformation $x \to \frac{\pi}{2} - x$, as said in [107]. Applying this transformation to the right-hand side of inequality (1.2) acquires

$$\sin x \le 1 - \frac{(\pi - 2x)^2}{4\pi}, \quad x \in \left[0, \frac{\pi}{2}\right], \tag{1.4}$$

which can not be compared with the right-hand side of (1.1) on $\left[0, \frac{\pi}{2}\right]$.

1.3. **Redheffer-Williams's inequality and Li-Li's refinement.**

1.3.1. *Redheffer-Williams's inequality.* In [80, 81], it was proposed that

$$\frac{\sin x}{x} \ge \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \ne 0. \tag{1.5}$$

In [92], the inequality (1.5) was verified as follows: For $x \ge 1$,

$$\begin{aligned}\frac{1 - x^2}{1 + x^2} - \frac{\sin(\pi x)}{\pi x} &= \frac{1 - x^2}{1 + x^2} + \frac{\sin[\pi(x-1)]}{\pi(x-1)}\frac{x-1}{x} \\ &\le \frac{1 - x^2}{1 + x^2} + \frac{x-1}{x} = -\frac{(1-x)^2}{x(1+x^2)} \le 0.\end{aligned}$$

For $0 < x < 1$, since

$$\frac{\sin(\pi x)}{\pi x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right),$$

it is enough to prove that $(1 + x^2)P_n \ge 1$ for $n \ge 2$, where

$$P_n = \prod_{k=2}^{n}\left(1 - \frac{x^2}{k^2}\right).$$

Actually, by a simple induction argument based on the relation

$$P_{n+1}=\biggl[1-\frac{x^2}{(n+1)^2}\biggr]P_n,$$

it is deduced that

$$\bigl(1+x^2\bigr)P_n\ge1+\frac{x^2}{n},\quad 0<x<1.$$

1.3.2. *Li-Li's refinement.* In [51, Theorem 4.1], the inequality (1.5) was refined for $0<x<1$ as

$$\frac{(1-x^2)(4-x^2)(9-x^2)}{x^6-2x^4+13x^2+36}\le\frac{\sin(\pi x)}{\pi x}\le\frac{1-x^2}{\sqrt{1+3x^4}}. \tag{1.6}$$

1.4. **Mercer-Caccia's inequality.** In [55], it was proposed that

$$\sin\theta\ge\frac{2}{\pi}\theta+\frac{1}{12\pi}\theta\bigl(\pi^2-4\theta^2\bigr) \tag{1.7}$$

for $\theta\in\bigl[0,\frac{\pi}{2}\bigr]$. By finding the minimum of the function

$$\begin{cases}1, & x=0,\\ x^{-1}\sin x+\dfrac{x^2}{3\pi}, & x\in\Bigl(0,\dfrac{\pi}{2}\Bigr],\end{cases}$$

the inequality (1.7) was not only proved but also improved in [1] as

$$\sin\theta\ge\frac{2}{\pi}\theta+\frac{1}{\pi^3}\theta\bigl(\pi^2-4\theta^2\bigr) \tag{1.8}$$

for $\theta\in\bigl[0,\frac{\pi}{2}\bigr]$. The inequality (1.8) is sharp in the sense that $\frac{1}{\pi^3}$ cannot be replaced by a larger constant.

1.5. **Prestin's inequality.** In [64], the following inequality was given: For $0<|x|\le\frac{\pi}{2}$,

$$\left|\frac{1}{\sin x}-\frac{1}{x}\right|\le1-\frac{2}{\pi}. \tag{1.9}$$

See also [48, p. 270].

For $0<x\le\frac{\pi}{2}$, the inequality (1.9) can be rewritten as

$$\frac{x}{\sin x}\le1+\left(1-\frac{2}{\pi}\right)x\quad\text{or}\quad\sin x\ge\frac{x}{1+(1-2/\pi)x}. \tag{1.10}$$

This inequality and the inequality (1.8) are not included each other on $(0,\frac{\pi}{2}]$.

1.6. **Some inequalities obtained from Taylor's formula.** In [43, pp. 101–102], [47, p. 313] and [48, p. 269], the following inequalities are listed: For $x\in\bigl[0,\frac{\pi}{2}\bigr]$,

$$x-\frac{1}{6}x^3\le\sin x\le x-\frac{1}{6}x^3+\frac{1}{120}x^5, \tag{1.11}$$

$$1-\frac{1}{2}x^2\le\cos x\le1-\frac{1}{2}x^2+\frac{1}{24}x^4, \tag{1.12}$$

$$(-1)^n\Biggl[\sin x-\sum_{k=1}^n(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}\Biggr]\le\frac{x^{2n+1}}{(2n+1)!}, \tag{1.13}$$

$$(-1)^{n+1}\Biggl[\cos x-\sum_{k=0}^n(-1)^k\frac{x^{2k}}{(2k)!}\Biggr]\le\frac{x^{2n+2}}{(2n+2)!}. \tag{1.14}$$

It is obvious that these inequalities are established basing on Taylor's formula.

In [54], the inequality (1.11) was applied to obtain the lower and upper estimations of $\zeta(3)$ by in virtue of

$$\sum_{i=0}^{\infty} \frac{1}{(2i+1)^3} = \frac{1}{4} \int_0^{\pi/2} \frac{x(\pi - x)}{\sin x} \, \mathrm{d}x = \frac{7}{8}\zeta(3). \tag{1.15}$$

In [37, Theorem 1.7], the very closer lower and upper bounds for $\zeta(3)$ are deduced by a different approach from [54] as a by-product.

1.7. **Cusa-Huygens's and related inequalities.** Nicolaus da Cusa (1401–1464) found by a geometrical method that

$$\frac{\sin x}{x} \le \frac{2 + \cos x}{3}, \tag{1.16}$$

for $0 < x \le \frac{\pi}{2}$. Christian Huygens (1629–1695) proved (1.16) explicitly when he approximated π. See [18, 38] and related references therein.

In [69], by using Techebysheff's integral inequality, it was constructed that

$$\frac{\sin x}{x} \ge \frac{1 + \cos x}{2}. \tag{1.17}$$

In [57, p. 238, 3.4.15], the following double inequality

$$\frac{2(1 + a\cos x)}{\pi} \le \frac{\sin x}{x} \le \frac{1 + a\cos x}{a + 1} \tag{1.18}$$

was given for $a \in \left(0, \frac{1}{2}\right]$ and $x \in \left[0, \frac{\pi}{2}\right]$.

1.8. **Some inequalities related to trigonometric functions.**

1.8.1. In [22, 32], the following inequalities were presented: For $0 < x < 1$,

$$\frac{2}{\pi} \cdot \frac{x}{1 - x^2} < \frac{1}{\pi x} - \cot(\pi x) < \frac{\pi}{3} \cdot \frac{x}{1 - x^2}, \tag{1.19}$$

$$\frac{\pi^2}{8} \cdot \frac{x}{1 - x^2} < \sec\frac{\pi x}{2} - 1 < \frac{4}{\pi} \cdot \frac{x}{1 - x^2}, \tag{1.20}$$

$$\frac{\pi}{6} \cdot \frac{x}{1 - x^2} < \csc(\pi x) - \frac{1}{\pi x} < \frac{2}{\pi} \cdot \frac{x}{1 - x^2}. \tag{1.21}$$

For $0 < |x| < 1$,

$$\ln\left(\frac{\pi x}{\sin(\pi x)}\right) < \frac{\pi^2}{6} \cdot \frac{x^2}{1 - x^2}, \tag{1.22}$$

$$\ln\left(\sec\frac{\pi x}{2}\right) < \frac{\pi^2}{8} \cdot \frac{x^2}{1 - x^2}, \tag{1.23}$$

$$\ln\left(\frac{\tan(\pi x/2)}{\pi x/2}\right) < \frac{\pi^2}{12} \cdot \frac{x^2}{1 - x^2}. \tag{1.24}$$

The constants $\frac{2}{\pi}$ and $\frac{\pi}{3}$ in (1.19), $\frac{\pi^2}{8}$ and $\frac{4}{\pi}$ in (1.20), $\frac{\pi}{6}$ and $\frac{2}{\pi}$ in (1.21) are the best possible. The constants $\frac{\pi^2}{6}$, $\frac{\pi^2}{8}$ and $\frac{\pi^2}{12}$ in (1.22), (1.23) and (1.24) respectively are the best possible.

1.8.2. For $x \in (0, \frac{\pi}{2})$ and $n \in \mathbb{N}$, it was proved in [21] that

$$\frac{2^{2(n+1)}[2^{2(n+1)}-1]B_{n+1}}{(2n+2)!}x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n}\tan x, \quad (1.25)$$

where

$$S_n(x) = \sum_{i=1}^{n} \frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-1} \quad (1.26)$$

and B_i for $i \in \mathbb{N}$ are the well-known Bernoulli's numbers defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j}\frac{x^{2j}}{(2j)!}, \quad |x| < 2\pi \quad (1.27)$$

and the first several Bernoulli's numbers are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}. \quad (1.28)$$

1.8.3. In [69], the double inequality (2.9) was verified as a lemma once again. Among other things, a lot of inequalities and integrals related to $\frac{\sin x}{x}$ and similar to (2.13), (2.14) and (2.16) are constructed by using the well-known Tchebysheff's integral inequality [57, p. 39, Theorem 8]. For examples,

$$\left(\frac{\sin t}{t}\right)^2 + 2\left(\frac{\sin t}{t}\right) \ge 4\left(\frac{1-\cos t}{t^2}\right) + \cos t, \quad t \in [0, \pi] \quad (1.29)$$

and

$$\int_0^t \left(\frac{x}{\sin x}\right)^2 \mathrm{d}x < 2\tan\left(\frac{t}{2}\right) + \frac{2}{3}\tan^3\left(\frac{t}{2}\right), \quad t \in \left(0, \frac{\pi}{2}\right]. \quad (1.30)$$

1.8.4. Let

$$p(\theta) = \begin{cases} \left(\frac{\pi^2}{8} - \frac{1}{2}\theta\right)\sec^2\theta - \theta\tan\theta - \frac{1}{2}, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & \theta = \pm\frac{\pi}{2}, \end{cases} \quad (1.31)$$

$$q(\theta) = \begin{cases} \frac{2}{\cos^2\theta}\int_\theta^{\pi/2} t\cos^2 t\,\mathrm{d}t, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & \theta = \pm\frac{\pi}{2}, \end{cases} \quad (1.32)$$

$$\phi(\theta) = \begin{cases} \frac{\pi}{4}\big(\theta\sec^2\theta + \tan\theta\big) - 2\tan\theta\sec\theta, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \pm1, & \theta = \pm\frac{\pi}{2}. \end{cases} \quad (1.33)$$

These functions originate from estimates of the eigenvalues of Laplace operator on compact Riemannian manifolds. Their monotonicity and estimates have been investigated by several mathematicians. For more detailed information, please refer to [36, 72, 77] and related references therein.

2. Refinements and generalizations of Jordan's and related inequalities

2.1. **Qi-Guo's refinements of Kober's and Jordan's inequality.**

2.1.1. *Refinements of Kober's inequality.* In [70], by the help of two auxiliary functions

$$\cos x - 1 + \frac{2}{\pi}x - \alpha x(\pi^2 - x^2) \quad \text{and} \quad \cos x - 1 + \frac{2}{\pi}x - \beta x(\pi - 2x) \tag{2.1}$$

with undetermined positive constants α and β for $x \in \left[0, \frac{\pi}{2}\right]$, Kober's inequality (1.2) was refined as

$$1 - \frac{2}{\pi}x + \frac{\pi - 2}{\pi^2}x(\pi - 2x) \le \cos x \le 1 - \frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi - 2x), \tag{2.2}$$

$$1 - \frac{2}{\pi}x + \frac{\pi - 2}{2\pi^3}x\left(\pi^2 - 4x^2\right) \le \cos x \le 1 - \frac{2}{\pi}x + \frac{2}{\pi^3}x\left(\pi^2 - 4x^2\right). \tag{2.3}$$

These two double inequalities are sharp in the sense that the constants $\frac{\pi-2}{\pi^2}$, $\frac{2}{\pi^2}$, $\frac{\pi-2}{2\pi^3}$ and $\frac{2}{\pi^3}$ connot be replaced by larger or smaller ones respectively.

The inequality (2.2) is better than (2.3). The inequality (2.2) may be rewritten as

$$1 - \frac{4 - \pi}{\pi}x - \frac{2(\pi - 2)}{\pi^2}x^2 \le \cos x \le 1 - \frac{4}{\pi^2}x^2. \tag{2.4}$$

The double inequality (2.4) is stronger than (1.2) on $\left[0, \frac{\pi}{2}\right]$.

Replacing x by $\frac{\pi}{2} - x$ in (2.4) gives

$$x - \frac{2(\pi - 2)}{\pi^2}x^2 \le \sin x \le \frac{4}{\pi}x - \frac{4}{\pi^2}x^2, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{2.5}$$

The lower bound in (2.5) is better than the corresponding one in (1.10) and it is not included in or includes the inequality (1.8).

2.1.2. *Refinements of Jordan's inequality.* In [73], by considering auxiliary functions

$$\sin x - \frac{2}{\pi}x - \alpha x\left(\pi^2 - 4x^2\right), \quad \sin x - \frac{2}{\pi}x - \beta x^2(\pi - 2x)$$

and

$$\sin x - \frac{2}{\pi}x - \theta x(\pi - 2x)$$

on $\left[0, \frac{\pi}{2}\right]$, the inequality (1.8) was recovered and the following inequalities were also obtained:

$$\sin x \le \frac{2}{\pi}x + \frac{\pi - 2}{\pi^3}x\left(\pi^2 - 4x^2\right), \tag{2.6}$$

$$\sin x \ge \frac{2}{\pi}x + \frac{4}{\pi^3}x^2(\pi - 2x), \tag{2.7}$$

$$\frac{2}{\pi}x + \frac{\pi - 2}{\pi^2}x(\pi - 2x) \le \sin x \le \frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi - 2x), \tag{2.8}$$

where the constants $\frac{\pi-2}{\pi^3}$, $\frac{4}{\pi^3}$, $\frac{\pi-2}{\pi^2}$ and $\frac{2}{\pi^2}$ are the best possible.

The inequality (2.8) may be rewritten as (2.5). Therefore, inequalities (2.4) and (2.8) are equivalent to each other.

Combination of (1.8) and (2.6) leads to

$$\frac{3}{\pi}x - \frac{4}{\pi^3}x^3 \le \sin x \le x - \frac{4(\pi - 2)}{\pi^3}x^3, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{2.9}$$

Inequalities (2.5) and (2.9) are not included each other on $\left[0, \frac{\pi}{2}\right]$. The inequality (2.7) is weaker than the left-hand side inequality in (2.9) and can not compare with the left-hand side inequality of (2.5).

