On a new Hardy-Littlewood-Polya's inequality with multi-parameters and its applications

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Abstract: The purpose of this paper is to build a new Hardy-Littlewood-Polya inequality with some parameters. As applications, we obtain some particular results. On the other hand, we give an application in Knopp's inequality.

Key words: Hardy-Littlewood-Polya's inequality; Hölder's inequality; Knopp's inequality; Beta function; Homogeneous -1 expression

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1 Introduction

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)$, $g(y)\geq 0$, $f(x)\in L^p(0,\infty)$, $g(y)\in L^q(0,\infty)$, $K(x,y)(\geq 0)$ is homogeneous -1 expression, and $\int_0^\infty K(x,1)x^{-1/p}dx=\int_0^\infty K(1,y)y^{-1/q}dy=k$, then

$$\int_0^\infty \left\{\int_0^\infty K(x,y)f(x)dx\right\}^p dy \leq k^p||f||_p^p \tag{1.1}$$

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \leq k||f||_p||g||_q. \tag{1.2}$$

(1.1) and (1.2) is the well known Hardy-Littlewood-Polya's inequality[1]. In connection with applications in analysis, their generalizations and variants have received considerable interest recent years. Firstly, by means of introducing a parameter, two forms of extended Hardy-Littlewood-Polya's inequality are obtained by Hu in [2] as follows.

(1) Let $\lambda>0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)$, $g(y)\geq 0$, $F(x)=x^{(1-\lambda)/q}f(x)\in L^p(0,\infty)$, $G(y)=y^{(1-\lambda)/p}g(y)\in L^q(0,\infty)$, $K(x,y)\geq 0$,$K^{1/\lambda}(x,y)$ is homogeneous -1 expression, and $\int_0^\infty K(\omega,1)\omega^{\lambda/p-1}d\omega=k$, then

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty K(x,y)f(x)dx\right\}^p dy \leq k^p||F||_p^p, \tag{1.3}$$

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \leq k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}, \tag{1.4}$$

where $m(p)=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $R(F,G)=\frac{(G^q,e)}{||G||_q^q}-\frac{(F^p,E)}{||F||_p^p}$, $E(x)=\int_0^\infty K(\omega,1)e(\frac{x}{\omega})\omega^{\lambda/q-1}d\omega$, $1-e(x)+e(y)\geq 0$, for x, $y\in(0,\infty)$. $(\bullet,*)$ in $R(F,G)$ is inner product.

(2) Let $\lambda>0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)$, $g(x)\geq 0$, $F(x)=x^{(1-\lambda)/p}f(x)\in L^p(0,\infty)$, $G(y)=y^{(1-\lambda)/q}g(y)\in L^q(0,\infty)$, $K(x,y)\geq 0$,$K^{1/\lambda}(x,y)$ is homogeneous -1 expression, and $\int_0^\infty K(\omega,1)\omega^{(\lambda-2)/p}d\omega=k$, then

$$\int_0^\infty y^{(\lambda-1)p/q}\left\{\int_0^\infty K(x,y)f(x)dx\right\}^p dy \leq k^p||F||_p^p, \tag{1.5}$$

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \leq k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}, \tag{1.6}$$

where $m(p)=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $R(F,G)=\frac{(G^q,e)}{||G||_q^q}-\frac{(F^p,E)}{||F||_p^p}$, $E(x)=\int_0^\infty K(\omega,1)e(\frac{x}{\omega})\omega^{(\lambda-2)/q}d\omega$, $1-e(x)+e(y)\geq 0$, for x, $y\in(0,\infty)$. $(\bullet,*)$ in $R(F,G)$ is inner product.

Remark 1 When $\lambda=1$ and $e(x)=1$, (1.3) and (1.5) are (1.1), (1.4) and (1.6) are (1.2).

Motivated by Yang[3], by introducing some parameters, we will give generalization of some Hardy-Littlewood-Polya's inequality included in (1.3) and (1.5). As applications, on the one hand, some particular results are considered. On the other hand, we give an application in Knopp's inequality.

2 Main results

Now we give our main result as follows in this paper.

