Refinements of some geometric inequalities

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Abstract: In this paper, based on the improved versions of certain classical analytic inequalities, some refinements of the inequalities involving the semi-perimeter,circumradius and inradius of a triangle are established.
Keywords: geometric inequality; analytic inequality; refinement; triangle; semi-perimeter; circumradius; inradius

2000 AMS Mathematics subject classification: 26D05, 26D15, 51M16

1 Main Results

For a given triangle ABC, we assume that A, B, C denote its angles, a, b, c denote the lengths of its corresponding sides, s, R and r denote the semi-perimeter, circumradius and inradius of a triangle respectively. We will customarily use the notations of cyclic sum and cyclic product such as

$$\sum f(a) = f(a) + f(b) + f(c), \quad \prod f(a) = f(a)f(b)f(b).$$

The inequalities related to the semi-perimeter,circumradius and inradius of a triangle have been widely studied in the literature, see for example [1–13]. The purpose of this paper is to establish some refinements of this type of geometric inequalities. To do this, we will present the improvements of certain classical analytic inequalities, these inequalities will be used as the main tool in the proof of the refinements of geometric inequalities.

Theorem 1. *If $x, y, z > 0$, then*

$$\frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz}. \tag{1}$$

Proof. Using the arithmetic-geometric means inequality gives

$$\begin{aligned}\left(\frac{x+y+z}{3}\right)^2 &= \frac{1}{9}\left(x^2+y^2+z^2+2xy+2yz+2zx\right)\\ &\geq \frac{1}{9}\left(xy+yz+zx+2xy+2yz+2zx\right)\\ &= \frac{1}{3}\left(xy+yz+zx\right),\end{aligned}$$

we hence have

$$\frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt{\frac{3\sqrt[3]{xy\cdot yz\cdot zx}}{3}} = \sqrt[3]{xyz}.$$

Application 1. *In all triangle ABC holds*

$$\left(\frac{2s}{3}\right)^6 \geq \left(\frac{s^2+r^2+4Rr}{3}\right)^3 \geq (4sRr)^2 \tag{2}$$

$$\left(\frac{s}{3}\right)^6 \geq \left(\frac{r(4R+r)}{3}\right)^3 \geq \left(sr^2\right)^2 \tag{3}$$

$$\left(\frac{s^2+r^2+4Rr}{6R}\right)^6 \geq \left(\frac{2s^2r}{3R}\right)^3 \geq \left(\frac{2s^2r^2}{R}\right)^2 \tag{4}$$

$$\left(\frac{4R+r}{3}\right)^6 \geq \left(\frac{s^2}{3}\right)^3 \geq \left(s^2r\right)^2 \tag{5}$$

$$\left(\frac{2R-r}{6R}\right)^6 \geq \left(\frac{s^2+r^2-8Rr}{48R^2}\right)^3 \geq \left(\frac{r}{4R}\right)^4 \tag{6}$$

$$\left(\frac{4R+r}{6R}\right)^6 \geq \left(\frac{s^2+(4R+r)^2}{48R^2}\right)^3 \geq \left(\frac{s}{4R}\right)^4 \tag{7}$$

Proof. In Theorem 1 we take $(x,y,z) \in \{(a,b,c),\ (s-a,s-b,s-c),\ (h_a,h_b,h_c),$ $(r_a,r_b,r_c),\ (\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}),\ (\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2})\}$ respectively, we have

$$\frac{\sum a}{3} \geq \sqrt{\frac{\sum ab}{3}} \geq \sqrt[3]{\prod a},$$

from $\frac{\sum a}{3} = \frac{2s}{3}, \sum ab = s^2+r^2+4Rr, \prod a = 4sRr$, we deduce the inequality (2).

$$\frac{\sum(s-a)}{3} \geq \sqrt{\frac{\sum(s-a)(s-b)}{3}} \geq \sqrt[3]{\prod(s-a)},$$

from $\frac{\sum(s-a)}{3} = \frac{s}{3}, \sum(s-a)(s-b) = r(r+4R), \prod(s-a) = sr^2$, we deduce the inequality (3).

$$\frac{\sum h_a}{3} \geq \sqrt{\frac{\sum h_ah_b}{3}} \geq \sqrt[3]{\prod h_a},$$

from $\sum h_a = \frac{s^2+r^2+4Rr}{2R}, \sum h_ah_b = \frac{2s^2r}{R}, \prod h_a = \frac{2s^2r^2}{R}$, we deduce the inequality (4).

$$\frac{\sum r_a}{3} \geq \sqrt{\frac{\sum r_ar_b}{3}} \geq \sqrt[3]{\prod r_a},$$

from $\sum r_a = 4R+r, \sum r_ar_b = s^2, \prod r_a = s^2r$, we deduce the inequality (5).