In [26], by the method used in [70, 73, 74], the following inequalities were deduced: For $x \in \left[0, \frac{\pi}{2}\right]$,

$$\sin x \ge \frac{2}{\pi}x + \frac{2}{\pi^4}x^2\left(\pi^2 - 4x^2\right), \tag{2.10}$$

$$\sin x \ge \frac{2}{\pi}x + \frac{8}{\pi^4}x^3(\pi - 2x), \tag{2.11}$$

$$\frac{2}{3\pi^4}x\left(\pi^3 - 8x^3\right) \le \sin x - \frac{2}{\pi}x \le \frac{\pi-2}{\pi^4}x\left(\pi^3 - 8x^3\right). \tag{2.12}$$

In [67], by constructing suitable auxiliary functions as above, the inequality (2.6) or the right-hand side inequality in (2.9), the double inequality (2.8) or (2.5), the inequality (2.7), the double inequality (2.2) or (2.4), the double inequality (2.3) and their sharpness are verified again. Employing these inequalities, it was derived in [67] that

$$\frac{4}{3} < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{\pi+1}{3} \tag{2.13}$$

and

$$\frac{1}{2} < \int_0^{\pi/2} \frac{1-\cos x}{x}\,\mathrm{d}x < \frac{6-\pi}{4}. \tag{2.14}$$

In [75], inequalities (1.8) and (2.6) or their variant (2.9) and the inequality (2.2) or (2.4) were proved once again by considering suitable auxiliary functions as above. From (2.9) and the symmetry and period of $\sin x$, it was deduced in [75] that

$$\begin{aligned}\frac{4}{\pi^3}x^3 - \frac{12}{\pi^2}x^2 + \frac{9}{\pi}x - 1 &\le \sin x\\ &\le \frac{4(\pi-2)}{\pi^3}x^3 - \frac{12(\pi-2)}{\pi^2}x^2 + \frac{11\pi-24}{\pi}x + 8 - 3\pi \end{aligned} \tag{2.15}$$

on $\left[\frac{\pi}{2}, \pi\right]$ and

$$\frac{7}{6} - \ln 2 < \int_{\pi/2}^{\pi} \frac{\sin x}{x} dx < \frac{13\pi - 32}{6} + (8 - 3\pi)\ln 2. \tag{2.16}$$

2.2. **Refinements of Jordan's inequality by L'Hôspital's rule.**

2.2.1. *L'Hôspital's rule.* The following monotonic form of the famous L'Hôspital's rule was put forward in [3, Theorem 1.25].

Lemma 1. *Let f and g be continuous on $[a,b]$ and differentiable in (a,b) such that $g'(x) \ne 0$ in (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) in (a,b), then so are the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ on (a,b).*

2.2.2. *Zhang-Wang-Chu's recoveries.* In [107], by using Lemma 1, inequalities (1.8), (2.2), (2.3), (2.6) and (2.8) were recovered once more.

2.3. **Li's power series expansion and refinements of Jordan's inequality.**

In [50], a power series expansion was established as follows: For $x > 0$,

$$\frac{\sin x}{x} = \frac{2}{\pi} + \sum_{k=1}^{\infty}(-1)^k \frac{R_k(\pi/2)}{k!\pi^{2k}}\left(\pi^2 - 4x^2\right)^k, \tag{2.17}$$

where

$$R_k(x)=\sum_{n=k}^{\infty}\frac{(-1)^n n!}{(2n+1)!(n-k)!}x^{2n} \tag{2.18}$$

satisfy $(-1)^k R_k(\pi/2)>0$ and

$$R_1(x)=\frac{x}{2}\left(\frac{\sin x}{x}\right)' \quad \text{and} \quad R_{k+1}(x)=-kR_k(x)+\frac{x}{2}R_k'(x) \tag{2.19}$$

for $k\in\mathbb{N}$.

As a direct consequence of above identity, the following lower bound for the function $\frac{\sin x}{x}$ was established in [50]:

$$\begin{aligned}\frac{\sin x}{x}&\ge\frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right)+\frac{12-\pi^2}{16\pi^5}\left(\pi^2-4x^2\right)^2\\&+\frac{10-\pi^2}{16\pi^7}\left(\pi^2-4x^2\right)^3+\frac{\pi^4-180\pi^2+1680}{3072\pi^9}\left(\pi^2-4x^2\right)^4,\quad 0<x<\frac{\pi}{2}.\end{aligned} \tag{2.20}$$

Equality in (2.20) is valid if and only if $x=\frac{\pi}{2}$. The constants $\frac{1}{\pi^3}$, $\frac{12-\pi^2}{16\pi^5}$, $\frac{10-\pi^2}{16\pi^7}$ and $\frac{\pi^4-180\pi^2+1680}{3072\pi^9}$ are the best possible.

Moreover, by employing

$$\frac{x}{\sin x}=\sum_{k=0}^{\infty}(-1)^{k+1}B_{2k}\frac{2^{2k}-2}{(2k)!}x^{2k} \tag{2.21}$$

for $|x|<\pi$, where B_{2k} for $0\le k<\infty$ is the well-known Bernoulli's numbers, it was presented in [50] that

$$\frac{x}{\sin x}\le 1+\frac{1}{6}x^2+\frac{7}{360}x^4+\frac{31}{15120}x^6,\quad |x|<\pi. \tag{2.22}$$

This refines (1.10) for $0<x\le\frac{\pi}{2}$.

2.4. **Li-Li's refinements and generalizations.** In [51], two seemingly general but not much significant results for refining or generalizing Jordan's inequality (1.1) were discovered.

2.4.1. The first result may be stated as follows: If the function $g:\left[0,\frac{\pi}{2}\right]\to[0,1]$ is continuous and

$$\frac{\sin x}{x}\ge g(x) \tag{2.23}$$

for $x\in\left[0,\frac{\pi}{2}\right]$, then the double inequality

$$\frac{2}{\pi}-h\left(\frac{\pi}{2}\right)+h(x)\le\frac{\sin x}{x}\le 1+h(x) \tag{2.24}$$

for $x\in\left[0,\frac{\pi}{2}\right]$ holds with equality if and only if $x=\frac{\pi}{2}$, where

$$h(x)=-\int_0^x\frac{1}{u^2}\int_0^u v^2g(v)\,\mathrm{d}v\,\mathrm{d}u,\quad x\in\left[0,\frac{\pi}{2}\right]. \tag{2.25}$$

Since $g(x)$ is positive, it is clear that the function $h(x)$ is decreasing and negative, therefore, the double inequality (2.24) refines Jordan's inequality (1.1).

It is remarked that the upper bound in (2.24) was not considered in [51], although it is implied in the arguments. On the other hand, if the inequality (2.23) is reversed, then so is the inequality (2.24).

Upon taking $g(x)=0$ in (2.23) and (2.25), Jordan's inequality (1.1) is derived from (2.24). If letting $g(x)=\frac{2}{\pi}$, then inequalities (1.7) and

$$\frac{\sin x}{x}\le 1-\frac{1}{3\pi}x^2,\quad x\in\left(0,\frac{\pi}{2}\right] \tag{2.26}$$

is deduced from (2.24). If choosing $g(x)$ as the function in the right-hand side of (1.7), then the inequality

$$\frac{\sin x}{x}\ge\frac{2}{\pi}+\frac{60+\pi^2}{720\pi}\left(\pi^2-4x^2\right)+\frac{1}{960\pi}\left(\pi^2-4x^2\right)^2,\quad x\in\left(0,\frac{\pi}{2}\right] \tag{2.27}$$

follows from the left-hand side of (2.24). These three examples given in [51] seemly show that, by using some lower bound for $\frac{\sin x}{x}$ on $\left(0,\frac{\pi}{2}\right]$, a corresponding stronger lower bound may be derived from the left-hand side inequality in (2.24). Actually, this is not always valid: By taking $g(x)$ as the function in the right-hand side of (1.8) or the one in the left-hand side of (2.9), it was obtained that

$$\frac{\sin x}{x}\ge\frac{2}{\pi}+\frac{1}{60\pi}\left(\pi^2-4x^2\right)+\frac{1}{80\pi^3}\left(\pi^2-4x^2\right)^2,\quad x\in\left(0,\frac{\pi}{2}\right]. \tag{2.28}$$

Unluckily, the inequality (2.28) is worse than both the inequality (1.8) and the left-hand side inequality in (2.9). This tells us that the inequality

$$\frac{2}{\pi}-h\left(\frac{\pi}{2}\right)+h(x)>g(x),\quad x\in\left(0,\frac{\pi}{2}\right] \tag{2.29}$$

is not always sound. Therefore, Theorem 2.1 in [51], one of the main results in [51], is not always significant and meaningful. This reminds us of proposing a question: Under what conditions on $0<g(x)<1$ for $x\in\left(0,\frac{\pi}{2}\right]$, the inequality (2.29) holds?

2.4.2. The second result in [51] is procured basing on Lemma 1. It can be summarized as follows: If the function $f(x)\in C^2\left[0,\frac{\pi}{2}\right]$ satisfies $f'(x)>0$ and $\left[x^2f'(x)\right]'\ne0$ for $x\in\left[0,\frac{\pi}{2}\right]$, then the double inequality

$$\begin{aligned}\lim_{x\to0^+}\frac{\sin x/x-2/\pi}{f(x)-f(\pi/2)}\left[f(x)-f\left(\frac{\pi}{2}\right)\right]&\le\frac{\sin x}{x}-\frac{2}{\pi}\\&\le\lim_{x\to(\pi/2)^-}\frac{\sin x/x-2/\pi}{f(x)-f(\pi/2)}\left[f(x)-f\left(\frac{\pi}{2}\right)\right],\quad x\in\left(0,\frac{\pi}{2}\right]\end{aligned} \tag{2.30}$$

is sharp in the sense that the limits before brackets in (2.30) can not be replaced by larger or smaller numbers. If $f'(x)<0$ and $\left[x^2f'(x)\right]'\ne0$, then the inequality (2.30) is reversed.

As an application, by taking $f(x)=x^n$ for $n\in\mathbb{N}$ in (2.30), the inequality (2.8) and

$$\frac{2}{\pi}+\frac{2}{n\pi^{n+1}}\left[\pi^n-(2x)^n\right]\le\frac{\sin x}{x}\le\frac{2}{\pi}+\frac{\pi-2}{\pi^{n+1}}\left[\pi^n-(2x)^n\right],\quad n\ge2 \tag{2.31}$$

were showed in [51, Theorem 3.2], where the equalities hold if and only if $x=\frac{\pi}{2}$ and the constants $\frac{2}{n\pi^{n+1}}$ and $\frac{\pi-2}{\pi^{n+1}}$ in (2.31) are the best possible.

If taking $n=2$ in (2.31), then the inequalities (1.8) and (2.6) are recovered.

It is worthwhile to remark that essentially established in Section 3 of [51] are sufficient conditions for the function $\frac{\sin x/x-2/\pi}{f(x)-f(\pi/2)}$ to be monotonic on $\left[0,\frac{\pi}{2}\right]$. Generally, more new sufficient conditions may be further found.

2.5. **Some generalizations of Redheffer-Williams's inequality.**

2.5.1. *Chen-Zhao-Qi's results.* In [24], the following three inequalities similar to (1.5) were established: If $|x| \leq \frac{1}{2}$, then

$$\cos(\pi x) \geq \frac{1-4x^2}{1+4x^2} \quad \text{and} \quad \cosh(\pi x) \leq \frac{1+4x^2}{1-4x^2}; \tag{2.32}$$

If $0 < |x| < 1$, then

$$\frac{\sinh(\pi x)}{\pi x} \leq \frac{1+x^2}{1-x^2}. \tag{2.33}$$

2.5.2. *Zhu-Sun's results.* In [121], by using Lemma 1 and other techniques, the above three inequalities are sharpened and some more results were demonstrated as follow:

(1) The double inequality

$$\left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\beta} \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\alpha} \tag{2.34}$$

holds for $0 < x < \pi$ if and only if $\alpha \leq \frac{\pi^2}{12}$ and $\beta \geq 1$.

(2) The double inequality

$$\left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\beta} \leq \cos x \leq \left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\alpha} \tag{2.35}$$

holds for $0 \leq x \leq \frac{\pi}{2}$ if and only if $\alpha \leq \frac{\pi^2}{16}$ and $\beta \geq 1$.

(3) The double inequality

$$\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\alpha} \leq \frac{\tan x}{x} \leq \left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\beta} \tag{2.36}$$

holds for $0 < x < \frac{\pi}{2}$ if and only if $\alpha \leq \frac{\pi^2}{24}$ and $\beta \geq 1$.

(4) The double inequality

$$\left(\frac{r^2+x^2}{r^2-x^2}\right)^{\alpha} \leq \frac{\sinh x}{x} \leq \left(\frac{r^2+x^2}{r^2-x^2}\right)^{\beta} \tag{2.37}$$

holds for $0 < x < r$ if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{12}$.

(5) The double inequality

$$\left(\frac{r^2+x^2}{r^2-x^2}\right)^{\alpha} \leq \cosh x \leq \left(\frac{r^2+x^2}{r^2-x^2}\right)^{\beta} \tag{2.38}$$

holds for $0 \leq x < r$ if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{4}$.

(6) The double inequality

$$\left(\frac{r^2-x^2}{r^2+x^2}\right)^{\beta} \leq \frac{\tanh x}{x} \leq \left(\frac{r^2-x^2}{r^2+x^2}\right)^{\alpha} \tag{2.39}$$

holds for $0 < x < r$ if and only if $\alpha \leq 0$ and $\beta \geq \frac{r^2}{6}$.

2.6. **Some generalizations and related results.**

2.6.1. In [94], it was obtained that

$$\frac{2}{\pi} \le \left|\frac{\sin(\lambda y)}{\lambda x \sin\frac{\pi y}{2x}}\right| \le \left|\frac{\sin(\lambda x)}{\lambda x}\right| \le \left|\frac{\sin(\lambda y)}{\lambda y}\right| < 1 \tag{2.40}$$

for $0 < |y| < |x|$ and $0 < |\lambda x| \le \frac{\pi}{2}$.

In [93], by considering the logarithmic concavity of $\frac{\sin x}{x}$ and the logarithmic convexity of $\frac{\tan x}{x}$ and by using Jensen's inequality for convex functions, it was obtained that

$$\begin{aligned}\left|\prod_{i=1}^{n}\tan x_i\right| &\ge \left|\prod_{i=1}^{n} x_i\left[\frac{\tan\frac{\sum_{i=1}^{n}|x_i|}{n}}{\frac{\sum_{i=1}^{n}|x_i|}{n}}\right]^n\right| \\ &> \left|\prod_{i=1}^{n} x_i\right| > \left|\prod_{i=1}^{n} x_i\left[\frac{\sin\frac{\sum_{i=1}^{n}|x_i|}{n}}{\frac{\sum_{i=1}^{n}|x_i|}{n}}\right]^n\right| \ge \left|\prod_{i=1}^{n}\sin x_i\right| \end{aligned} \tag{2.41}$$

for $0 < |x_i| < \frac{\pi}{2}$, $1 \le i \le n$ and $n \in \mathbb{N}$ and that

$$\frac{|\tan(\alpha x)|}{\alpha|x|} > \frac{|\tan(\beta x)|}{\beta|x|} > 1 > \frac{|\sin(\beta x)|}{\beta|x|} > \frac{|\sin(\alpha x)|}{\alpha|x|} > \frac{|\sin(\beta x)|}{\alpha|x|}\csc\frac{\beta\pi}{2\alpha} \tag{2.42}$$

for $0 < \beta < \alpha$ and $0 < |\alpha x| < \frac{\pi}{2}$.