Theorem 2.1 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$ and $(2-\min\{r,s\})t<\lambda$. $f(x)$, $g(x)\geq 0$, $F(x)=x^{[(1-t)+(2t-\lambda)/r]-1/p}f(x)\in L^p(0,\infty)$, $G(y)=y^{[(1-t)+(2t-\lambda)/s]-1/q}g(y)\in L^q(0,\infty)$, $K(x,y)\geq 0$,$K^{1/\lambda}(x,y)$ is homogeneous -1 expression, and $\int_0^\infty K(\omega,1)\omega^{-[(1-t)r+2t-\lambda]/r}d\omega=k$, then

$$\int_0^\infty y^{p[t-(2t-\lambda)/s]-1}\left\{\int_0^\infty K(x,y)f(x)dx\right\}^p dy\leq k^p||F||_p^p, \tag{2.1}$$

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\leq k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}, \tag{2.2}$$

where $m(p)=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $R(F,G)=\frac{(G^q,e)}{||G||_q^q}-\frac{(F^p,E)}{||F||_p^p}$, $E(x)=\int_0^\infty K(\omega,1)e(\frac{x}{\omega})\omega^{-[(1-t)r+2t-\lambda]/r}d\omega$, $1-e(x)+e(y)\geq 0$, for x, $y\in(0,\infty)$. $(\bullet,*)$ in $R(F,G)$ is inner product.

In particular, (a) for $s=p$, $r=q$, $t=0$, $0<\lambda$, $F(x)=x^{(1-\lambda)/q}f(x)$, $G(y)=y^{(1-\lambda)/p}g(y)$, we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty K(x,y)f(x)dx\right\}^p dy\leq k^p||F||_p^p, \tag{2.3}$$

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\leq k||F||_p||G||_q\{1-R(F,G)\}^{m(p)/2}. \tag{2.4}$$

(b) for $s=q$, $r=p$, $t=1$, $2-\min\{p,q\}<\lambda$, $F(x)=x^{(1-\lambda)/p}f(x)$, $G(y)=y^{(1-\lambda)/q}g(y)$, we have

$$\int_0^\infty y^{(\lambda-1)p/q}\left\{\int_0^\infty K(x,y)f(x)dx\right\}^p dy\leq k^p||F||_p^p, \tag{2.5}$$

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\leq k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}. \tag{2.6}$$

Remark 2 Inequalities (2.3)-(2.6) are (1.3)-(1.6), respectively.

The idea of the proof of Theorem 2.1 comes from Hu[2]. To prove Theorem 2.1, we need an inequality from Hu [6].

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $f(x)$, $g(y)\geq 0$, $1-e(x)-e(y)\geq 0$, then

$$\begin{aligned}\int_0^\infty f(x)g(y)dxdy &\leq \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty f^q(y)dy\right\}^{\frac{1}{q}}\\ &\times\left\{1-\left(\frac{\int_0^\infty g^q(x)e(x)dx}{\int_0^\infty g^q(x)dx}-\frac{\int_0^\infty f^p(x)e(x)dx}{\int_0^\infty f^p(x)dx}\right)\right\}^{\frac{m(p)}{2}},\end{aligned} \tag{2.7}$$

where $m(p)=\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof According to Hölder's inequality,

$$\begin{aligned}J(y) &= \int_0^\infty K(x,y)f(x)dx\\ &= \int_0^\infty\left\{K^{1/p}(x,y)f(x)\frac{x^{[(1-t)r+(2t-\lambda)]/(qr)}}{y^{[(1-t)s+(2t-\lambda)]/(ps)}}\right\}\left\{K^{1/q}(x,y)\frac{y^{[(1-t)s+(2t-\lambda)]/(ps)}}{x^{[(1-t)r+(2t-\lambda)]/(qr)}}\right\}dx\\ &= y^{1-\lambda}\int_0^\infty\left\{K^{1/p}(\omega,1)f(y\omega)\omega^{[(1-t)r+(2t-\lambda)]/(qr)}\right\}\left\{K^{1/q}(\omega,1)\omega^{-[(1-t)r+(2t-\lambda)]/(qr)}\right\}d\omega\\ &\leq y^{1-\lambda}\left\{\int_0^\infty K(\omega,1)f^p(y\omega)\omega^{p[(1-t)r+(2t-\lambda)]/(qr)}d\omega\right\}^{\frac{1}{p}}\left\{K(\int_0^\infty\omega,1)\omega^{-[(1-t)r+(2t-\lambda)]/r}d\omega\right\}^{\frac{1}{q}}\\ &= k^{\frac{1}{q}}y^{1-\lambda}\left\{\int_0^\infty K(\omega,1)f^p(y\omega)\omega^{p[(1-t)r+(2t-\lambda)]/(qr)}d\omega\right\}^{\frac{1}{p}},\end{aligned} \tag{2.8}$$

where in the third equation we set $\omega = \frac{x}{y}$.

$$
\begin{aligned}
& \int_0^\infty y^{p[t-(2t-\lambda)/s]-1} J^p(y)dy \\
\leq & \int_0^\infty y^{p[t-(2t-\lambda)/s]-1} k^{\frac{p}{q}} y^{p(1-\lambda)} \left\{ \int_0^\infty K(\omega,1) f^p(y\omega) \omega^{p[(1-t)r+(2t-\lambda)]/(qr)} d\omega \right\} dy \\
= & k^{\frac{p}{q}} \int_0^\infty x^{p[(1-t)+(2t-\lambda)/r]-1} f^p(x)dx \int_0^\infty K(\omega,1) \omega^{-[(1-t)r+2t-\lambda]/r} d\omega \\
= & k^p \int_0^\infty x^{p[(1-t)+(2t-\lambda)/r]-1} f^p(x)dx = k^p ||F||_p^p,
\end{aligned}
$$

where in the third equation we set that $y = \frac{x}{\omega}$. From this we obtain (2.1).