$$\frac{\sum\sin^2\frac{A}{2}}{3} \geq \sqrt{\frac{\sum\sin^2\frac{C}{2}\sin^2\frac{B}{2}}{3}} \geq \sqrt[3]{\prod\sin^2\frac{A}{2}}$$

from $\sum\sin^2\frac{A}{2} = \frac{2R-r}{2R}, \sum\sin^2\frac{A}{2}\sin^2\frac{B}{2} = \frac{s^2+r^2-8Rr}{16R^2}, \prod\sin^2\frac{A}{2} = \frac{r^2}{16R^2}$, we deduce the inequality (6).

$$\frac{\sum \cos^2 \frac{A}{2}}{3} \geq \sqrt{\frac{\sum \cos^2 \frac{C}{2} \cos^2 \frac{B}{2}}{3}} \geq \sqrt[3]{\prod \cos^2 \frac{A}{2}},$$

from $\sum \cos^2 \frac{A}{2} = \frac{4R+r}{2R}, \sum \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} = \frac{s^2+(4R+r)^2}{16R^2}, \prod \cos^2 \frac{A}{2} = \frac{s^2}{16R^2}$, we deduce the inequality (7).

Theorem 2. *If* $x, y, z > 0$, *then*

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 3\left(1+\sqrt[3]{\frac{(x+y)(y+z)(z+x)}{xyz}}\right) \geq 9. \tag{8}$$

Proof. Using the arithmetic-geometric means inequality gives

$$\begin{aligned}
(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) &= 3+\frac{y}{x}+\frac{z}{x}+\frac{x}{y}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z} \\
&= 3+\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z} \\
&\geq 3+3\sqrt[3]{\frac{y+z}{x}\cdot\frac{x+z}{y}\cdot\frac{x+y}{z}} \\
&= 3\left(1+\sqrt[3]{\frac{(x+y)(y+z)(z+x)}{xyz}}\right),
\end{aligned}$$

$$\begin{aligned}
3\left(1+\sqrt[3]{\frac{(x+y)(y+z)(z+x)}{xyz}}\right) &\geq 3\left(1+\sqrt[3]{\frac{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}}{xyz}}\right) \\
&= 9.
\end{aligned}$$

The inequality (8) is proved.

Application 2. *In all triangle ABC holds*

$$\frac{s^2+r^2+4Rr}{2Rr} \geq 3\left(1+\sqrt[3]{\frac{s^2+r^2+2Rr}{2Rr}}\right) \geq 9 \tag{9}$$

$$\frac{4R+r}{r} \geq 3\left(1+\sqrt[3]{\frac{4R}{r}}\right) \geq 9 \tag{10}$$

$$\frac{s^2+r^2+4Rr}{2Rr} \geq 3\left(1+\sqrt[3]{\frac{s^2+r^2+2Rr}{2Rr}}\right) \geq 9 \tag{11}$$

$$\frac{(2R-r)\left(s^2+r^2-8Rr\right)}{2Rs^2} \geq 3\left(1+\sqrt[3]{\frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{2Rr^2}}\right) \geq 9 \tag{12}$$

$$\frac{(4R+r)\left(s^2+(4R+r)^2\right)}{2Rs^2} \geq 3\left(1+\sqrt[3]{\frac{(4R+r)^3+s^2\left(2R+r\right)}{2Rs^2}}\right) \geq 9 \tag{13}$$

Proof. In Theorem 2 we take $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c),\ (h_a,h_b,h_c),$ $(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}),\ (\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2})\}$ respectively, we have

$$(\sum a)\sum\frac{1}{a}\geq 3\left(1+\sqrt[3]{\frac{\prod(a+b)}{\prod c}}\right)\geq 9\Longleftrightarrow(9),$$

since

$$(\sum a)\sum\tfrac{1}{a}=\tfrac{4Rr+r^2+s^2}{2Rr},\ \tfrac{\prod(a+b)}{\prod c}=\tfrac{2Rr+r^2+s^2}{2Rr}.$$

$$(\sum(s-a))\sum\frac{1}{s-a}\geq 3\left(1+\sqrt[3]{\frac{\prod[(s-a)+(s-b)]}{\prod(s-c)}}\right)\geq 9\Longleftrightarrow(10),$$

since

$$\sum(s-a)=s,\ \sum\tfrac{1}{s-a}=\tfrac{4R+r}{sr},\ \tfrac{\prod[(s-a)+(s-b)]}{\prod(s-c)}=\tfrac{4R}{r}.$$

$$(\sum h_a)\sum\frac{1}{h_a}\geq 3(1+\sqrt[3]{\frac{\prod(h_a+h_b)}{\prod h_c}})\geq 9\Longleftrightarrow(11),$$

since

$$\sum h_a=\tfrac{s^2+r^2+4Rr}{2R},\ \sum\tfrac{1}{h_a}=\tfrac{1}{r},\ \tfrac{\prod(h_a+h_b)}{\prod h_c}=\tfrac{s^2+r^2+2Rr}{2Rr}.$$