In [83], it was proved that a positive and concave function is logarithmically concave and that the function $\frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$ is a concave function. As a corollary, the following inequality was obtained:

$$\frac{\sin x}{x} \ge 1 + \frac{2(2-\pi)}{\pi^2}x \ge \frac{2}{\pi}, \quad 0 < x \le \frac{\pi}{2}. \tag{2.43}$$

This inequality is better than (1.10) and it is not included in or includes (1.8).

In passing it is pointed out that the the above relationship between concave functions and logarithmically concave functions was also verified much simply in [56, p. 85].

2.6.2. Some results obtained in [52, 82, 88] and the related references therein may be also interesting.

2.6.3. In [46, 47], [48, pp. 269–288] and [57, pp. 235–265], a large amount of inequalities involving trigonometric functions are collected.

3. Refinements of Jordan's inequality and Yang's inequality

3.1. **Yang's inequality.** In [105, pp. 116–118], an inequality due to L. Yang states that the inequality

$$\cos^2(\lambda A) + \cos^2(\lambda B) - 2\cos(\lambda A)\cos(\lambda B)\cos(\lambda\pi) \ge \sin^2(\lambda\pi) \tag{3.1}$$

is valid for $0 \le \lambda \le 1$, $A > 0$ and $B > 0$ with $A + B \le \pi$, where the equality holds if and only if $\lambda = 0$ or $A + B = \pi$.

The inequality (3.1) has been generalized in [109, 111] and related references therein.

3.2. **Zhao's result.** In [110, Theorem 1 and Theorem 2], by using inequalities (1.1) and (1.5) respectively, it was concluded that

$$4\binom{n}{2}\lambda^2\cos^2\left(\frac{\pi}{2}\lambda\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\pi^2\lambda^2 \tag{3.2}$$

and

$$\binom{n}{2}\left(\frac{1-\lambda^2}{1+\lambda^2}\right)^2 \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\pi^2\lambda^2, \tag{3.3}$$

where

$$H_{ij} = \cos^2(\lambda A_i) + \cos^2(\lambda A_j) - 2\cos(\lambda A_i)\cos(\lambda A_j)\cos(\lambda\pi) \tag{3.4}$$

for $0 \le \lambda \le 1$ and $A_i > 0$ with $\sum_{i=1}^n A_i \le \pi$ for $n \ge 2$.

This generalizes Yang's inequality (3.1).

3.3. **Debnath-Zhao's result.** In [25], inequalities (1.7) and (1.8) or the left-hand side inequality in (2.9) were recovered once again. However, it seems that the authors of the paper [25] did not compare (1.7) and (1.8) explicitly.

As an application of (1.8), with the help of

$$\sin^2(\lambda\pi) \le H_{ij} \le 4\sin^2\left(\frac{\lambda}{2}\pi\right) \tag{3.5}$$

in [109] and [111, (2.13)], Yang's inequality (3.1) was generalized in [25] to

$$\binom{n}{2}\lambda^2\left(3-\lambda^2\right)^2\cos^2\left(\frac{\lambda}{2}\pi\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\lambda^2\pi^2. \tag{3.6}$$

3.4. **Özban's result.** In [62], a new refined form of Jordan's inequality was given for $0 < x \le \frac{\pi}{2}$ as follows:

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \frac{\pi-3}{\pi^3}(2x-\pi)^2 \tag{3.7}$$

with equality if and only if $x = \frac{\pi}{2}$. As an application of (3.7) as in [25], the lower bound in (3.6) was refined as

$$\sum_{1\le i<j\le n} H_{ij} \ge \binom{n}{2}\lambda^2\left[\pi + (6-2\pi)\lambda + (\pi-4)\lambda^2\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right). \tag{3.8}$$

3.5. **Jiang-Hua's result.**

3.5.1. Motivated the papers [26, 67], it was procured in [41] that

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{8x}{\pi^3}\left(\frac{\pi}{2} - x\right) + \frac{4(\pi-2)}{\pi^3}\left(\frac{\pi}{2} - x\right)^2 \tag{3.9}$$

for $x \in \left(0, \frac{\pi}{2}\right]$. Equality in (3.9) holds if and only if $x = \frac{\pi}{2}$.

As an application of (3.9), Yang's inequality (3.1) is generalized and refined as

$$4\binom{n}{2}\lambda^2\left[\frac{\pi-2}{2}(\lambda-1)^2 + \lambda(1-\lambda) + 1\right]^2\cos^2\left(\frac{\pi\lambda}{2}\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\pi^2\lambda^2. \tag{3.10}$$

3.5.2. In [42], by Lemma 1, the inequality

$$\frac{1}{2\pi^5}\left(\pi^4-16x^4\right)\le\frac{\sin x}{x}-\frac{2}{\pi}\le\frac{\pi-2}{\pi^5}\left(\pi^4-16x^4\right) \tag{3.11}$$

for $0<x\le\frac{\pi}{2}$, a refinement of Jordan's inequality (1.1), was presented. Meanwhile, Yang's inequality was refined as

$$\binom{n}{2}\frac{\lambda^2\left(5-\lambda^4\right)^2}{4}\cos^2\left(\frac{\lambda}{2}\pi\right)\le\sum_{1\le i<j\le n}H_{ij}\le\binom{n}{2}\lambda^4\left[1+2\lambda^3-\lambda^4\right]^2. \tag{3.12}$$

3.6. **Agarwal-Kim-Sen's result.** In [2], inequalities (3.7) and (3.19) were refined as follows: For $0<x\le\frac{\pi}{2}$, the double inequality

$$1+B_1x-B_2x^2+B_3x^3\le\frac{\sin x}{x}\le1+C_1x-C_2x^2+B_3x^3 \tag{3.13}$$

holds with equalities if and only if $x=\frac{\pi}{2}$ and

$$B_1=\frac{4}{\pi^2}\left(66-43\pi+7\pi^2\right),\quad B_2=\frac{4}{\pi^3}\left(124-83\pi+14\pi^2\right),\quad B_3=\frac{16}{\pi^4}(\pi-3),$$
$$C_1=\frac{4}{\pi^2}\left(75-49\pi+8\pi^2\right),\quad C_2=\frac{4}{\pi^3}\left(142-95\pi+16\pi^2\right).$$

By using (3.13), Yang's inequality was refined in [2, Theorem 3.1] as

$$U(\lambda)\le\sum_{1\le i<j\le n}\sin^2(\lambda\pi)\le\sum_{1\le i<j\le n}H_{ij}\le\sum_{1\le i<j\le n}\lambda^2\pi^2, \tag{3.14}$$

where

$$U(\lambda)=\frac{n(n-1)}{2}\lambda^2[B(\lambda;\pi)]^2\cos^2\left(\frac{\lambda}{2}\pi\right) \tag{3.15}$$

with

$$B(\lambda;\pi)=\pi+2\left(66-43\pi+7\pi^2\right)\lambda-\left(124-83\pi+14\pi^2\right)\lambda^2+2(\pi-3)\lambda^3. \tag{3.16}$$

3.7. **Zhu's results.**

3.7.1. In [119], inequalities (1.8) and (2.6), eqivalently the double inequality (2.9), and their sharpness were recovered once more by using Lemma 1.

As an application of (2.6), the upper bound in (3.6) was refined as

$$\sum_{1\le i<j\le n}H_{ij}\le4\binom{n}{2}\left[\lambda^3+\frac{\lambda\left(1-\lambda^2\right)\pi}{2}\right]^2. \tag{3.17}$$

3.7.2. In [120], by using Lemma 1, the inequality (3.7) and the following two refined forms of Jordan's inequality were established:

$$\frac{12-\pi^2}{16\pi^5}\left(\pi^2-4x^2\right)^2\le\frac{\sin x}{x}-\frac{2}{\pi}-\frac{1}{\pi^3}\left(\pi^2-4x^2\right)\le\frac{\pi-3}{\pi^5}\left(\pi^2-4x^2\right)^2, \tag{3.18}$$

$$\frac{\sin x}{x}\le\frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right)+\frac{12-\pi^2}{\pi^3}\left(x-\frac{\pi}{2}\right)^2. \tag{3.19}$$

The inequality (3.18) and the right-hand side inequality in (3.19) were also applied to obtain

$$N_3(\lambda)\le\sum_{1\le i<j\le n}H_{ij}\le\min\left\{M_3(\lambda),M_3'(\lambda)\right\}, \tag{3.20}$$

where

$$N_3(\lambda) = \binom{n}{2}\lambda^2\left[3-\lambda^2+\frac{12-\pi^2}{16}\left(1-\lambda^2\right)^2\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right),$$

$$M_3(\lambda) = \binom{n}{2}\lambda^2\left[3-\lambda^2+(\pi-3)\left(1-\lambda^2\right)^2\right]^2,$$

$$M_3'(\lambda) = \binom{n}{2}\lambda^2\left[3-\lambda^2+\frac{12-\pi^2}{4}(1-\lambda)^2\right]^2.$$

3.7.3. In [113], a general refinement of Jordan's inequality (1.1) was presented by a different approach from that used in [60, 61] as follows: For $0 < x \le \frac{\pi}{2}$ and any non-negative integer $n \ge 0$, the inequality

$$a_{n+1}\left(\pi^2-4x^2\right)^{n+1} \le \frac{\sin x}{x} - P_{2n}(x) \le \frac{1-\sum_{k=0}^{n} a_k\pi^{2k}}{\pi^{2(n+1)}}\left(\pi^2-4x^2\right)^{n+1} \tag{3.21}$$

is valid with the equalities if and only if $x = \frac{\pi}{2}$, where

$$P_{2n}(x) = \sum_{k=0}^{n} a_k\left(\pi^2-4x^2\right)^k \tag{3.22}$$

and a_k satisfies the recurrent formula

$$a_0 = \frac{2}{\pi}, \quad a_1 = \frac{1}{\pi^3}, \quad a_{k+1} = \frac{2k+1}{2(k+1)\pi^2}a_k - \frac{1}{16k(k+1)\pi^2}a_{k-1} \tag{3.23}$$

for $k \in \mathbb{N}$. Furthermore, the constants a_{n+1} and $\frac{1-\sum_{k=0}^{n} a_k\pi^{2k}}{\pi^{2(n+1)}}$ in (3.21) are the best possible.

Moreover, the following series expansion for $\frac{\sin x}{x}$ was also deduced in [113]: For $0 < x \le \frac{\pi}{2}$ and $n \ge 0$, we have

$$\frac{\sin x}{x} = P_{2n}(x) + Q_{2n+2}, \tag{3.24}$$

where the reminder term is

$$Q_{2n+2} = \frac{1}{2^{3(n+1)}(n+1)!(2n+3)!!}\cdot\frac{\sin\eta}{\eta}\left(\pi^2-4x^2\right)^{n+1}, \quad 0 < \eta < \frac{\pi}{2}. \tag{3.25}$$

If taking $n \to \infty$ in (3.24), since $\lim_{n\to\infty} Q_{2n+2} = 0$, then

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} a_k\left(\pi^2-4x^2\right)^k, \quad 0 < |x| \le \frac{\pi}{2}, \tag{3.26}$$

which implies

$$\sum_{k=0}^{\infty} a_k\pi^{2k} = 1. \tag{3.27}$$

As an application of (3.21), a general improvement of Yang's inequality (3.1) was deduced in [113] as

$$\binom{n}{2}(\lambda\pi)^2\left[P_{2n}\left(\frac{\lambda}{2}\pi\right) + a_{n+1}\pi^{2(n+1)}\left(1-\lambda^2\right)^{n+1}\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right) \le \sum_{1\le i<j\le n} H_{ij}$$
$$\le \binom{n}{2}(\lambda\pi)^2\left[P_{2n}\left(\frac{\lambda}{2}\pi\right) + \left(1-\sum_{k=0}^{n} a_k\pi^{2k}\right)\left(1-\lambda^2\right)^{n+1}\right]^2. \tag{3.28}$$

3.8. **Niu-Huo-Cao-Qi's result.** In [60, 61], the following general refinement of Jordan's inequality was presented: For $0 < x \le \frac{\pi}{2}$ and $n \in \mathbb{N}$, the inequality

$$\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le \frac{\sin x}{x} \le \frac{2}{\pi} + \sum_{k=1}^{n} \beta_k \left(\pi^2 - 4x^2\right)^k \tag{3.29}$$

holds with the equalities if and only if $x = \frac{\pi}{2}$, where the constants

$$\alpha_k = \frac{(-1)^k}{(4\pi)^k k!} \sum_{i=1}^{k+1} \left(\frac{2}{\pi}\right)^i a_{i-1}^k \sin\left(\frac{k+i}{2}\pi\right) \tag{3.30}$$

and

$$\beta_k = \begin{cases} \dfrac{1 - 2/\pi - \sum_{i=1}^{n-1} \alpha_i \pi^{2i}}{\pi^{2n}}, & k = n \\ \alpha_k, & 1 \le k < n \end{cases} \tag{3.31}$$

with

$$a_i^k = \begin{cases} (i + k - 1) a_{i-1}^{k-1} + a_i^{k-1}, & 0 < i \le k \\ 1, & i = 0 \\ 0, & i > k \end{cases} \tag{3.32}$$

in (3.29) are the best possible.

As an application of inequality (3.29), a refinement and generalization of Yang's inequality (3.1) is obtained: For $0 \le \lambda \le 1$ and $A_i > 0$ such that $\sum_{i=1}^{n} A_i \le \pi$, if $m \in \mathbb{N}$ and $n \ge 2$, then

$$L_m(n, \lambda) \le \sum_{1 \le i < j \le n} H_{ij} \le R_m(n, \lambda), \tag{3.33}$$

where

$$L_m(n, \lambda) = \binom{n}{2} \lambda^2 \left[2 + \sum_{k=1}^{m} \alpha_k \pi^{2k+1} \left(1 - \lambda^2\right)^k\right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right), \tag{3.34}$$

$$R_m(n, \lambda) = \binom{n}{2} \lambda^2 \left[2 + \sum_{k=1}^{m} \beta_k \pi^{2k+1} \left(1 - \lambda^2\right)^k\right]^2. \tag{3.35}$$

4. Generalizations of Jordan's inequality and applications

4.1. **Qi-Niu-Cao's generalization and application.** In [60, 78], a general generalization of Jordan's inequality was established: For $0 < x \le \theta < \pi$, $n \in \mathbb{N}$ and $t \ge 2$, the inequality

$$\sum_{k=1}^{n} \mu_k \left(\theta^t - x^t\right)^k \le \frac{\sin x}{x} - \frac{\sin \theta}{\theta} \le \sum_{k=1}^{n} \omega_k \left(\theta^t - x^t\right)^k \tag{4.1}$$

holds with the equalities if and only if $x = \theta$, where the constants

$$\mu_k = \frac{(-1)^k}{k! t^k} \sum_{i=1}^{k+1} a_{i-1}^k \theta^{k-i-kt} \sin\left(\theta + \frac{k+i-1}{2}\pi\right) \tag{4.2}$$

and

$$\omega_k = \begin{cases} \dfrac{1-\sin\theta/\theta-\sum_{i=1}^{n-1}\mu_i\theta^{ti}}{\theta^{tn}}, & k=n \\ \mu_k, & 1\le k<n \end{cases} \tag{4.3}$$

with

$$a_i^k = \begin{cases} a_i^{k-1}+[i+(k-1)(t-1)]a_{i-1}^{k-1}, & 0<i\le k \\ 1, & i=0 \\ 0, & i>k \end{cases} \tag{4.4}$$

in (4.1) are the best possible.