Now we prove the inequality (2.2). From (2.8), we have

$$
\begin{aligned}
\int_0^\infty g(y)J(y)dy \leq & k^{\frac{1}{q}} \int_0^\infty y^{[(1-t)+(2t-\lambda)/s]-1/q} g(y) \\
& \times \ y^{[(1-t)+(2t-\lambda)/r]-1/p} \left\{ \int_0^\infty K(\omega,1) f^p(y\omega) \omega^{p[(1-t)r+(2t-\lambda)]/(qr)} d\omega \right\}^{\frac{1}{p}} dy \\
= & k^{\frac{1}{q}} \int_0^\infty G(y) F_\lambda(y) dy. \qquad (2.9)
\end{aligned}
$$

According to the inequality (2.7), one obtains

$$
\int_0^\infty G(y) F_\lambda(y) dy \leq ||F_\lambda||_p ||G||_q \{1 - R^2(F_\lambda, G)\}^{m(p)/2}, \qquad (2.10)
$$

where $m(p) = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $R(F_\lambda, G) = \frac{(F_\lambda, e)}{||F_\lambda||_p^p} - \frac{(G,e)}{||G||_q^q}$, $1 - e(x) + e(y) \geq 0$, for x, $y \in (0, \infty)$.

Since

$$
\begin{aligned}
\int_0^\infty F_\lambda^p(y)dy = & \int_0^\infty y^{p[(1-t)+(2t-\lambda)/r]-1} \left\{ \int_0^\infty K(\omega,1) f^p(y\omega) \omega^{p[(1-t)r+(2t-\lambda)]/(qr)} d\omega \right\} dy \\
= & \int_0^\infty x^{p[(1-t)+(2t-\lambda)/r]-1} f^p(x)dx \int_0^\infty K(\omega,1) \omega^{-[(1-t)r+2t-\lambda]/r} d\omega \\
= & k \int_0^\infty x^{p[(1-t)+(2t-\lambda)/r]-1} f^p(x)dx = k||F||_p^p, \qquad (2.11)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\infty F_\lambda^p(y)e(y)dy = & \int_0^\infty y^{p[(1-t)+(2t-\lambda)/r]-1} \left\{ \int_0^\infty K(\omega,1) f^p(y\omega) \omega^{p[(1-t)r+(2t-\lambda)]/(qr)} d\omega \right\} de(y)y \\
= & \int_0^\infty x^{p[(1-t)+(2t-\lambda)/r]-1} f^p(x)dx \int_0^\infty K(\omega,1) e(\frac{x}{\omega}) \omega^{-[(1-t)r+2t-\lambda]/r} d\omega \\
= & ||F||_p^p \int_0^\infty K(\omega,1) e(\frac{x}{\omega}) \omega^{-[(1-t)r+2t-\lambda]/r} d\omega. \qquad (2.12)
\end{aligned}
$$

Substituting (2.10)-(2.12) into (2.9), one obtains

$$
\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dx dy \leq k||F||_p ||G||_q \{1 - R^2(F,G)\}^{m(p)/2},
$$

where $m(p) = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $R(F,G) = \frac{(G^q,e)}{||G||_q^q} - \frac{(F^p,E)}{||F||_p^p}$, $E(x) = \int_0^\infty K(\omega,1) e(\frac{x}{\omega}) \omega^{-[(1-t)r+2t-\lambda]/r} d\omega$, $1 - e(x) + e(y) \geq 0$, for x, $y \in (0, \infty)$. The proof of Theorem 2.1 is completed.

3 Some particular results

Remark 3 $B(\bullet,*)$ is Beta function and $(\bullet,*)$ in $R(F,G)$ is inner product in this paper.

(1) Setting

$$K(x,y)=\frac{1}{(x^c+y^c)^{\lambda/c}}\ \ (c>0,\ (2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}).$$

It is clearly that $K(x,y)\ge0$ for $x,y\in(0,\infty)$, $K^{1/\lambda}(x,y)$ is homogeneous -1 expression.