$$\sum\sin^2\frac{A}{2}\sum\frac{1}{\sin^2\frac{A}{2}}\geq 3\left(1+\sqrt{\frac{\prod(\sin^2\frac{A}{2}+\sin^2\frac{B}{2})}{\prod\sin^2\frac{C}{2}}}\right)\geq 9\Longleftrightarrow(12),$$

since

$$\sum\sin^2\tfrac{A}{2}\sum\tfrac{1}{\sin^2\frac{A}{2}}=\tfrac{(2R-r)(s^2+r^2-8Rr)}{2Rr^2},\ \tfrac{\prod(\sin^2\frac{A}{2}+\sin^2\frac{B}{2})}{\prod\sin^2\frac{C}{2}}=\tfrac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{2Rr^2}.$$

$$\sum\cos^2\frac{A}{2}\sum\frac{1}{\cos^2\frac{A}{2}}\geq 3(1+\sqrt[3]{\frac{\prod(\cos^2\frac{A}{2}+\cos^2\frac{B}{2})}{\prod\cos^2\frac{A}{2}}})\geq 9\Longleftrightarrow(13),$$

since

$$\sum\cos^2\tfrac{A}{2}\sum\tfrac{1}{\cos^2\frac{A}{2}}=\tfrac{(4R+r)\left(s^2+(4R+r)^2\right)}{2Rs^2},\ \tfrac{\prod(\cos^2\frac{A}{2}+\cos^2\frac{B}{2})}{\prod\cos^2\frac{A}{2}}=\tfrac{(4R+r)^3+s^2(2R+r)}{2s^2R}.$$

Theorem 3. *If* $x,y,z>0$, *then*

$$2\sum x\leq\sum\frac{x^2+y^2}{z}\leq 2\sum\frac{x^3}{yz}. \tag{14}$$

Proof. Using the arithmetic-geometric means inequality gives

$$
\begin{aligned}
\sum \frac{x^2+y^2}{z} &\geq \frac{2xy}{z}+\frac{2yz}{x}+\frac{2zx}{y}\\
&= \left(\frac{xy}{z}+\frac{yz}{x}\right)+\left(\frac{xy}{z}+\frac{zx}{y}\right)+\left(\frac{yz}{x}+\frac{zx}{y}\right)\\
&\geq 2\sqrt{\frac{xy}{z}\cdot\frac{yz}{x}}+2\sqrt{\frac{xy}{z}\cdot\frac{zx}{y}}+2\sqrt{\frac{yz}{x}\cdot\frac{zx}{y}}\\
&= 2(y+x+z)\\
&= 2\sum x.
\end{aligned}
$$

On the other hand, we have

$$\frac{x^3}{yz}+\frac{y^3}{zx}=\frac{x^4+y^4}{xyz}\geq\frac{\frac{1}{2}(x^2+y^2)^2}{xyz}\geq\frac{x^2+y^2}{z},$$

thus

$$2\sum\frac{x^2}{yz}\geq\sum\frac{x^2+y^2}{z}.$$

The Theorem 3 is proved.
Application 3. *In all triangle ABC holds*

$$4s\leq\sum\frac{a^2+b^2}{c}\leq\frac{2\sum a^4}{abc}. \tag{15}$$

Proof. Putting in the inequality (14) $x=a,\ y=b,\ z=c$, the inequality (15) follows immediately.

Theorem 4. *If $x,y,z>0$, then*

$$\frac{x+y+z}{3}-\sqrt[3]{xyz}\geq\frac{1}{3}\max\left\{(\sqrt{x}-\sqrt{y})^2,\ \left(\sqrt{y}-\sqrt{z}\right)^2,\ \left(\sqrt{z}-\sqrt{x}\right)^2\right\}. \tag{16}$$

Proof. Direct computation gives

$$
\begin{aligned}
\frac{x+y+z}{3}-\sqrt[3]{xyz}-\frac{1}{3}(\sqrt{x}-\sqrt{y})^2 &= \frac{z+\sqrt{xy}+\sqrt{xy}}{3}-\sqrt[3]{xyz}\\
&\geq \sqrt[3]{xyz}-\sqrt[3]{xyz}\\
&= 0,
\end{aligned}
$$

thus

$$\frac{x+y+z}{3}-\sqrt[3]{xyz}\geq\frac{1}{3}(\sqrt{x}-\sqrt{y})^2.$$

The Theorem 4 is proved.
Application 4. *In all triangle ABC holds*

$$\frac{2s}{3}-\sqrt[3]{4sRr}\geq\frac{1}{3}\max\left\{(\sqrt{a}-\sqrt{b})^2,\ (\sqrt{b}-\sqrt{c})^2,\ (\sqrt{c}-\sqrt{a})^2\right\} \tag{17}$$

$$\frac{s}{3}-\sqrt[3]{sr^2}\geq\frac{1}{3}\max\left\{\left(\sqrt{s-a}-\sqrt{s-b}\right)^2,\ \left(\sqrt{s-b}-\sqrt{s-c}\right)^2,\ \left(\sqrt{s-c}-\sqrt{s-a}\right)^2\right\} \tag{18}$$