As an application of inequality (4.1), Yang's inequality was refined as follows: Let $0\le\lambda\le1$, $0<x\le\theta<\pi$, $t\ge2$ and $A_i>0$ with $\sum_{i=1}^n A_i\le\pi$ for $n\in\mathbb{N}$. If $m\in\mathbb{N}$ and $n\ge2$, then

$$L_m(n,\lambda)\le\sum_{1\le i<j\le n}H_{ij}\le R_m(n,\lambda), \tag{4.5}$$

where

$$L_m(n,\lambda)=\binom{n}{2}\lambda^2\pi^2\left[\frac{\sin\theta}{\theta}+\sum_{k=1}^m 2^{-kt}\mu_k\left(2^t\theta^t-\lambda^t\pi^t\right)^k\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right), \tag{4.6}$$

$$R_m(n,\lambda)=\binom{n}{2}\lambda^2\pi^2\left[\frac{\sin\theta}{\theta}+\sum_{k=1}^m 2^{-kt}\omega_k\left(2^t\theta^t-\lambda^t\pi^t\right)^k\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right), \tag{4.7}$$

and μ_k and ω_k are defined by (4.2).

4.2. **Zhu's generalizations and applications.**

4.2.1. In [118], by making use of Lemma 1, the author obtained the following generalization of Jordan's inequality: If $0<x\le r\le\frac{\pi}{2}$, then

$$\frac{\sin r}{r}+\frac{\sin r-r\cos r}{2r^3}\left(r^2-x^2\right)\le\frac{\sin x}{x}\le\frac{\sin r}{r}+\frac{r-\sin r}{r^3}\left(r^2-x^2\right). \tag{4.8}$$

As an application of (4.8), in virtue of (3.5), Yang's inequality (3.1) was sharpened and generalized as

$$4\binom{n}{2}\left[\frac{\lambda\pi\sin r}{2r}+\frac{\sin r-r\cos r}{2r^3}\left(\frac{\lambda\pi r^2}{2}-\frac{(\lambda\pi)^3}{8}\right)\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right)$$
$$\le\sum_{1\le i<j\le n}H_{ij}\le4\binom{n}{2}\left[\frac{\lambda\pi\sin r}{2r}+\frac{r-\sin r}{r^3}\left(\frac{\lambda\pi r^2}{2}-\frac{(\lambda\pi)^3}{8}\right)\right]^2. \tag{4.9}$$

4.2.2. In [116], the double inequality (3.21) was extended by using the method in [113] as

$$A_{2n,r}(x)+\alpha_{n,r}\left(r^2-x^2\right)^{n+1}\le\frac{\sin x}{x}\le A_{2n,r}(x)+\beta_{n,r}\left(r^2-x^2\right)^{n+1} \tag{4.10}$$

and

$$A_{2m,r}(x)+\mu_{m,r}(r-x)^{m+1}\le\frac{\sin x}{x}\le A_{2m,r}(x)+\nu_{m,r}(r-x)^{m+1} \tag{4.11}$$

with the equalities in (4.10) and (4.11) if and only if $x = r$, where $0 < x \le r \le \frac{\pi}{2}$, $n \ge 0$, $m \in \mathbb{N}$ and

$$A_{2n,r}(x) = \sum_{k=0}^{n} a_{k,r}\bigl(r^2 - x^2\bigr)^k \tag{4.12}$$

with

$$a_{0,r} = \frac{\sin r}{r}, \quad a_{1,r} = \frac{\sin r - r\cos r}{2r^3}, \tag{4.13}$$

$$a_{k+1,r} = \frac{2k+1}{2(k+1)r^2} a_{k,r} - \frac{1}{4k(k+1)r^2} a_{k-1,r}, \quad k \in \mathbb{N}. \tag{4.14}$$

The constants $\alpha_{n,r} = a_{n+1}$ and

$$\beta_{n,r} = \frac{1 - \sum_{k=0}^{n} a_k r^{2k}}{r^{2(n+1)}}$$

in (4.10) and the constants

$$\mu_{m,r} = \frac{1 - \sum_{k=0}^{m} a_{k,r} r^{2k}}{r^{n+1}}$$

and $\nu_{m,r} = (2r)^{m+1} a_{m+1}$ in (4.11) are the best possible.

As an application of inequalities in (4.10), Yang's inequality (3.1) was extended or generalized as follows: If $A_i > 0$ for $i \in \mathbb{N}$ with $\sum_{i=1}^{n} A_i \le r$ for $0 < r \le \pi$ and $n \ge 2$, then

$$\max\{L_1(r), L_2(r)\} \le (n-1)\sum_{k=1}^{n} \cos^2 A_k - 2\cos r \sum_{1 \le i < j \le n} \cos A_i \cos A_j \le \min\{R_1(r), R_2(r)\}, \tag{4.15}$$

where

$$L_1(r) = \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + a_{n+1}\bigl(\pi^2 - r^2\bigr)^{n+1}\right]^2 \cos^2\frac{r}{2}, \tag{4.16}$$

$$L_2(r) = \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + \frac{1 - \sum_{k=0}^{n} a_k \pi^{2k}}{\pi^{2(n+1)}}\left(\frac{\pi - r}{2}\right)^{n+1}\right]^2 \cos^2\frac{r}{2}, \tag{4.17}$$

$$R_1(r) = \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + \frac{1 - \sum_{k=0}^{n} a_k \pi^{2k}}{\pi^{2(n+1)}}\bigl(\pi^2 - r^2\bigr)^{n+1}\right]^2, \tag{4.18}$$

$$R_2(r) = \binom{n}{2} r^2 \left[P_{2n}\left(\frac{r}{2}\right) + a_{n+1}\left(\frac{\pi - r}{2}\right)^{n+1}\right]^2. \tag{4.19}$$

4.2.3. In [112], the double inequality (4.10) was recovered by a similar method as in [113, 116].

The series expansion (3.24) was generalized in [112, Theorem 8] as follows: If $0 < x \le r \le \frac{\pi}{2}$ and $n \ge 0$, then

$$\frac{\sin x}{x} = S_{2n}(x) + R_{2n+2}, \tag{4.20}$$

where

$$S_{2n}(x) = \sum_{k=0}^{n} a_k\bigl(r^2 - x^2\bigr)^k \tag{4.21}$$

and

$$R_{2n+2} = \frac{1}{2^{n+1}(n+1)!(2n+3)!!} \cdot \frac{\sin\eta}{\eta}\left(r^2 - x^2\right)^{n+1}, \quad 0 < \eta < r \le \frac{\pi}{2} \tag{4.22}$$

with

$$a_0 = \frac{\sin r}{r}, \quad a_1 = \frac{\sin r - r\cos r}{2r^3} \tag{4.23}$$

and

$$a_{k+1} = \frac{2k+1}{2(n+1)r^2}a_k - \frac{1}{4k(k+1)r^2}a_{k-1}, \quad k \in \mathbb{N}. \tag{4.24}$$

The series expansion (3.26) was also generalized in [112, Theorem 9]: If $0 < |x| \le r \le \frac{\pi}{2}$, then

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} a_k\left(r^2 - x^2\right)^k, \tag{4.25}$$

where a_k for $k \ge 0$ are defined by (4.23) and (4.24).

As applications of the above inequalities, the following general improvement of Yang's inequality was established in [112, Theorem 11]:

$$\begin{aligned}\binom{n}{2}(\lambda\pi)^2\left[S_{2n}\left(\frac{\pi\lambda}{2}\right) + a_{n+1}\left(r^2 - \frac{1}{4}\pi^2\lambda^2\right)^{n+1}\right]^2 \cos^2\left(\frac{\pi\lambda}{2}\right) \le \sum_{1\le i<j\le n} H_{ij} \\ \le \binom{n}{2}(\lambda\pi)^2\left[S_{2n}\left(\frac{\pi\lambda}{2}\right) + \frac{1 - \sum_{k=0}^{n} a_k r^{2k}}{r^{2(n+1)}}\left(r^2 - \frac{1}{4}\pi^2\lambda^2\right)^{n+1}\right]^2,\end{aligned} \tag{4.26}$$

for $n \ge 2$ and $0 < r \le \frac{\pi}{2}$.

4.3. **Wu's generalization and applications.** In [95], Jordan's inequality (1.1) was generalized as

$$\begin{aligned}\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) + \left[1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\right]\left(1 - \frac{x}{\theta}\right)^\lambda \\ \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} \le \left(1 - \frac{\sin\theta}{\theta}\right)\left(1 - \frac{x^\lambda}{\theta^\lambda}\right),\end{aligned} \tag{4.27}$$

where $0 < x \le \theta \le \pi$ and $\lambda \ge 2$.

As an application of (4.27), Yang's inequality (3.1) was generalized as follows: If $A_i \ge 0$ for $1 \le i \le n$ and $n \ge 2$ satisfy $\sum_{i=1}^{n} A_i \le \theta \in [0, \pi]$, then

$$\begin{aligned}\binom{n}{2}\left[\left(\pi - 2 - \frac{2}{\lambda}\right)\left(1 - \frac{\theta}{\pi}\right)^\lambda - \frac{2}{\lambda}\left(\frac{\theta}{\pi}\right)^\lambda + \frac{2}{\lambda} + 2\right]^2\left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2 \\ \le (n-1)\sum_{k=1}^{n}\cos^2 A_k - 2\cos\theta\sum_{1\le i<j\le n}\cos A_i\cos A_j \\ \le \binom{n}{2}\left[2\left(\frac{\theta}{\pi}\right)^{\lambda+1} - \theta\left(\frac{\theta}{\pi}\right)^\lambda + \theta\right], \quad \lambda \ge 2.\end{aligned} \tag{4.28}$$

Moreover, the right-hand side inequality in (2.13) was recovered and the left-hand side inequality in (2.13) was improved in [95].

4.4. **Wu-Debnath's generalizations and applications.**

4.4.1. In [96], the following generalizations of Jordan's inequality was established:

$$\max\left\{\frac{3}{2}\varphi_1(\theta)\left(1-\frac{x}{\theta}\right)^2, \frac{3}{8}\varphi_2(\theta)\left(1-\frac{x^2}{\theta^2}\right)^2\right\} \\ \le \frac{\sin x}{x}-\frac{\sin\theta}{\theta}-\frac{1}{2}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\left(1-\frac{x^2}{\theta^2}\right) \\ \le \min\left\{\frac{3}{2}\varphi_2(\theta)\left(1-\frac{x}{\theta}\right)^2, \frac{3}{2}\varphi_1(\theta)\left(1-\frac{x^2}{\theta^2}\right)^2\right\} \quad (4.29)$$

for $0<x\le\theta$ and $\theta\in(0,\pi]$, where

$$\varphi_1(\theta)=\frac{2}{3}+\frac{\cos\theta}{3}-\frac{\sin\theta}{\theta} \quad \text{and} \quad \varphi_2(\theta)=\frac{\sin\theta}{\theta}-\frac{1}{3}\theta\sin\theta-\cos\theta. \quad (4.30)$$

The equalities in (4.29) hold if and only if $x=\theta$ and the coefficients of the factors $\left(1-\frac{x}{\theta}\right)^2$ and $\left(1-\frac{x^2}{\theta^2}\right)^2$ are the best possible.

If taking $\theta=\frac{\pi}{2}$ then inequalities (3.18) and (3.19) are deduced from (4.29).

Integrating on both sides of (4.29) yields

$$\max\left\{\frac{5\sin\theta-\theta\cos\theta+2\theta}{6}, \frac{23\sin\theta-8\theta\cos\theta-\theta^2\sin\theta}{15}\right\} < \int_0^\theta \frac{\sin x}{x}\,\mathrm{d}x \\ < \min\left\{\frac{11\sin\theta-5\theta\cos\theta-\theta^2\sin\theta}{6}, \frac{8\sin\theta-\theta\cos\theta+8\theta}{15}\right\}. \quad (4.31)$$

If taking $\theta=\frac{\pi}{2}$ in (4.31), then

$$\frac{92-\pi^2}{60} < \int_0^{\pi/2}\frac{\sin x}{x}\,\mathrm{d}x < \frac{8+4\pi}{15} \quad (4.32)$$

which is better than (2.13).

The basic tool for proving (4.29) is also Lemma 1.

As another application of (4.29), a generalization of Yang's inequality (3.1) was obtained: If $A_i>0$ for $1\le i\le n$ and $n\ge 2$ such that $\sum_{i=1}^n A_i\le\theta\in[0,\pi]$, then

$$\max\{N_1(\theta),N_2(\theta)\} \le \binom{n}{2}\sin^2\theta \\ \le (n-1)\sum_{k=1}^n\cos^2 A_k - 2\cos\theta\sum_{1\le i<j\le n}\cos A_i\cos A_j \\ \le 4\binom{n}{2}\sin^2\frac{\theta}{2} \le \min\{M_1(\theta),M_2(\theta)\}, \quad (4.33)$$

where

$$N_1(\theta)=\binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+(\pi-3)\left(1-\frac{\theta}{\pi}\right)^2\right]^2\left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2, \quad (4.34)$$

$$N_2(\theta)=\binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+\frac{12-\pi^2}{16}\left(1-\frac{\theta^2}{\pi^2}\right)^2\right]^2\left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2, \quad (4.35)$$

$$M_1(\theta)=\binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+\frac{12-\pi^2}{4}\left(1-\frac{\theta}{\pi}\right)^2\right]^2\left(\frac{\theta}{\pi}\right)^2, \quad (4.36)$$

$$M_2(\theta)=\binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+(\pi-3)\left(1-\frac{\theta^2}{\pi^2}\right)^2\right]^2\left(\frac{\theta}{\pi}\right)^2. \tag{4.37}$$

If substituting A_i by λA_i and θ by $\lambda\pi$ in (4.33), then inequalities (3.8) and (3.20) are deduced.

4.4.2. In [97], as a generalization of inequality (4.29), the following sharp inequality

$$\begin{aligned}\frac{1}{2\tau^2}&\left[(1+\lambda)\left(\frac{\sin\theta}{\theta}-\cos\theta\right)-\theta\sin\theta\right]\left(1-\frac{x^\tau}{\theta^\tau}\right)^2\\ &\le\frac{\sin x}{x}-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\left(1-\frac{x^\lambda}{\theta^\lambda}\right)\\ &\le\left[1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right]\left(1-\frac{x^\tau}{\theta^\tau}\right)^2\end{aligned} \tag{4.38}$$

was obtained for $0<x\le\theta\in\left(0,\frac{\pi}{2}\right]$, $\tau\ge2$ and $\tau\le\lambda\le2\tau$ by employing Lemma 1. The equalities in (4.38) holds if and only if $x=\theta$. The coefficients of the term $\left(1-\frac{x^\tau}{\theta^\tau}\right)^2$ are the best possible. If $1\le\tau\le\frac{5}{3}$ and either $\lambda\ne0$ or $\lambda\ge2\tau$ then the inequality (4.38) is reversed. Specially, when $\theta=\frac{\pi}{2}$, the inequality (4.38) becomes

$$\begin{aligned}\frac{4\lambda+4-\pi^2}{4\tau^2\pi^{2\tau+1}}\left(\pi^\tau-2^\tau x^\tau\right)^2&\le\frac{\sin x}{x}-\frac{2}{\pi}-\frac{2}{\lambda\pi^{\lambda+1}}\left(\pi^\lambda-2^\lambda x^\lambda\right)\\ &\le\frac{\lambda\pi-2\lambda-2}{\lambda\pi^{2\tau+1}}\left(\pi^\tau-2^\tau x^\tau\right)^2\end{aligned} \tag{4.39}$$

for $0<x\le\frac{\pi}{2}$, $\tau\ge2$ and $\tau\le\lambda\le2\tau$. If $1\le\tau\le\frac{5}{3}$ and either $\lambda\ne0$ or $\lambda\ge2\tau$ then the inequality (4.39) is reversed.

If taking $(\tau,\lambda)=(2,2)$ and $(\tau,\lambda)=(1,2)$, then inequalities (3.7), (3.18) and (3.19) are derived.

If $\lambda\ge2$ and $A_i\ge0$ with $\sum_{i=1}^n A_i\le\theta\in[0,\pi]$ for $n\ge2$, then the following generalization of Yang's inequality was obtained by using the inequality (4.38) in [97]:

$$\begin{aligned}\max\{K_1(\lambda,\theta),K_2(\lambda,\theta)\}\le(n-1)\sum_{k=1}^n\cos^2A_k-2\cos\theta\sum_{1\le i<j\le n}\cos A_i\cos A_j\\ \le\min\{Q_1(\lambda,\theta),Q_2(\lambda,\theta)\},\end{aligned} \tag{4.40}$$

where

$$K_1(\lambda,\theta)=\binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{\lambda\pi-2\lambda-2}{2}\left(1-\frac{\theta}{\pi}\right)^2\right]\frac{2\theta}{\lambda\pi}\cos\frac{\theta}{2}\right\}^2, \tag{4.41}$$

$$K_2(\lambda,\theta)=\binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{4\lambda\pi+4-\pi^2}{8\lambda}\left(1-\frac{\theta^\lambda}{\pi^\lambda}\right)^2\right]\frac{2\theta}{\lambda\pi}\cos\frac{\theta}{2}\right\}^2, \tag{4.42}$$

$$Q_1(\lambda,\theta)=\binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{4\lambda+4\lambda^2-\lambda\pi^2}{8}\left(1-\frac{\theta}{\pi}\right)^2\right]\frac{2\theta}{\lambda\pi}\right\}^2, \tag{4.43}$$

$$Q_2(\lambda,\theta)=\binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{\lambda\pi-2\lambda-2}{2}\left(1-\frac{\theta^\lambda}{\pi^\lambda}\right)^2\right]\frac{2\theta}{\lambda\pi}\right\}^2. \tag{4.44}$$

Note that inequalities (3.8), (3.20) and (4.33) can be deduced from (4.40).

4.4.3. By analytic techniques, the following inequalities are presented in [98]:

(1) If $0 < x \le \theta \le \pi$, then
$$\frac{\sin x}{x} \ge \frac{\sin\theta}{\theta} - \frac{\theta - \sin\theta}{\theta^2}(x-\theta). \tag{4.45}$$
(2) If $0 < x \le \pi$ and $0 < \theta \le \frac{\pi}{2}$, then
$$\frac{\sin x}{x} \le \frac{\sin\theta}{\theta} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x-\theta). \tag{4.46}$$
(3) Equalities in (4.45) and (4.46) hold if and only if $x = \theta$.

These two inequalities extend the double inequality obtained by applying $n = 1$ to the inequality (4.52).

As applications of inequalities in (4.45) and (4.46), the following double inequalities were gained: If $x_i > 0$ for $1 \le i \le n$ and $n \ge 2$ satisfying $\sum_{i=1}^n x_i = \theta$ for $0 < \theta \le \pi$, then
$$\frac{\sin\theta}{\theta} + n - 1 < \sum_{i=1}^n \frac{\sin x_i}{x_i} \le \frac{n^2}{\theta}\sin\frac{\theta}{n}, \tag{4.47}$$
$$\sum_{i=1}^n \frac{\sin x_i}{\theta - x_i} > 1 + \frac{1}{n-1}\cdot\frac{\sin\theta}{\theta} \tag{4.48}$$
and
$$\begin{aligned} 1 + (n-1)\biggl(\frac{\sin\theta}{\theta}\biggr) &< \sum_{i=1}^n \frac{\sin(\theta - x_i)}{\theta - x_i} \\ &< (n^2 - 3n + 1)\cos\frac{\theta}{n-1} - \frac{(n-1)(n^2-4n+1)}{\theta}\sin\frac{\theta}{n-1}, \quad n \ge 3. \end{aligned} \tag{4.49}$$
The equality in (4.47) holds if and only if $x_i = \frac{\theta}{n}$ for all $1 \le i \le n$.

The inequality (4.47) generalizes Janous-Klamkin's inequality [39, 44]:
$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi}, \tag{4.50}$$
where $A > 0$, $B > 0$ and $C > 0$ satisfy $A + B + C = \pi$. Meanwhile, the inequalities (4.48) and (4.49) generalize and improve Tsintsifas–Murty-Henderson's double inequality [58, 87]:
$$\frac{3}{\pi} < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{3\sqrt{3}}{\pi}, \tag{4.51}$$
where $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$ and $0 < C < \frac{\pi}{2}$ satisfy $A + B + C = \pi$.

4.5. **Wu-Srivastava's generalizations and applications.** By using Lemma 1 and other techniques, a double inequality was obtained in [100], which can be simplified as follows: Let i be a nonnegative integer and $0 < x \le \theta \le \frac{\pi}{2}$.

(1) For $n = 4i + 1$ or $n = 4i + 2$,
$$\begin{aligned} &\frac{(\theta - x)^n}{\theta^n}\Biggl[1 - \sum_{k=0}^{n-1}\sum_{\ell=0}^{k} \frac{(-1)^\ell \theta^{\ell-1}}{\ell!}\sin\biggl(\theta + \frac{\ell\pi}{2}\biggr)\Biggr] \\ &\qquad \le \frac{\sin x}{x} - \sum_{k=0}^{n-1}\sum_{\ell=0}^{k} \frac{(-1)^{k+\ell}(x-\theta)^k}{\ell!\,\theta^{k-\ell+1}}\sin\biggl(\theta + \frac{\ell\pi}{2}\biggr) \end{aligned}$$

$$\leq \frac{(\theta-x)^n}{\theta^{n+1}} \sum_{\ell=0}^{n} \frac{(-1)^\ell \theta^\ell}{\ell!} \sin\left(\theta + \frac{\ell\pi}{2}\right). \quad (4.52)$$

(2) For $n = 4i+3$ or $n = 4i+4$, the inequality (4.52) is reversed.
(3) The equalities in (4.52) hold true if and only if $x = \theta$.

Upon letting $n = 2$ in (4.52), the following inequality is derived:

$$\begin{aligned}\frac{\theta - 2\sin\theta + \theta\cos\theta}{\theta^3}&(x-\theta)^2 \\ &\leq \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \frac{\theta\cos\theta - \sin\theta}{\theta^2}(x-\theta) \\ &\leq \frac{2\sin\theta - 2\theta\cos\theta - \theta^2\sin\theta}{2\theta^3}(x-\theta)^2 \quad (4.53)\end{aligned}$$

for $0 < x \leq \theta \leq \pi$.

Upon taking $n = 2$ and $\theta = \frac{\pi}{2}$, the inequality (3.19) follows.

As a consequence of (4.52), a double inequality for estimating the definite integral $\int_0^{\pi/2} \frac{\sin x}{x} \, \mathrm{d}x$ was established in [100], which refines the double inequality (2.13).

Finally, the inequality (4.52) for $n = 5$ and $\theta = \frac{\pi}{2}$ was applied to refine and generalize Yang's inequality (3.1).

4.6. **Wu-Debnath's general generalizations and applications.** In [99], the inequality (4.52) was generalized to a general form which can be recited as follows: Let f be a real-valued $(n+1)$-time differentiable function on $[0,\theta]$ with $f(0) = 0$.

(1) If n is either a positive even number such that $f^{(n+1)}$ is increasing on $[0,\theta]$ or a positive odd number such that $f^{(n+1)}$ is decreasing on $[0,\theta]$, then the following double inequality is valid for $x \in (0,\theta]$:

$$\begin{aligned}\frac{(-1)^n}{\theta^n}&\left[f'(0^+) + \sum_{k=0}^{n-1}\sum_{i=0}^{k} \frac{(-1)^{i-1}\theta^{i-1}}{i!} f^{(i)}(\theta)\right] \\ &\leq \frac{f(x)}{x} - \sum_{k=0}^{n-1}\sum_{i=0}^{k} \frac{(-1)^i(\theta-x)^k}{i!\theta^{k-i+1}} f^{(i)}(\theta) \leq \sum_{i=0}^{n} \frac{(-1)^i(\theta-x)^n}{i!\theta^{n-i+1}} f^{(i)}(\theta). \quad (4.54)\end{aligned}$$

(2) If n is either a positive even number such that $f^{(n+1)}$ is decreasing on $[0,\theta]$ or a positive odd number such that $f^{(n+1)}$ is increasing on $[0,\theta]$, then the inequality (4.54) is reversed.
(3) The equalities in (4.54) hold if and only if $x = \theta$.

Upon taking $f(x) = \sin x$, the inequality (4.52) follows straightforwardly.

The tool of the paper [99] is Lemma 1. The authors also used their techniques to present similar inequalities for the functions

$$\frac{\sinh x}{x} \quad \text{and} \quad \frac{\ln(1+x)}{x}. \quad (4.55)$$

As consequences of the above inequalities, a double inequality for bounding the definite integral $\int_0^a \frac{\ln(1+x)}{x} \, \mathrm{d}x$ for $a > 0$ and some known inequalities were derived.

4.7. **Wu-Srivastava-Debnath's generalization and applications.** In virtue of Lemma 1, the following conclusion for bounding the function $\frac{\sin x}{x}$ was gained in [101]: For $n \in \mathbb{N}$, $0 < x \le \theta \le \pi$ and $f(x) = \frac{\sin \sqrt{x}}{\sqrt{x}}$, we have

$$\frac{f^{(n)}(\theta^2)}{n!}\left(x^2-\theta^2\right)^n \le \frac{\sin x}{x} - \sum_{k=0}^{n-1}\frac{f^{(k)}(\theta^2)}{k!}\left(x^2-\theta^2\right)^k \le \frac{1}{\theta^{2n}}\left[1-\sum_{k=0}^{n-1}\frac{(-1)^k\theta^{2k}f^{(k)}(\theta^2)}{k!}\right]\left(\theta^2-x^2\right)^n. \quad (4.56)$$

The equalities in (4.56) hold true if and only if $x = \theta$.

In [101, Lemma 3], the function $f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$ was proved to be completely monotonic on $(0, \pi^2]$. For detailed information on the class of completely monotonic functions, please see the survey paper [66] and related references therein.

In the final of [101], Yang's inequality (3.1) was generalized by virtute of the inequality (4.56) for $n = 4$ and $\theta = \frac{\pi}{2}$.

5. Refinements of Kober's inequality

5.1. **Niu's results.** As a direct consequence of (3.29), the following general refinements of Kober's inequality was obtained in [60]: For $0 < x \le \frac{\pi}{2}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$, inequalities

$$\left(\frac{\pi}{2}-x\right)\left[\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k(4x)^k(\pi-x)^k\right] \le \cos x \le \left(\frac{\pi}{2}-x\right)\left[\frac{2}{\pi}+\sum_{k=1}^{n}\beta_k(4x)^k(\pi-x)^k\right], \quad (5.1)$$

which may be deduced by replacing x with $x - \frac{\pi}{2}$ in (3.29), and

$$\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^i\binom{k}{i}\alpha_k\pi^{2k-2i}}{2i+2}x^{2i+2} \le 1-\cos x-\frac{x^2}{\pi} \le \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^i\binom{k}{i}\beta_k\pi^{2k-2i}}{2i+2}x^{2i+2}, \quad (5.2)$$

which follows from integrating (3.29) from 0 to $x \in \left[0, \frac{\pi}{2}\right]$, hold with constants α_k and β_k defined by (3.30) and (3.31) respectively.

5.2. **Zhu's result.** By an utilization of the inequality (3.21) and a simple transformation of variables, the following Kober type inequality was deduced in [112, Theorem 13]: Let

$$R(u) = \sum_{k=0}^{n}\frac{a_k}{2}\pi^{2k+1}(1-u)u^k(2-u)^k \quad (5.3)$$

and

$$S(u) = \frac{1}{2}\pi^{2n+3}(1-u)u^{n+1}(2-u)^{n+1} \quad (5.4)$$

for $n \ge 0$, where a_k for $k \ge 0$ are defined by (3.23). Then the inequality

$$R(u) + \lambda S(u) \le \cos\left(\frac{\pi u}{2}\right) \le R(u) + \mu S(u) \tag{5.5}$$

holds if either $0 \le u \le 1$, $\lambda = a_{n+1}$ and $\mu = \frac{1-\sum_{k=0}^{n} a_k \pi^{2k}}{\pi^{2(n+1)}}$ or $1 \le u \le 2$ and $\lambda = \frac{1-\sum_{k=0}^{n} a_k \pi^{2k}}{\pi^{2(n+1)}}$ and $\mu = a_{n+1}$.

6. Niu's applications and analysis of coefficients

6.1. **An application to the gamma function.** In [60], combining

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z} \tag{6.1}$$

with (3.29) yields that if $0 < x < \frac{\pi}{2}$ and $n \in \mathbb{N}$ then

$$\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le \frac{1}{\Gamma(1+x/\pi)\Gamma(1-x/\pi)} \le \frac{2}{\pi} + \sum_{k=1}^{n} \beta_k \left(\pi^2 - 4x^2\right)^k, \tag{6.2}$$

where $\Gamma(x)$ is the classical Euler gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, \mathrm{d}t. \tag{6.3}$$

6.2. **Applications to definite integrals.** In [60], as applications of (3.29), the following conclusions were also obtained:

(1) For $0 < x \le \frac{\pi}{2}$ and $k, n \in \mathbb{N}$,

$$\begin{aligned} \frac{2}{\pi}x + \sum_{k=1}^{n}\sum_{i=0}^{k} \frac{(-4)^i \binom{k}{i} \alpha_k \pi^{2k-2i}}{2i+1} x^{2i+1} &\le \int_0^x \frac{\sin t}{t} \, \mathrm{d}t \\ &\le \frac{2}{\pi}x + \sum_{k=1}^{n}\sum_{i=0}^{k} \frac{(-4)^i \binom{k}{i} \beta_k \pi^{2k-2i}}{2i+1} x^{2i+1}. \end{aligned} \tag{6.4}$$

(2) Let $f(x)$ be continuous on $[a,b]$ such that $f(x) \not\equiv 0$ and $0 \le f(x) \le M$. If $0 < b - a < \pi$ and $n \in \mathbb{N}$, then

$$\begin{aligned} 0 < \left(\int_a^b f(x)\,\mathrm{d}x\right)^2 - \left(\int_a^b f(x)\cos x\,\mathrm{d}x\right)^2 - \left(\int_a^b f(x)\sin x\,\mathrm{d}x\right)^2 \\ \le M^2(b-a)^2\left\{1 - \left[\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left(\pi^2 - a^2 - b^2 + 2ab\right)^k\right]^2\right\}. \end{aligned} \tag{6.5}$$

6.3. **Analysis of coefficients.** The coefficients α_k and β_k defined by (3.30) and (3.31) were estimated in [60] as follows: For $k > 1$,

$$-\frac{\sqrt{\pi}}{\pi^{2k}\sqrt{4k+1}} < \alpha_k < \frac{1}{\pi^{2k}\sqrt{4k+1}}, \tag{6.6}$$

$$\beta_k < \frac{1 - 2/\pi + \sqrt{\pi}\left(\sqrt{k-1} - 1/2\right)}{\pi^{2k}}, \tag{6.7}$$

$$0 \le \beta_k - \alpha_k < \frac{1 - 2/\pi + \sqrt{\pi}\left(\sqrt{k} - 1/2\right)}{\pi^{2k}}. \tag{6.8}$$

6.4. **A power series.** The inequality (3.29) can be rearranged as

$$0 \le \frac{\sin x}{x} - \frac{2}{\pi} - \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le \sum_{k=1}^{n} (\beta_k - \alpha_k)\left(\pi^2 - 4x^2\right)^k \to 0$$

as $n \to \infty$, this implies that

$$\sin x = \frac{2}{\pi} x - \sum_{k=1}^{\infty} \alpha_k x \left(\pi^2 - 4x^2\right)^k. \tag{6.9}$$

This gives an alternative power series expansion similar to (2.17) and (3.26).