$$\frac{s^2+r^2+4Rr}{6R}-\sqrt[3]{\frac{2s^2r^2}{R}}\geq\frac{1}{3}\max\left\{(\sqrt{h_a}-\sqrt{h_b})^2,\ (\sqrt{h_b}-\sqrt{h_c})^2,\ (\sqrt{h_c}-\sqrt{h_a})^2\right\} \tag{19}$$

$$\frac{4R+r}{3}-\sqrt[3]{s^2r}\geq\frac{1}{3}\max\left\{(r_a-r_b)^2,\ (r_b-r_c)^2,\ (r_c-r_a)^2\right\} \tag{20}$$

$$\frac{2R-r}{6R}-\sqrt[3]{\frac{r^2}{16R^2}}\geq\frac{1}{3}\max\left\{\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)^2,\ \left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)^2,\ \left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)^2\right\} \tag{21}$$

$$\frac{4R+r}{6R}-\sqrt[3]{\frac{s^2}{16R^2}}\geq\frac{1}{3}\max\left\{\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)^2,\ \left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)^2,\ \left(\cos\frac{C}{2}-\cos\frac{A}{2}\right)^2\right\} \tag{22}$$

Proof. We take $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c),\ (h_a,h_b,h_c),\ (r_a,r_b,r_c),$ $(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}),\ (\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2})\}$ in Theorem 4 respectively, and apply the following identities:

$\sum a=2s, \prod a=4sRr,$
$\sum(s-a)=s, \prod(s-a)=sr^2,$
$\sum h_a=\frac{s^2+r^2+4Rr}{2R}, \prod h_a=\frac{2s^2r^2}{R},$
$\sum r_a=4R+r, \prod r_a=s^2r,$
$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}, \prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2},$
$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R}, \prod\cos^2\frac{A}{2}=\frac{s^2}{16R^2},$
we obtain the desired inequalities (17)–(22).

Theorem 5. *If* $x,y,z>0$, *then*

$$\max\{x,y,z\}-\min\{x,y,z\}\leq\sqrt{\frac{4}{3}}\sqrt{x^2+y^2+z^2-xy-yz-zx} \tag{23}$$

Proof. Without loss of generality, we assume that $x=\max\{x,y,z\}$, $z=\min\{x,y,z\}$. Then, we have

$$\begin{aligned}x^2+y^2+z^2-xy-yz-zx &= \frac{(x-y)^2+(y-z)^2+(z-x)^2}{2}\\ &\geq \frac{\frac{1}{2}(x-y+y-z)^2+(z-x)^2}{2}\\ &= \frac{3(x-z)^2}{4},\end{aligned}$$

thus

$$x-z\leq\sqrt{\frac{4}{3}}\sqrt{x^2+y^2+z^2-xy-yz-zx}.$$

The inequality (23) is proved.

Application 5. *In all triangle ABC holds*

$$\max\{a,b,c\}-\min\{a,b,c\}\leq\sqrt{\frac{4}{3}}\sqrt{s^2-3r^2-12Rr} \tag{24}$$

$$\max\{s-a,s-b,s-c\}-\min\{s-a,s-b,s-c\}\le\sqrt{\frac{4}{3}}\sqrt{s^2-3r^2-12Rr} \tag{25}$$

$$\max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\}\le\sqrt{\frac{4}{3}}\sqrt{\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-\frac{6s^2r}{R}} \tag{26}$$

$$\max\{r_a,r_b,r_c\}-\min\{r_a,r_b,r_c\}\le\sqrt{\frac{4}{3}}\sqrt{(4R+r)^2-3s^2} \tag{27}$$

$$\max\left\{\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right\}-\min\left\{\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right\}\le\sqrt{\frac{4}{3}}\sqrt{\frac{16R^2+r^2-3s^2+8Rr}{16R^2}} \tag{28}$$

$$\max\left\{\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right\},-\min\left\{\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right\}\le\sqrt{\frac{4}{3}}\sqrt{\frac{(4R+r)^2-3s^2}{16R^2}} \tag{29}$$

Proof. We take $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c),\ (h_a,h_b,h_c),\ (r_a,r_b,r_c),$ $(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}),\ (\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2})\}$ in Theorem 5 respectively, and apply the following identities:

$a^2+b^2+c^2-ab-bc-ca=s^2-3r^2-12Rr$,
$(s-a)^2+(s-b)^2+(s-c)^2-(s-a)(s-b)-(s-b)(s-c)-(s-c)(s-a)=s^2-3r^2-12Rr$,
$h_a^2+h_b^2+h_c^2-h_ah_b-h_bh_c-h_ch_a=\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-\frac{6s^2r}{R}$,
$r_a^2+r_b^2+r_c^2-r_ar_b-r_br_c-r_cr_a=(4R+r)^2-3s^2$,
$\sin^4\frac{A}{2}+\sin^4\frac{B}{2}+\sin^4\frac{C}{2}-\sin^2\frac{A}{2}\sin^2\frac{B}{2}-\sin^2\frac{B}{2}\sin^2\frac{C}{2}-\sin^2\frac{C}{2}\sin^2\frac{A}{2}=\frac{16R^2+r^2-3s^2+8Rr}{16R^2}$,
$\cos^4\frac{A}{2}+\cos\sin^4\frac{B}{2}+\cos^4\frac{C}{2}-\cos^2\frac{A}{2}\cos^2\frac{B}{2}-\cos^2\frac{B}{2}\cos^2\frac{C}{2}-\cos^2\frac{C}{2}\cos^2\frac{A}{2}=\frac{(4R+r)^2-3s^2}{16R^2}$.