6.5. **A remark.** It is natural to consider that the series (2.17), (3.26) and (6.9) should be the same one, although they seems to have different expressions.

7. Generalizations of Jordan's inequality to Bessel functions

For $x \in \mathbb{R}$, some Bessel functions are defined by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \tag{7.1}$$

$$I_p(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \tag{7.2}$$

$$\lambda_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n \Gamma\left(p + \frac{b+1}{2}\right)}{n!\Gamma\left(p + \frac{b+1}{2} + n\right)} \left(\frac{x}{2}\right)^{2n}, \tag{7.3}$$

$$\mathcal{J}_p(x) = 2^p \Gamma(p+1) x^{-p} J_p(x), \tag{7.4}$$

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x). \tag{7.5}$$

It is well-known that

$$\mathcal{J}_{-1/2}(x) = \cos x, \quad \mathcal{I}_{-1/2}(x) = \cosh x, \tag{7.6}$$

$$\mathcal{J}_{1/2}(x) = \frac{\sin x}{x}, \quad \mathcal{I}_{1/2}(x) = \frac{\sinh x}{x}. \tag{7.7}$$

7.1. **Neuman's generalizations of Jordan's inequality.** In [59], it was established for $p \ge \frac{1}{2}$ and $|x| \le \frac{\pi}{2}$ that

$$\frac{1}{3(p+1)}\left[2p+1+(p+2)\cos\left(\sqrt{\frac{3}{2(p+2)}}\,x\right)\right] \ge \mathcal{J}_p(x) \ge \cos\left(\frac{x}{\sqrt{2(p+1)}}\right). \tag{7.8}$$

When $p = -\frac{1}{2}$ equality in (7.8) validates.

Taking in (7.8) $p = \frac{1}{2}$ leads to

$$\frac{2}{9}\left[2 + \frac{5}{2}\cos\left(\sqrt{\frac{3}{5}}\,x\right)\right] \ge \frac{\sin x}{x} \ge \cos\left(\frac{x}{\sqrt{3}}\right), \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{7.9}$$

By employing Lemma 1, inequalities (2.2) and (2.9) are generalized in [10] as

$$\left[1 - \lambda_p\left(\frac{\pi}{2}\right)\right] \frac{\pi - 2x}{\pi} \le \lambda_p(x) - \lambda_p\left(\frac{\pi}{2}\right) \le \left[\left(\frac{c\pi}{2k}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right)\right] \frac{\pi - 2x}{\pi} \tag{7.10}$$

for $k \ge \frac{1}{2}$ and $0 \le c \le 1$ and

$$\left[\left(\frac{c}{4k}\right)\lambda_{p+1}\left(\frac{\pi}{2}\right)\right] \frac{\pi^2 - 4x^2}{4} \le \lambda_p(x) - \lambda_p\left(\frac{\pi}{2}\right) \le \left[1 - \lambda_p\left(\frac{\pi}{2}\right)\right] \frac{\pi^2 - 4x^2}{\pi^2} \tag{7.11}$$

for $k \geq 0$ and $0 \leq c \leq 1$.

In [8], inequalities (7.10) and (7.11) were further improved.

7.2. **Niu's generalizations of Jordan's inequality.** In [60], the following two conclusions were established:

(1) For $n \in \mathbb{N}$ and $x \in (0, \frac{\pi}{2}]$, if $k \geq \frac{1}{2}$ and $0 \leq c \leq 1$, then

$$\sum_{i=0}^{n} \gamma_i \big(\pi^2 - 4x^2\big)^i \leq \lambda_p(x) \leq \sum_{i=0}^{n} \eta_i \big(\pi^2 - 4x^2\big)^i, \tag{7.12}$$

where

$$\gamma_i = \left(\frac{c}{16}\right)^i \frac{\Gamma(k)}{i!\Gamma(k+i)} \lambda_{i+p}\big(\pi/2\big), \quad 0 \leq i \leq n, \tag{7.13}$$

and

$$\eta_i = \begin{cases} \gamma_i, & 0 \leq i \leq n-1, \\ \dfrac{1 - \sum_{\ell=0}^{n-1} \gamma_\ell \pi^{2\ell}}{\pi^{2n}}, & i = n, \end{cases} \tag{7.14}$$

are the best possible. For $k > 0$ and $c \leq 0$ and $0 < x < \theta < \infty$, when n is odd the inequality (7.12) holds, when n is even the inequality (7.12) is reversed.

(2) For $n \in \mathbb{N}$ and $0 < x \leq \theta \leq \frac{\pi}{2}$, if $k \geq \frac{1}{2}$ and $0 \leq c \leq 1$, then

$$\sum_{i=0}^{n} \sigma_i \big(\theta^2 - x^2\big)^i \leq \lambda_p(x) \leq \sum_{i=0}^{n} \nu_i \big(\theta^2 - x^2\big)^i, \tag{7.15}$$

where

$$\sigma_i = \left(\frac{c}{4}\right)^i \frac{\Gamma(k)}{i!\Gamma(k+i)} \lambda_{i+p}(\theta), \quad 0 \leq i \leq n \tag{7.16}$$

and

$$\nu_i = \begin{cases} \sigma_i, & 0 \leq i \leq n-1, \\ \dfrac{1 - \sum_{\ell=0}^{n-1} \sigma_\ell \theta^{2\ell}}{\theta^{2n}}, & i = n \end{cases} \tag{7.17}$$

are the best possible. For $k > 0$, $c \leq 0$ and $0 < x < \theta < \infty$, if n is odd the inequality (7.15) holds true, if n is even the inequality (7.15) is reversed.

7.3. **Baricz's generalizations of Cusa-Huygens's inequality.** The inequality (1.18) was generalized in [10] to

$$\frac{1 + 2ak\lambda_p(x)}{a(2k-1) + \pi/2} \leq \lambda_{p+1}(x) \leq \frac{1 + 2ak\lambda_p(x)}{a + 1 + a(2k-1)}, \tag{7.18}$$

where $|x\sqrt{c}\,| \leq \frac{\pi}{2}$, $a \in (0, \frac{1}{2}]$, $c \geq 0$, and $k \geq \frac{1}{2}$.

By making use of the inequality (1.16) and (1.17), the inequality (7.18) was further strengthened as

$$\frac{1 + 2k\lambda_p(x)}{2k+1} \leq \lambda_{p+1}(x) \leq \frac{1 + k\lambda_p(x)}{k+1}. \tag{7.19}$$

7.4. **Baricz's generalizations of Redheffer-Williams's inequality.** In [9], inequalities (1.5), (2.32) and (2.33) were generalized to the case of Bessel functions. The motivation of the paper [9] comes from [24, 68, 69, 75] and other related references.

7.5. **Lazarević's inequality and generalizations.** An inequality due to [49] states that

$$\left(\frac{\sinh t}{t}\right)^3 > \cosh t \tag{7.20}$$

for $t \neq 0$. The exponent 3 in (7.20) is the best possible. See also [20, p. 131], [48, p. 300] and [57, p. 270].

The inequality (7.20) was generalized in [5] to modified Bessel functions.

7.6. **Oppenheim's problem.** Considering inequalities stated in Section 1.7 and Section 7.3, it is natural to ask the following problems:

(1) What are the best possible positive constants a, b, c, r and α, β, γ, λ such that

$$\alpha + \beta \cos^\gamma(\lambda x) \leq \frac{\sin x}{x} \leq a + b\cos^c(rx) \tag{7.21}$$

for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ with $x \neq 0$ and

$$\alpha + \beta \cosh^\gamma(\lambda x) \leq \frac{\sinh x}{x} \leq a + b\cosh^c(rx) \tag{7.22}$$

for $-\infty < x < \infty$ with $x \neq 0$ hold respectively?

(2) What about the analogues of Bessel functions or other special functions?

These problems are similar to Oppenheim's problem which has been investigated in [5, 17, 10, 115].

7.7. **Some inequalities of Bessel functions.** For more information on inequalities of Bessel functions and some other special functions, please refer to [6, 7, 13, 14, 16] and related references therein.

8. Wilker-Anglesio's inequality and its generalizations

8.1. **Wilker's inequality and generalizations.** In [91], J. B. Wilker proved

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{8.1}$$

and proposed that there exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3\tan x \tag{8.2}$$

for $0 < x < \frac{\pi}{2}$.

In recent years, Wilker's inequality (8.1) has been proved once and again in papers such as [23, 29, 53, 86, 103, 114].

In [102], the inequality (8.1) was generalized as: If $q > 0$ or $q \leq \min\{-1, -\frac{\lambda}{\mu}\}$, then

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q > 1 \tag{8.3}$$

holds for $0 < x < \frac{\pi}{2}$, where $\lambda > 0$, $\mu > 0$ and $p \leq \frac{2q\mu}{\lambda}$. As an application of the inequality (8.3), an inequality posed as an open problem in [84] was solved and improved.

In [117], the inequality (8.1) was generalized as

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2. \tag{8.4}$$

In [5], inequalities (8.1) and (8.4) were generalized and extended naturally to the cases of Bessel function. Recently, the inequality (8.3) and all results in [102] were extended in [15] to Bessel functions.

8.2. **Wilker-Anglesio's inequality.** In [85], the best constant c in (8.2) was found and it was proved that

$$2+\frac{8}{45}x^3\tan x>\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+\left(\frac{2}{\pi}\right)^4x^3\tan x \tag{8.5}$$

for $0<x<\frac{\pi}{2}$. The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ in the inequality (8.5) are the best possible.

In [30, 31, 34, 108], several proofs of Wilker-Anglesio's inequality (8.5) were given.

In [63], a new proof of the inequality (8.5) was provided by using Lemma 1 and compared with [34].

In [40, 103, 104], three lower bounds for $\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2$ were presented, but they are weaker than $\left(\frac{2}{\pi}\right)^4x^3\tan x$ in (8.5).

In [89, 90], the following Wilker type inequality was obtained:

$$2+\frac{2}{45}x^3\sin x<\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}<2+\left(\frac{2}{\pi}-\frac{16}{\pi^3}\right)x^3\sin x \tag{8.6}$$

for $x\in\left(0,\frac{\pi}{2}\right)$. The constants $\frac{2}{45}$ and $\frac{2}{\pi}-\frac{16}{\pi^3}$ in (8.6) are the best possible.

8.3. **An open problem.** It is clear that to generalize Wilker-Anglesio's inequality (8.5) is more significant than to generalize Wilker's inequality (8.1).

It is conjectured that Wilker-Anglesio's inequality (8.5) may be generalized as follows: Let α, β, λ and μ be positive real numbers satisfying $\alpha\lambda=2\beta\mu$, then

$$\begin{aligned}\frac{16\mu}{\pi^4}x^4\left(\frac{\tan x}{x}\right)^\beta&<\lambda\left(\frac{\sin x}{x}\right)^\alpha+\mu\left(\frac{\tan x}{x}\right)^\beta-(\lambda+\mu)\\&<\frac{\lambda\alpha[5\lambda\alpha+\mu(12+5\alpha)]}{360\mu}x^4\left(\frac{\tan x}{x}\right)^\beta\end{aligned} \tag{8.7}$$

holds for $0<x<\frac{\pi}{2}$.

9. Applications of a method of auxiliary functions

In Section 2.1 of this paper, a method constructing auxiliary functions to refine Jordan's inequality (1.1) in [65, 67, 70, 73, 75] is introduced. Now the aim of this section is to summarize some other applications of this method, including estimation of some complete elliptic integrals and construction of inequalities for the exponential function e^x.

The complete elliptic integrals are classed into three kinds and defined for $0<k<1$ as

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta, \tag{9.1}$$

$$F(k)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{9.2}$$

$$\mathit{\Pi}(k,h)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}. \tag{9.3}$$

9.1. **Estimates for a discrete complete elliptic integral.** In [79], it was posed that

$$\frac{\pi}{6}<\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x<\frac{\pi\sqrt{2}}{8}. \tag{9.4}$$

In [28], the inequality (9.4) was verified by using $4-x^2>4-x^2-x^3>4-2x^2$ on the unit interval $[0,1]$.

In [74], by considering monotonicity and convexity of the function

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^4+\alpha x^3(1-x) \tag{9.5}$$

on $[0,1]$ for undetermined constant $\alpha\ge0$, the inequality

$$\frac{1}{\sqrt{4-x^2-x^3}}\ge\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^4+\biggl(\frac{11\sqrt{2}}{8}-2\biggr)(1-x)x^3 \tag{9.6}$$

for $x\in[0,1]$ was established, and then the lower bound in (9.4) was improved to

$$\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x>\frac{3}{10}+\frac{27\sqrt{2}}{160}. \tag{9.7}$$

It was also remarked in [74] that if discussing the auxiliary functions

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^2+\beta(1-x)x^2 \tag{9.8}$$

and

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^4+\theta(1-x^3)x \tag{9.9}$$

on $[0,1]$, then inequalities

$$\frac{1}{\sqrt{4-x^2-x^3}}\ge\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2+\biggl(\frac{3\sqrt{2}}{8}-1\biggr)(1-x)x^2 \tag{9.10}$$

and

$$\frac{1}{\sqrt{4-x^2-x^3}}\ge\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^4+\biggl(\frac{2}{3}-\frac{11\sqrt{2}}{24}\biggr)(x^3-1)x \tag{9.11}$$

can be obtained, and then, by integrating on both sides of above two inequalities, the lower bound in (9.4) may be improved to

$$\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x>\frac{1}{4}+\frac{19\sqrt{2}}{96} \tag{9.12}$$

and

$$\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x>\frac{1}{5}+\frac{19\sqrt{2}}{80}. \tag{9.13}$$

Numerical computation shows that the lower bound in (9.7) is better than those in (9.12) and (9.13).

In [106], by directly proving the inequality (9.6) and

$$\frac{1}{\sqrt{4-x^2-x^3}}\le\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2+\frac{5-4\sqrt{2}}{8}x^2(1-x)\biggl(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\biggr), \tag{9.14}$$

the inequality (9.7) and an improved upper bound in (9.4),

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x < \frac{79}{192} + \frac{\sqrt{2}}{10}, \tag{9.15}$$

were obtained.