Theorem 6. *If* $x,y,z>0$, *then*

$$\frac{3}{2}\min\left\{(x-y)^2,(y-z)^2,(z-x)^2\right\}\le x^2+y^2+z^2-xy-yz-zx\le\frac{3}{2}\max\left\{(x-y)^2,(y-z)^2,(z-x)^2\right\}. \tag{30}$$

Proof. Note that

$$x^2+y^2+z^2-xy-yz-zx=\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2},$$

thus we have

$$\begin{aligned}\frac{3}{2}\min\left\{(x-y)^2,(y-z)^2,(z-x)^2\right\}&\le x^2+y^2+z^2-xy-yz-zx\\&\le\frac{3}{2}\max\left\{(x-y)^2,(y-z)^2,(z-x)^2\right\}.\end{aligned}$$

The inequality (30) is proved.

Application 6. *In all triangle ABC holds*

$$\frac{3}{2}\min\left\{(a-b)^2,(b-c)^2,(c-a)^2\right\}$$

$$\leq s^2 - 3r^2 - 12Rr \leq \max\left\{(a-b)^2, (b-c)^2, (c-a)^2\right\} \tag{31}$$

$$\frac{3}{2}\min\left\{(h_a-h_b)^2,(h_b-h_c)^2(h_c-h_a)^2\right\}$$
$$\leq \left(\frac{s^2+r^2+4Rr}{2R}\right)^2 - \frac{6s^2r}{R} \leq \frac{3}{2}\max\left\{(h_a-h_b)^2,(h_b-h_c)^2(h_c-h_a)^2\right\} \tag{32}$$

$$\frac{3}{2}\min\left\{(r_a-r_b)^2,(r_b-r_c)^2(r_c-r_a)^2\right\}$$
$$\leq (4R+r)^2 - 3s^2 \leq \frac{3}{2}\max\left\{(r_a-r_b)^2,(r_b-r_c)^2(r_c-r_a)^2\right\} \tag{33}$$

$$\frac{3}{2}\min\left\{\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)^2, \left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)^2, \left(\sin^2\frac{C}{2}-\sin^2\frac{A}{2}\right)^2\right\}$$
$$\leq \frac{16R^2+r^2-3s^2+8Rr}{16R^2} \leq \frac{3}{2}\max\left\{\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)^2, \left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)^2, \left(\sin^2\frac{C}{2}-\sin^2\frac{A}{2}\right)^2\right\} \tag{34}$$

$$\frac{3}{2}\min\left\{\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)^2, \left(\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right)^2 \left(\cos^2\frac{C}{2}-\cos^2\frac{A}{2}\right)^2\right\}$$
$$\leq \frac{(4R+r)^2-3s^2}{16R^2} \leq \frac{3}{2}\max\left\{\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)^2, \left(\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right)^2 \left(\cos^2\frac{C}{2}-\cos^2\frac{A}{2}\right)^2\right\} \tag{35}$$

Proof. The proof of inequalities (31)–(35) are similar to the proof of inequalities (24)–(29).
Theorem 7. *If* $x, y, z > 0$, *then*

$$\left(\frac{2}{3}\sum x^2 + \frac{1}{3}\sum xy\right)^3 \geq \prod\left(x^2+xy+y^2\right) \geq \left(\sum xy\right)^3 \tag{36}$$

Proof. By applying the arithmetic-geometric means inequality, we have

$$\prod\left(x^2+xy+y^2\right) \leq \left(\frac{\sum\left(x^2+xy+y^2\right)}{3}\right)^3$$
$$= \left(\frac{2}{3}\sum x^2 + \frac{1}{3}\sum xy\right)^3.$$

Setting $a = \sqrt{x^2+xy+y^2}$, $b = \sqrt{y^2+yz+z^2}$, $c = \sqrt{z^2+zx+x^2}$, it is easy to verify that there is a triangle with the sides length a, b, c, according to the above substitution, the right-hand side inequality of (36) is equivalent to the well-known Pólya-Szegö inequality in a triangle [14]:

$$(abc)^2 \geq \left(\frac{4F}{\sqrt{3}}\right)^3, \tag{37}$$

where F denotes the area of a triangle whose sides are a, b, c. This completes the proof of Theorem 7.

Application 7. *In all triangle ABC holds*

$$\frac{1}{27}(5s^2-3r^2-12Rr)\geq\prod\left(a^2+ab+b^2\right)\geq(s^2+r^2+4Rr)^3 \tag{38}$$

$$\frac{1}{27}(2s^2-3r^2-12Rr)\geq\prod\left[(s-a)^2+(s-a)(s-b)+(s-b)^2\right]\geq(r^2+4Rr)^3 \tag{39}$$

$$\frac{1}{27}\left[2\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-\frac{6s^2r}{R}\right]^3\geq\prod\left(h_a^2+h_ah_b+h_b^2\right)\geq\left(\frac{2s^2r}{R}\right)^6 \tag{40}$$

$$\frac{1}{27}\left[2(4R+r)^2-3s^2\right]^3\geq\prod\left(r_a^2+r_ar_b+r_b^2\right)\geq s^6 \tag{41}$$

$$\frac{1}{27}\left[2\left(\frac{2R-r}{2R}\right)^2-\frac{3(s^2+r^2-8Rr)}{16R^2}\right]^3\geq\prod\left(\sin^4\frac{A}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{B}{2}\right)\geq\left(\frac{s^2+r^2-8Rr}{16R^2}\right)^3 \tag{42}$$

$$\frac{1}{27}\left[2\left(\frac{4R+r}{2R}\right)^2-\frac{3\left[s^2+(4R+r)^2\right]}{16R^2}\right]^3\geq\prod\left(\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}\right)\geq\left[\frac{s^2+(4R+r)^2}{16R^2}\right]^3 \tag{43}$$

Proof. In Theorem 7 we take $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c),\ (h_a,h_b,h_c),$ $(r_a,r_b,r_c),\ (\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}),\ (\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2})\}$ respectively, we have

$$\frac{1}{27}\left(2\sum a^2+\sum ab\right)^3\geq\prod\left(a^2+ab+b^2\right)\geq\left(\sum ab\right)^3,$$

from $\sum a=2s, \sum ab=s^2+r^2+4Rr$, we deduce the inequality (38).

$$\frac{1}{27}\left[2\sum(s-a)^2+\sum(s-a)(s-b)\right]^3\geq\prod\left[(s-a)^2+(s-a)(s-b)+(s-b)^2\right]\geq\left[\sum(s-a)(s-b)\right]^3,$$

from $\sum(s-a)=s, \sum(s-a)(s-b)=r(r+4R)$, we deduce the inequality (39).

$$\frac{1}{27}\left(2\sum h_a^2+\sum h_ah_b\right)^3\geq\prod\left(h_a^2+h_ah_b+h_b^2\right)\geq\left(\sum h_ah_b\right)^3,$$

from $\sum h_a=\frac{s^2+r^2+4Rr}{2R}, \sum h_ah_b=\frac{2s^2r}{R}$, we deduce the inequality (40).

$$\frac{1}{27}\left[2\sum {r_a}^2+\sum r_ar_b\right]^3\geq\prod\left(r_a^2+r_ar_b+r_b^2\right)\geq\left(\sum r_ar_b\right)^3,$$

from $\sum r_a=4R+r, \sum r_ar_b=s^2$, we deduce the inequality (41).

$$\frac{1}{27}\left[2\sum\sin^4\frac{A}{2}+\sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right]^3\geq\prod\left(\sin^4\frac{A}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{B}{2}\right)\geq\left(\sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)^3,$$

from $\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}, \sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}=\frac{s^2+r^2-8Rr}{16R^2}$, we deduce the inequality (42).

$$\frac{1}{27}\left[2\sum\cos^4\frac{A}{2}+\sum\cos^2\frac{A}{2}\cos^2\frac{B}{2}\right]^3\geq\prod\left(\cos^4\frac{A}{2}^2+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}\right)\geq\left(\sum\cos^2\frac{A}{2}\cos^2\frac{B}{2}\right)^3,$$

from $\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R}$, $\sum\cos^2\frac{A}{2}\cos^2\frac{B}{2}=\frac{s^2+(4R+r)^2}{16R^2}$, we deduce the inequality (43).

Theorem 8. *If* $x,y,z>0$, *then*

$$\frac{9xyz}{2\left(x+y+z\right)}\leq\sum\frac{xy^2}{x+y}\leq\frac{1}{4}\left(\sum x^2+\sum xy\right). \tag{44}$$

Proof. By applying the arithmetic-geometric means inequality, we have

$$\begin{aligned}\sum\frac{xy^2}{x+y}&\geq 3\prod\frac{\sqrt[3]{xy^2}}{\sqrt[3]{x+y}}\\&=\frac{3xyz}{\prod\sqrt[3]{x+y}}\\&\geq\frac{3xyz}{\frac{1}{3}\sum\left(x+y\right)}\\&=\frac{9xyz}{2\left(x+y+z\right)},\end{aligned}$$

on the other hand, we have

$$\begin{aligned}\sum\frac{xy^2}{x+y}&\leq\frac{1}{4}\sum\frac{(x+y)^2y}{x+y}\\&=\frac{1}{4}\left(\sum x^2+\sum xy\right).\end{aligned}$$