In [71], by considering an auxiliary function

$$\frac{1}{\sqrt{4-x^2-x^3}} - \frac{1}{2} + \frac{1-\sqrt{2}}{2}x^2 + \alpha x^2(1-x)\left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10} + x\right) \tag{9.16}$$

on $[0,1]$, inequalities (9.14) and

$$\frac{1}{\sqrt{4-x^2-x^3}} \ge \frac{1}{2} + \frac{\sqrt{2}-1}{2}x^2 - \frac{1137\bigl(4\sqrt{2}-5\bigr)}{64\bigl(64-39\sqrt{2}\bigr)}(1-x)\left(\frac{8\sqrt{2}-9}{8\sqrt{2}-10} + x\right) \tag{9.17}$$

were demonstrated to be sharp, and then, by integrating on both sides of (9.14), the inequality (9.15) was recovered.

9.2. **Estimates for the first kind of complete elliptic integrals.** In [33], by discussing

$$\sqrt{1+k^2\cos^2 t} - \sqrt{1+k^2} + \frac{4}{\pi^2}\left(\sqrt{1+k^2}-1\right)t^2 + \theta\left(\frac{\pi}{2}-t\right)t \tag{9.18}$$

or

$$\sqrt{1+k^2\cos^2 t} - \sqrt{1+k^2} + \frac{2}{\pi}\left(\sqrt{1+k^2}-1\right)t + \beta\left(\frac{\pi}{2}-t\right)t \tag{9.19}$$

on $\left[0,\frac{\pi}{2}\right]$, where θ and β are undetermined constants, the inequality

$$-\frac{8}{\pi^2}\left(\sqrt{1+k^2}-1\right)t\left(\frac{\pi}{2}-t\right) \le \sqrt{1+k^2\cos^2 t} - \left[\sqrt{1+k^2} - \frac{4}{\pi^2}\left(\sqrt{1+k^2}-1\right)t^2\right] \le 0 \tag{9.20}$$

for $t \in \left[0,\frac{\pi}{2}\right]$ was obtained, where $k^2 = \frac{b^2}{a^2} - 1$ and $a, b > 0$. Integrating (9.20) yields

$$\frac{\pi}{6}(2a+b) < \int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,\mathrm{d}t \le \frac{\pi}{6}(a+2b). \tag{9.21}$$

When $b \ge 7a$, the right-hand side of the inequality (9.21) is stronger than the well-known result

$$\frac{\pi}{4}(a+b) \le \int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,\mathrm{d}t \le \frac{\pi}{4}\sqrt{2(a^2+b^2)} \tag{9.22}$$

which can be obtained by using some properties of definite integral.

9.3. **Inequalities for the remainder of power series expansion of e^x.** In [35, 65], by considering the auxiliary function

$$e^x - S_n(x) - \alpha_n x^{n+1} + \theta(b-x)x^{n+1} \tag{9.23}$$

for $0 \le x \le b \in (0,\infty)$, where $\alpha_{-1} = e^b$ and $\alpha_n = \frac{1}{b}\left(\alpha_{n-1} - \frac{1}{n!}\right)$, the following inequalities of the reminder

$$R_n(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \tag{9.24}$$

for $n \ge 0$ and $x \in [0, \infty)$ were established:

$$\frac{n+2-(n+1)x}{(n+2)!}x^{n+1}e^x \le R_n(x) \le \frac{n+1+e^x}{(n+2)!}x^{n+1} \le \frac{e^x}{(n+1)!}x^{n+1}, \tag{9.25}$$

$$\frac{(n+2)!}{(n-k+2)!}R_n(x) \le x^k R_{n-k}(x) + \frac{k}{(n-k+2)!}x^{n+1}, \quad 0 \le k \le n \tag{9.26}$$

and, for $n \ge k \ge 1$,

$$x^k R_{n-k}(x) \le \frac{kx^{n+1}e^x}{(n+1)(n-k+2)!} - \frac{n! - (n-k+2)(n+1)!}{(n-k+2)!}R_n(x). \tag{9.27}$$

10. Estimates and inequalities for complete elliptic integrals

By the way, we would like to collect some estimates and inequalities for complete elliptic integrals and their new developments in recent years.

10.1. **Inequalities between three kinds of complete elliptic integrals.** By using Tchebycheff's integral inequality [57, p. 39, Theorem 9], the following inequalities between three kinds of complete elliptic integrals were derived in [76]:

$$\frac{\pi \arcsin k}{2k} < F(k) < \frac{\pi \ln((1+k)/(1-k))}{4k}; \tag{10.1}$$

$$E(k) < \frac{16 - 4k^2 - 3k^4}{4(4+k^2)}F(k); \tag{10.2}$$

$$F(k) < \Big(1 + \frac{h}{2}\Big)\mathit{\Pi}(k,h), \quad -1 < h < 0 \quad \text{or} \quad h > \frac{k^2}{2-3k^2} > 0; \tag{10.3}$$

$$\mathit{\Pi}(k,h)E(k) > \frac{\pi^2}{4\sqrt{1+h}}, \quad -2 < 2h < k^2; \tag{10.4}$$

$$E(k) \ge \frac{16 - 28k^2 + 9k^4}{4(4-5k^2)}F(k), \quad k^2 \le \frac{2}{3}. \tag{10.5}$$

For $0 < 2h < k^2$, the inequality (10.3) is reversed. For $h > \frac{k^2}{2-3k^2} > 0$, the inequality (10.4) is reversed.

As concrete examples, the following estimates of the complete elliptic integrals are also deduced in [76]:

$$\frac{\pi^2}{4\sqrt{2}} < \int_0^{\pi/2}\Big(1 - \frac{\sin^2 x}{2}\Big)^{-1/2} \,\mathrm{d}x < \frac{\pi \ln(1+\sqrt{2}\,)}{\sqrt{2}}, \tag{10.6}$$

$$\int_0^{\pi/2}\Big(1 + \frac{\cos x}{2}\Big)^{-1} \,\mathrm{d}x < \frac{\pi(\ln 3 - \ln 2)}{2}, \tag{10.7}$$

$$\int_0^{\pi/2}\Big(1 - \frac{\sin x}{2}\Big)^{-1} \,\mathrm{d}x = \int_{\pi/2}^{\pi}\Big(1 + \frac{\cos x}{2}\Big)^{-1} \,\mathrm{d}x > \frac{\pi \ln 2}{2}. \tag{10.8}$$

These results are better than those in [47, p. 607].

10.2. **Bracken's inequality.** In [19, Theorem 4], the following inequality was proved:

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \le \frac{\ln b - \ln a}{b-a}. \tag{10.9}$$

Equality holds if and only if $a = b$.

There are two natural questions on bounding the complete elliptic integral in (10.9) to ask:

(1) What are the best constants $\beta > \alpha > 0$ such that the inequality

$$\left(\frac{\ln b - \ln a}{b - a}\right)^{\alpha} \le \frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \le \left(\frac{\ln b - \ln a}{b - a}\right)^{\beta} \tag{10.10}$$

holds for all positive numbers a and b with $a \ne b$?

(2) Is the lower bound for (10.9) the reciprocal of the exponential mean

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \tag{10.11}$$

for positive numbers a and b with $a \ne b$?

Since the complete elliptic integral in (10.9) tends to infinity as the ratio $\frac{b}{a}$ for $a > b > 0$ tends to zero, so we think that the former question is more significant.

10.3. **Some recent results of elliptic integrals.** It is noted that some new results on complete elliptic integrals are obtained in [11] recently. It was pointed in [11] that the right-hand side inequality in (10.1) is a recovery of [4, Theorem 3.10]. In [11], the inequality (10.1) was also generalized to the case of generalized complete elliptic integrals by the same method as in [69, 76].

In [12], some of the results in [11] were further improved.

Acknowledgements. The authors appreciate Árpád Baricz in Romania and many mathematicians in China, such as Chao-Ping Chen, Wei-Dong Jiang, Jian-Lin Li, Shan-He Wu, Xiao-Ming Zhang and Ling Zhu, for their valuable comments on this manuscript and timely supply of several newly published articles of their own.

This paper was revised while the first author was visiting the RGMIA, School of Computer Science and Mathematics, Victoria University, Australia between March 2008 and February 2009. Therefore, the first author would like to express thanks to local members of the RGMIA for their invitation and hospitality throughout the period.

References

[1] U. Abel and D. Caccia, *A sharpening of Jordan's inequality*, Amer. Math. Monthly **93** (1986), no. 7, 568–569.

[2] R. P. Agarwal, Y.-H. Kim and S. K. Sen, *A new refined Jordan's inequality and its application*, Math. Inequal. Appl. **11** (2008), no. 4, in press.

[3] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasi-conformal Maps*, John Wiley & Sons, New York, 1997.

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Functional inequalities for hypergeometric functions and complete elliptic integrals*, SIAM J. Math. Anal. **23** (1992), 512–524.

[5] Á. Baricz, *Functional inequalities involving Bessel and modified Bessel functions of the first kind*, Expo. Math. (2008), in press.

[6] Á. Baricz, *Functional inequalities involving special functions*, J. Math. Anal. Appl. **319** (2006), no 2, 450–459.

[7] Á. Baricz, *Functional inequalities involving special functions, II*, J. Math. Anal. Appl. **327** (2007), no. 2, 1202–1213.

[8] Á. Baricz, *Jordan-type inequalities for generalized Bessel functions*, J. Inequal. Pure Appl. Math. **9** (2008), no. 2, Art. 39; Available online at `http://jipam.vu.edu.au/article.php?sid=971`.

[9] Á. Baricz, *Redheffer type inequality for Bessel functions*, J. Inequal. Pure Appl. Math. **8** (2007), no. 1, Art. 11; Availble online at `http://jipam.vu.edu.au/article.php?sid=824`.
[10] Á Baricz, *Some inequalities involving generalized Bessel functions*, Math. Inequal. Appl. **10** (2007), no. 4, 827–842.
[11] Á Baricz, *Turán type inequalities for generalized complete elliptic integrals*, Math. Z. **256** (2007), no. 4, 895–911.
[12] Á Baricz, *Turán type inequalities for hypergeometric functions*, Proc. Amer. Math. Soc. **136** (2008), no. 9, 3223–3229.
[13] Á. Baricz and E. Neuman, *Inequalities involving generalized Bessel functions*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art 126; Available online at `http://jipam.vu.edu.au/article.php?sid=600`.
[14] Á. Baricz and E. Neuman, *Inequalities involving modified Bessel functions of the first kind II*, J. Math. Anal. Appl. **332** (2007), no. 1, 265–271.
[15] Á. Baricz and J. Sándor, *Extensions of the genralized Wilker inequality to Bessel functions*, Integral Transforms Spec. Funct. (2008), submitted.
[16] Á. Baricz and Sh.-H. Wu, *Sharp Jordan-type inequalities for Bessel functions*, submitted.
[17] Á. Baricz and L. Zhu, *Extension of Oppenheim's problem to Bessel functions*, J. Inequal. Appl. **2007** (2007), Article ID 82038, 7 pages; Available onine at `http://dx.doi.org/10.1155/2007/82038`.
[18] M. Bencze and J. Sándor, *On Hungens's trigonometric inequality*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 14; Available online at `http://www.staff.vu.edu.au/rgmia/v8n3.asp`.
[19] P. Bracken, *An arithmetic-geometric mean inequality*, Expo. Math. **19** (2001), no. 3, 273–279.
[20] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.
[21] Ch.-P. Chen and F. Qi, *A double inequality for remainder of power series of tangent function*, Tamkang J. Math. **34** (2003), no. 3, 351–355. RGMIA Res. Rep. Coll. **5** (2002), Suppl., Art. 2; Available online at `http://www.staff.vu.edu.au/rgmia/v5(E).asp`.
[22] Ch.-P. Chen and F. Qi, *Inequalities of some trigonometric functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **15** (2004), 72–79. RGMIA Res. Rep. Coll. **6** (2003), no. 3, Art. 2, 419–429; Available online at `http://www.staff.vu.edu.au/rgmia/v6n3.asp`.
[23] Ch.-P. Chen and F. Qi, *On two new proofs of Wilker's inequality*, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) **5** (2002), no. 4, 38–39. (Chinese)
[24] Ch.-P. Chen, J.-W. Zhao, and F. Qi, *Three inequalities involving hyperbolically trigonometric functions*, Octogon Math. Mag. **12** (2004), no. 2, 592–596. RGMIA Res. Rep. Coll. **6** (2003), no. 3, Art. 4, 437–443; Available online at `http://www.staff.vu.edu.au/rgmia/v6n3.asp`.
[25] L. Debnath and Ch.-J. Zhao, *New strengthened Jordan's inequality and its applications*, Appl. Math. Lett. **16** (2003), no. 4, 557–560.
[26] K. Deng, *The noted Jordan's inequality and its extensions*, Xiāngtán Kuàngyè Xuéyuàn Xuébào (Journal of Xiangtan Mining Institute) **10** (1995), no. 4, 60–63. (Chinese)
[27] Y.-F. Feng (Feng Yuefeng), *Proof without words: Jordan's inequality* $\frac{2x}{\pi} \le \sin x \le x$, $0 \le x \le \frac{\pi}{2}$, Math. Mag. **69** (1996), 126.
[28] R. H. Garstang, *Bounds on an elliptic integral*, Amer. Math. Monthly **94** (1987), no. 6, 556–557.
[29] Y.-H. Guo, *Two new proofs of Wilker's inequality*, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) **9** (2006), no. 4, 79. (Chinese)
[30] B.-N. Guo, W. Li and F. Qi, *Proofs of Wilker's inequalities involving trigonometric functions*, Inequality Theory and Applications, Volume **2** (Chinju/Masan, 2001), 109–112, Nova Science Publishers, Hauppauge, NY, 2003.
[31] B.-N. Guo, W. Li, B.-M. Qiao and F. Qi, *On new proofs of inequalities involving trigonometric functions*, RGMIA Res. Rep. Coll. **3** (2000), no. 1, Art. 15, 167–170; Available online at `http://www.staff.vu.edu.au/rgmia/v3n1.asp`.
[32] B.-N. Guo and F. Qi, *Alternative proofs for inequalities of some trigonometric functions*, Internat. J. Math. Ed. Sci. Tech. **39** (2008), no. 3, 384–389.