This completes the proof of Theorem 8.
Application 8. *In all triangle ABC holds*

$$9Rr\leq\sum\frac{ab^2}{a+b}\leq\frac{1}{4}(3s^2-r^2-4Rr) \tag{45}$$

$$\frac{9r^2}{2}\leq\sum\frac{\left(s-a\right)\left(s-b\right)^2}{c}\leq\frac{1}{4}(s^2-r^2-4Rr) \tag{46}$$

$$\frac{9s^2r^2}{s^2+r^2+4Rr}\leq\sum\frac{h_ah_b^2}{h_a+h_b}\leq\frac{1}{4}\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-\frac{2s^2r}{R}\right] \tag{47}$$

$$\frac{9s^2r}{2\left(4R+r\right)}\leq\sum\frac{r_ar_b^2}{r_a+r_b}\leq\frac{1}{4}\left[(4R+r)^2-s^2\right] \tag{48}$$

$$\frac{9r^2}{16R(2R-r)}\leq\sum\frac{\sin^2\frac{A}{2}\sin^4\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}\leq\frac{1}{4}\left[\left(\frac{2R-r}{2R}\right)^2-\frac{s^2+r^2-8Rr}{16R^2}\right] \tag{49}$$

$$\frac{9s^2}{16R(4R+r)}\leq\sum\frac{\cos^2\frac{A}{2}\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}\leq\frac{1}{4}\left[\left(\frac{4R+r}{2R}\right)^2-\frac{s^2+(4R+r)^2}{16R^2}\right] \tag{50}$$

Proof. In Theorem 8 we take $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c),\ (h_a,h_b,h_c),$ $(r_a,r_b,r_c),\ (\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}),\ (\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2})\}$ respectively, we have

$$\frac{9\prod a}{2\sum a}\le\sum\frac{ab^2}{a+b}\le\frac{1}{4}(\sum a^2+\sum ab),$$

from $\sum a=2s,\sum ab=s^2+r^2+4Rr,\prod a=4sRr$, we deduce the inequality (45).

$$\frac{9\prod(s-a)}{2\sum(s-a)}\le\sum\frac{(s-a)(s-b)^2}{c}\le\frac{1}{4}(\sum(s-a)^2+\sum(s-a)(s-b)),$$

from $\sum(s-a)=s,\sum(s-a)(s-b)=r(r+4R),\prod(s-a)=sr^2$, we deduce the inequality (46).

$$\frac{9\prod h_a}{2\sum h_a}\le\sum\frac{h_ah_b^2}{h_a+h_b}\le\frac{1}{4}\left(\sum h_a^2+\sum h_ah_b\right),$$

from $\sum h_a=\frac{s^2+r^2+4Rr}{2R},\sum h_ah_b=\frac{2s^2r}{R},\prod h_a=\frac{2s^2r^2}{R}$, we deduce the inequality (47).

$$\frac{9\prod r_a}{2\sum r_a}\le\sum\frac{r_ar_b^2}{r_a+r_b}\le\frac{1}{4}\left(\sum r_a^2+\sum r_ar_b\right),$$

from $\sum r_a=4R+r,\sum r_ar_b=s^2,\prod r_a=s^2r$, we deduce the inequality (48).

$$\frac{9\prod\sin^2\frac{A}{2}}{2\sum\sin^2\frac{A}{2}}\le\sum\frac{\sin^2\frac{A}{2}\sin^4\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}\le\frac{1}{4}\left[\sum\sin^4\frac{A}{2}+\sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right],$$

from $\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R},\sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}=\frac{s^2+r^2-8Rr}{16R^2},\prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}$, we deduce the inequality (49).

$$\frac{9\prod\cos^2\frac{A}{2}}{2\sum\cos^2\frac{A}{2}}\le\sum\frac{\cos^2\frac{A}{2}\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}\le\frac{1}{4}\left[\sum\cos^4\frac{A}{2}+\sum\cos^2\frac{A}{2}\cos^2\frac{B}{2}\right],$$

from $\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},\sum\cos^2\frac{A}{2}\cos^2\frac{B}{2}=\frac{s^2+(4R+r)^2}{16R^2},\prod\cos^2\frac{A}{2}=\frac{s^2}{16R^2}$, we deduce the inequality (50).