[33] B.-N. Guo, F. Qi and Sh.-J. Jing, *Improvement for the upper bound of a class of elliptic integral*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **14** (1995), no. 6, 125–128. (Chinese)
[34] B.-N. Guo, B.-M. Qiao, F. Qi and W. Li, *On new proofs of Wilker's inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), no. 1, 19–22.
[35] Q.-D. Hao, *On construction of several sharp inequalities for the exponential function* e^x, Kuàng Yè (Mining) (1995), no. 1, 39–42. (Chinese)
[36] Q.-D. Hao and B.-N. Guo, *A method of finding extremums of composite functions of trigonometric functions*, Kuàng Yè (Mining) (1993), no. 4, 80–83. (Chinese)
[37] A. Hoorfar and F. Qi, *Some new bounds for Mathieu's series*, Abstr. Appl. Anal. **2007** (2007), Article ID 94854, 10 pages; Available online at http://dx.doi.org/10.1155/2007/94854.
[38] A. P. Iuskevici, *History of Mathematics in 16th and 17th Centuries*, Moskva, 1961.
[39] W. Janous, *Propblem 1216*, Crux Math. **13** (1987), 53.
[40] W.-D. Jiang and Y. Hua, *Note on Wilker's inequality and Huygens's inequality*, Bùděngshì Yānjiū Tōngxùn (Communications in Studies on Inequalities) **13** (2006), no. 1, 149–151.
[41] W.-D. Jiang and Y. Hua, *An improvement and application of Jordan's inequality*, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) **9** (2006), no. 5, 60–61.
[42] W.-D. Jiang and Y. Hua, *Sharpening of Jordan's inequality and its applications*, J. Inequal. Pure Appl. Math. **7** (2006), no. 3, Art. 102; Available online at http://jipam.vu.edu.au/article.php?sid=719. Bùděngshì Yānjiū Tōngxùn (Communications in Studies on Inequalities) **12** (2005), no. 3, 288–290.
[43] G. Klambauer, *Problems and Propositions in Analysis*, Marcel Dekker, New York and Basel, 1979.
[44] M. S. Klamkin, *Solution of Problem 1216*, Crux Math. **14** (1988), 120–122.
[45] H. Kober, *Approximation by integral functions in the complex domain*, Trans. Amer. Math. Soc. **56** (1944), no. 1, 7–31.
[46] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), Húnán Jiàoyù Chūbǎn Shè (Hunan Education Press), Changsha, China, June 1989. (Chinese)
[47] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd ed., Húnán Jiàoyù Chūbǎn Shè (Hunan Education Press), Changsha, China, May 1993. (Chinese)
[48] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shāndōng Kēxué Jìshù Chūbǎn Shè (Shandong Science and Technology Press), Jinan City, Shandong Province, China, 2004. (Chinese)
[49] I. Lazarević, *Sur une inégalité de Lochs*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **230–241** (1968), 55–56.
[50] J.-L. Li, *An identity related to Jordan's inequality*, Intern. J. Math. Math. Sci. **2006** (2006), Article ID 76782, 6 pages; Availble online at http://dx.doi.org/10.1155/IJMMS/2006/76782.
[51] J.-L. Li and Y.-L. Li, *On the strengthened Jordan's inequality*, J. Inequal. Appl. **2007** (2007), Article ID 74328, 8 pages; Available online at http://dx.doi.org/10.1155/2007/74328.
[52] F.-M. Li and Y.-F. Ma, *Notes on Jordan inequality*, Xī'ān Shíyóu Dàxué Xuébào (Zìrán Kēxué Bǎn) (Journal of Xi'an Shiyou University (Natural Science Edition)) **19** (2004), no. 5, 85–86. (Chinese)
[53] A.-Q. Liu, G. Wang, and W. Li, *New proofs of Wilker's inequalities involving trigonometric functions*, Jiāozuò Gōngxuéyuàn Xuébào (Zìrán Kēxué Bǎn) (Journal of Jiaozuo Institute of Technology (Natural Science)) **21** (2002), no. 5, 401–403. (Chinese)
[54] Q.-M. Luo, Z.-L. Wei and F. Qi, *Lower and upper bounds of* $\zeta(3)$, Adv. Stud. Contemp. Math. (Kyungshang) **6** (2003), no. 1, 47–51. RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 7, 565–569; Available online at http://www.staff.vu.edu.au/rgmia/v4n4.asp.
[55] A. McD. Mercer, *Problem E 2952*, Amer. Math. Monthly **89** (1982), no. 6, 424.
[56] M. Merkle, *Inequalities for the gamma function via convexity*, In: P. Cerone, S. S. Dragomir (Eds.), "Advances in Inequalities for Special Functions", Nova Science Publishers, New York, 2008, 81–100.
[57] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, 1970.
[58] V. N. Murty and G. P. Henderson, *Solution of Problem 1365*, Crux Math. **15** (1989), 251–256.

[59] E. Neuman, *Inequalities involving Bessel functions of the first kind*, J. Inequal. Pure Appl. Math. **5** (2004), no 4, Art. 94; Available online at `http://jipam.vu.edu.au/article.php?sid=449`.

[60] D.-W. Niu, *Generalizations of Jordan's Inequality and Applications*, Thesis supervised by Professor Feng Qi and submitted for the Master Degree of Science at Henan Polytechnic University in June 2007. (Chinese)

[61] D.-W. Niu, Zh.-H. Huo, J. Cao, and F. Qi, *A general refinement of Jordan's inequality and a refinement of L. Yang's inequality*, Integral Transforms Spec. Funct. **19** (2008), no. 3, 157–164.

[62] A. Y. Özban, *A new refined form of Jordan's inequality and its applications*, Appl. Math. Lett. **19** (2006), no. 2, 155–160.

[63] I. Pinelis, *L'Hospital rules for monotonicity and the Wilker-Anglesio inequality*, Amer. Math. Monthly **111** (2004), 905–909.

[64] J. Prestin, *Trigonometric interpolation in Hölder spaces*, J. Approx. Theory **53** (1988), no. 2, 145–154.

[65] F. Qi, *A method of constructing inequalities about* e^x, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 16–23.

[66] F. Qi, *Bounds for the ratio of two gamma functions*, RGMIA Res. Rep. Coll. **11** (2008), no. 2, Art. ??; Available online at `http://www.staff.vu.edu.au/rgmia/v11n2.asp`.

[67] F. Qi, *Extensions and sharpenings of Jordan's and Kober's inequality*, Gōngkē Shùxué (Journal of Mathematics for Technology) **12** (1996), no. 4, 98–102. (Chinese)

[68] F. Qi, *Jordan's inequality: Refinements, generalizations, applications and related problems*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art. 12; Available online at `http://www.staff.vu.edu.au/rgmia/v9n3.asp`. Bùděngshì Yānjiū Tōngxùn (Communications in Studies on Inequalities) **13** (2006), no. 3, 243–259.

[69] F. Qi, L.-H. Cui, and S.-L. Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (1999), no. 4, 517–528.

[70] F. Qi and B.-N. Guo, *Extensions and sharpenings of the noted Kober's inequality*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **12** (1993), no. 4, 101–103. (Chinese)

[71] F. Qi and B.-N. Guo, *Estimate for upper bound of an elliptic integral*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory) **26** (1996), no. 3, 285–288. (Chinese)

[72] F. Qi and B.-N. Guo, *Lower bound of the first eigenvalue for the Laplace operator on compact Riemannian manifold*, Chinese Quart. J. Math. **8** (1993), no. 2, 40–49.

[73] F. Qi and B.-N. Guo, *On generalizations of Jordan's inequality*, Méitàn Gāoděng Jiàoyù (Coal Higher Education), Suppl., November/1993, 32–33. (Chinese)

[74] F. Qi and B.-N. Guo, *The estimation of inferior bound for an ellipse integral*, Gōngkē Shùxué (Journal of Mathematics for Technology) **10** (1994), no. 1, 87–90. (Chinese)

[75] F. Qi and Q.-D. Hao, *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (1998), no. 3, 116–120.

[76] F. Qi and Zh. Huang, *Inequalities of the complete elliptic integrals*, Tamkang J. Math. **29** (1998), no. 3, 165–169.

[77] F. Qi, H.-Ch. Li, B.-N. Guo and Q.-M. Luo, *Inequalities and estimates of the eigenvalue for Laplace operator*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **13** (1994), no. 3, 89–95. (Chinese)

[78] F. Qi, D.-W. Niu, and J. Cao, *A general generalization of Jordan's inequality and a refinement of L. Yang's inequality*, RGMIA Res. Rep. Coll. **10** (2007), Suppl., Art. 3; Available online at `http://www.staff.vu.edu.au/rgmia/v10(E).asp`.

[79] Th. M. Rassias, *Problem E 3111*, Amer. Math. Monthly **92** (1985), no. 9, 665.

[80] R. Redheffer, *Problem 5642*, Amer. Math. Monthly **75** (1968), no. 10, 1125.

[81] R. Redheffer, *Correction*, Amer. Math. Monthly **76** (1969), no. 4, 422.

[82] J. Sándor, *A note on certain Jordan type inequalities*, RGMIA Res. Rep. Coll. **10** (2001), no. 1, Art. 1; Available online at `http://www.staff.vu.edu.au/rgmia/v10n1.asp`.

[83] J. Sándor, *On the concavity of* $\frac{sin x}{x}$, Octogon Math. Mag. **13** (2005), no. 1, 406–407.

[84] J. Sándor and M. Bencze, *On Huygens's trigonometric inequality*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 14; Available online at `http://www.staff.vu.edu.au/rgmia/v8n3.asp`.

[85] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), no. 3, 264–267.

[86] J.-Sh. Sun, *Two simple proof of Wiker's inequality involving trigonometric functions*, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) **7** (2004), no. 4, 43. (Chinese)
[87] G. Tsintsifas, *Problem 1365*, Crux Math. **14** (1988), 202.
[88] L.-Ch. Wang, *Further extension of the Jordan inequality*, Dáxiàn Shīfàn Gǎoděng Zhuānkē Xuéxiào Xuébào (Zìrán Kēxué Bǎn) (Journal of Daxian Teachers College (Natural Science Edition)) **14** (2004), no. 5, 3–4. (Chinese)
[89] Z.-H. Wang, *A new Wilker-type inequality*, J. Yinbin Univ. **7** (2007), no. 6, 21–22. (Chinese)
[90] Z.-H. Wang and X.-M. Zhang, *Introduction to a Wilker type inequality*, Bùděngshì Yānjiū Tōngxùn (Communications in Studies on Inequalities) **13** (2006), no. 3, 272–274. (Chinese)
[91] J. B. Wilker, *Problem E 3306*, Amer. Math. Monthly **96** (1989), no. 1, 55.
[92] J. P. Williams, *A delightful inequality*, Amer. Math. Monthly **76** (1969), no. 10, 1153–1154.
[93] Sh.-H. Wu, *On generalizations and refinements of Jordan type inequality*, Octogon Math. Mag. **12** (2004), no. 1, 267–272. RGMIA Res. Rep. Coll. **7** (2004), Suppl., Art. 2; Available online at `http://www.staff.vu.edu.au/rgmia/v7(E).asp`
[94] Sh.-H. Wu, *Refinement and generalization of the Jordan's inequality*, Chéngdū Dàxué Xuébào (Zìrán Kēxué Bǎn) (Journal of Chengdu University (Natural Science)) **23** (2004), no. 2, 37–40. (Chinese)
[95] Sh.-H. Wu, *Sharpness and generalization of Jordan's inequality and its application*, Taiwanese J. Math. **12** (2008), no. 2, 325–336.
[96] Sh-H. Wu and L. Debnath, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality*, Appl. Math. Lett. **19** (2006), 1378–1384.
[97] Sh-H. Wu and L. Debnath, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality, II*, Appl. Math. Lett. **20** (2007), no. 5, 532–538.
[98] Sh-H. Wu and L. Debnath, *Generalizations of a parameterized Jordan-type inequality, Janouss inequality and Tsintsifass inequality*, Appl. Math. Lett. **22** (2008), in press.
[99] Sh-H. Wu and L. Debnath, *Jordan-type inequalitities for differentiable functions and their applications*, Appl. Math. Lett. **21** (2008), no. 8, 803–809.
[100] Sh.-H. Wu and H. M. Srivastava, *A further refinment of a Jordan type inequalty and its application*, Appl. Math. Comput. **197** (2008), no. 2, 914–923.
[101] Sh.-H. Wu, H. M. Srivastava and L. Debnath, *Some refined families of Jordan type inequalities and their applications*, Integral Transforms Spec. Funct. **19** (2008), 183–193.
[102] Sh.-H. Wu and H. M. Srivastava, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transforms Spec. Funct. **18** (2007), no. 8, 529–535.
[103] Y.-F. Wu and X.-S. Xu, *Simple proofs and a refinement of Wilker's inequality*, Tónglíng Xuéyuàn Xuébào (Journal of Tongling College) **5** (2006), no. 2, 72–88. (Chinese)
[104] Sh.-Ch. Yang, *A sharpening of Wilker inequality*, Ābà Shīfàn Gāoděng Zhuānkē Xuéxiào Xuébào (Journal of Aba Teachers College) (2003), no. 3, 104–105. (Chinese)
[105] L. Yang, *Zhí Fēnbù Lǐlùn Jíqí Xīn Yánjiū* (*Theory of Distribution of Values of Functions and Recent Researches*), Science Press, Beijing, 1982. (Chinese)
[106] L.-Q. Yu, F. Qi and B.-N. Guo, *Estimates for upper and lower bounds of a complete elliptic integral*, Kuàng Yè (Mining) (1995), no. 1, 35–38. (Chinese)
[107] X.-H. Zhang, G.-D. Wang, Y.-M. Chu, *Extensions and sharpenings of Jordan's and Kober's inequalities*, J. Inequal. Pure Appl. Math. **7** (2006), no. 2, Art. 63; Available online at `http://jipam.vu.edu.au/article.php?sid=680`.
[108] L. Zhang and L. Zhu, *A new elementary proof of Wilker's inequalities*, Math. Inequal. Appl. **11** (2008), no. 1, 149–151.
[109] Ch.-J. Zhao, *Generalization and strengthening of the Yang Le inequality*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory) **30** (2000), no. 4, 493–497 (Chinese).
[110] Ch.-J. Zhao, *On several new inequalities*, Chinese Quart. J. Math. **16** (2001), no. 2, 41–45.
[111] Ch.-J. Zhao and L. Debnath, *On generalizations of L. Yang's inequality*, J. Inequal. Pure Appl. Math. **3** (2002), no. 4, Art. 56; Available online at `http://jipam.vu.edu.au/article.php?sid=208`.
[112] L. Zhu, *A general form of Jordan's inequalities and its applications*, Math. Inequal. Appl. **11** (2008), no. 3, in press.
[113] L. Zhu, *A general refinement of Jordan-type inequality*, Comput. Math. Appl. **55** (2008) 2498–2505.

[114] L. Zhu, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. **8** (2005), no. 4, 749–750.
[115] L. Zhu, *A solution of a problem of Oppenheim*, Math. Inequal. Appl. **10** (2007), no. 1, 57–61.
[116] L. Zhu, *General forms of Jordan and Yang Le inequality*, Appl. Math. Lett. (2008), in press; Available online at http://dx.doi.org/10.1016/j.aml.2008.03.017.
[117] L. Zhu, *On Wilker-type inequalities*, Math. Inequal. Appl. **10** (2007), no. 4, 727–731.
[118] L. Zhu, *Sharpening of Jordan's inequalities and its applications*, Math. Inequal. Appl. **9** (2006), no. 1, 103–106.
[119] L. Zhu, *Sharpening Jordan's inequality and the Yang Le inequality*, Appl. Math. Lett. **19** (2006), no. 3, 240–243.
[120] L. Zhu, *Sharpening Jordan's inequality and the Yang Le inequality, II*, Appl. Math. Lett. **19** (2006), no. 9, 990–994.
[121] L. Zhu and J. Sun, *Six new Redheffer-type inequalities for circular and hyperbolic functions*, Comput. Math. Appl. (2008), in press; Available online at http://dx.doi.org/10.1016/j.camwa.2008.01.012.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: http://qifeng618.spaces.live.com

(D.-W. Niu) College of Information and Business, Zhongyuan University of Technology, Zhengzhou City, Henan Province, 450007, China

E-mail address: nnddww@gmail.com, nnddww@hotmail.com