Theorem 9. *Let* $E(m)=\left(\sum x^m\right)\left(\sum\frac{1}{x^m}\right)$ $(m=1,2,\ldots)$, *then for* $x,y,z>0$ *the following inequality holds*

$$E(m+1)\ge E(m)\ge\cdots\ge E(1)\ge 9. \tag{51}$$

Proof. By using power means inequality and Cauchy-Schwarz inequality [15], we deduce that

$$\begin{aligned}
E(m+1) &= \left(\sum x^{m+1}\right)\left(\sum\frac{1}{x^{m+1}}\right)\\
&= \left[\sum(x^m)^{\frac{m+1}{m}}\right]\left[\sum(\frac{1}{x^m})^{\frac{m+1}{m}}\right]\\
&\ge 3^{1-\frac{m+1}{m}}\left(\sum x^m\right)^{\frac{m+1}{m}}3^{1-\frac{m+1}{m}}\left(\sum\frac{1}{x^m}\right)^{\frac{m+1}{m}}\\
&= \left(\sum x^m\right)\left(\sum\frac{1}{x^m}\right)\left[\frac{1}{9}\left(\sum x^m\right)\left(\sum\frac{1}{x^m}\right)\right]^{\frac{1}{m}}\\
&\ge \left(\sum x^m\right)\left(\sum\frac{1}{x^m}\right)\\
&= E(m),
\end{aligned}$$

and

$$E(1)=\left(\sum x\right)\left(\sum \frac{1}{x}\right)\geq 9.$$

The Theorem 9 is proved.

Application 9. *In all triangle ABC holds*

$$\frac{\left(s^2-r^2-4Rr\right)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)}{8s^2R^2r^2}\geq \frac{s^2+r^2+4Rr}{2Rr}\geq 9, \tag{52}$$

$$\frac{\left(s^2-2r^2-8Rr\right)\left((4R+r)^2-2s^2\right)}{s^2r^2}\geq \frac{4R+r}{r}\geq 9. \tag{53}$$

Proof. Putting $m=2$ in Theorem 9 gives

$$\left(x^2+y^2+z^2\right)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\geq (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9. \tag{54}$$

In the inequality (54) we take $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c)\}$, respectively, it follows that

$$\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)\geq\left(\sum a\right)\left(\sum\frac{1}{a}\right)\geq 9, \tag{55}$$

$$\left(\sum (s-a)^2\right)\left(\sum \frac{1}{(s-a)^2}\right)\geq\left(\sum (s-a)\right)\left(\sum\frac{1}{(s-a)}\right)\geq 9. \tag{56}$$

In inequalities (55) and (56), we apply the following identities:

$$\begin{aligned}
&(\sum a^2)(\sum\frac{1}{a^2})=\frac{\left(s^2-r^2-4Rr\right)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)}{8s^2R^2r^2},\\
&(\sum a)(\sum\frac{1}{a})=\frac{s^2+r^2+4Rr}{2Rr},\\
&(\sum (s-a)^2)(\sum\frac{1}{(s-a)^2})=\frac{\left(s^2-2r^2-4Rr\right)\left((4R+r)^2-2s^2\right)}{s^2r^2},\\
&(\sum (s-a))(\sum\frac{1}{(s-a)})=\frac{4R+r}{r},
\end{aligned}$$

giving the desired inequalities (52) and (53).

Acknowledgments. The research was supported by the Science and Technology Project of Longyan University of China.

References

[1] O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović and P. M. Vasić, Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.

[2] D. S. Mitrinović, J. E. Pečarić and V. Volenec, Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.

[3] D. S. Mitrinović, J. E. Pečarić, V. Volenec and J. Chen, Addenda to the Monograph: Recent Advances in Geometric Inequalities (I), J. Ningbo Univ., **4** (2) (1991),79–145.

[4] J. Sándor, Selected Chapters of Geometry, Analysis and Number Theory, RGMIA Monographs, Victoria Univ., 2005 [Available online at URL: http://rgmia.vu.edu.au/monographs].

[5] J. C. Kuang, Applied Inequalities, third ed., Shandong Science and Technology Press, Jinan, China, 2004 (in Chinese).

[6] X. Z. Yang, Research in Inequality, Tibet People's Publishing House, Lasha, China, 2000 (in Chinese).

[7] Y. D. Wu. The best constant for a geometric inequality, J. Inequal. Pure Appl. Math., **6** (4) (2005), Article 111, pp.1–17 (electronic).

[8] J. L. Díaz-Barrero, Some cyclical inequalities for the triangle, J. Inequal. Pure Appl. Math., **6** (1) (2005), Article 20, pp.1–11 (electronic).

[9] Y.N. Aliyev, The best constant for an algebraic inequality, J. Inequal. Pure Appl. Math., **8** (2) (2007), Article 79, pp.1–13 (electronic).

[10] R. A. Satnoiau, General power inequalities between the sides and the circumscribed and inscribed radii related to the fundamental triangle inequality. Math. Inequal. Appl., **5** (4) (2002), 745–751.

[11] R. A. Satnoiau, Refined geometric inequalities between two or more triangles obtained by dedublation. Math. Inequal. Appl., **7** (2) (2004), 289–298.

[12] R. A. Satnoiau, A universal method for establishing geometric inequalities in a triangle. Amer. Math. Monthly, **108** (2001), 360–364.

[13] R. B. Nelsen, Euler's triangle inequality via proofs without words, Math. Mag., 81 (1) (2008), 58–61.

[14] G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, II, Leipzig, 1925.

[15] D. S. Mitrinović and P. M. Vasić, Analytic Inequalities, Springer-Verlag, New York, 1970.