By Theorem 2.1, we have

Corollary 3.1 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$ and $(2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}$. $f(x),g(x)\ge0$, $F(x)=x^{[(1-t)+(2t-\lambda)/r]-1/p}f(x)\in L^p(0,\infty)$, $G(y)=y^{[(1-t)+(2t-\lambda)/s]}\cdot y^{-1/q}g(y)\in L^q(0,\infty)$, $k=\frac{1}{c}B\left(\frac{1}{c}\left(\frac{\lambda}{r}+\frac{2t}{s}-t\right),\frac{1}{c}\left(\frac{\lambda}{s}+\frac{2t}{r}-t\right)\right)$, then

$$\int_0^\infty y^{p[t-(2t-\lambda)/s]-1}\left\{\int_0^\infty\frac{f(x)}{(x^c+y^c)^{\lambda/c}}dx\right\}^p dy\le k^p||F||_p^p, \tag{3.1}$$

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x^c+y^c)^{\lambda/c}}dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}, \tag{3.2}$$

where $m(p)=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $R(F,G)=\frac{(G^q,e)}{||G||_q^q}-\frac{(F^p,E)}{||F||_p^p}$, $E(x)=\int_0^\infty\frac{1}{(\omega^c+1)^{\lambda/c}}e(\frac{x}{\omega})\omega^{-[(1-t)r+2t-\lambda]/r}d\omega$, $1-e(x)+e(y)\ge0$, for x, $y\in(0,\infty)$.

In particular, (a) for $s=p$, $r=q$, $t=0$, $0<\lambda\le\min\{p,q\}$, $F(x)=x^{(1-\lambda)/q}f(x)$, $G(y)=y^{(1-\lambda)/p}g(y)$, $k=\frac{1}{c}B\left(\frac{\lambda}{cp},\frac{\lambda}{cq}\right)$, we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty\frac{f(x)}{(x^c+y^c)^{\lambda/c}}dx\right\}^p dy\le k^pk^p||F||_p^p, \tag{3.3}$$

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x^c+y^c)^{\lambda/c}}dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}. \tag{3.4}$$

(b) for $s=q$, $r=p$, $t=1$, $2-\min\{p,q\}<\lambda\le2$, $F(x)=x^{(1-\lambda)/p}f(x)$, $G(y)=y^{(1-\lambda)/q}g(y)$, $k=\frac{1}{c}B\left(\frac{1}{c}\left(\frac{\lambda}{p}+\frac{2}{q}-1\right),\frac{1}{c}\left(\frac{\lambda}{q}+\frac{2}{p}-1\right)\right)$, we have

$$\int_0^\infty y^{(\lambda-1)p/q}\left\{\int_0^\infty\frac{f(x)}{(x^c+y^c)^{\lambda/c}}dx\right\}^p dy\le k^pk^p||F||_p^p, \tag{3.5}$$

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x^c+y^c)^{\lambda/c}}dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}. \tag{3.6}$$

(2) Setting

$$K(x,y)=\frac{\ln(x/y)}{x^\lambda-y^\lambda}\ \ ((2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}).$$

It is clear that $K(x,y)\ge0$ for $x,y\in(0,\infty)$, $K^{1/\lambda}(x,y)$ is homogeneous -1 expression.

By Theorem 2.1, we have

Corollary 3.2 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$ and $(2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}$. $f(x),g(x)\ge0$, $F(x)=x^{[(1-t)+(2t-\lambda)/r]-1/p}f(x)\in L^p(0,\infty)$, $G(y)=y^{[(1-t)+(2t-\lambda)/s]}\cdot y^{-1/q}g(y)\in L^q(0,\infty)$, $k=\frac{1}{\lambda^2}B^2\left(\frac{1}{\lambda}\left(\frac{\lambda}{r}+\frac{2t}{s}-t\right),\frac{1}{\lambda}\left(\frac{\lambda}{s}+\frac{2t}{r}-t\right)\right)$, then

$$\int_0^\infty y^{p[t-(2t-\lambda)/s]-1}\left\{\int_0^\infty\frac{\ln(x/y)}{x^\lambda-y^\lambda}f(x)dx\right\}^p dy\le k^p||F||_p^p, \tag{3.7}$$

$$\int_0^\infty\int_0^\infty\frac{\ln(x/y)}{x^\lambda-y^\lambda}f(x)g(y)dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}, \tag{3.8}$$

where $m(p) = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $R(F,G) = \frac{(G^q,e)}{||G||_q^q} - \frac{(F^p,E)}{||F||_p^p}$, $E(x) = \int_0^\infty \frac{\ln \omega}{(\omega^c-1)^{\lambda/c}} e(\frac{x}{\omega})\omega^{-[(1-t)r+2t-\lambda]/r} d\omega$, $1 - e(x) + e(y) \geq 0$, for x, $y \in (0,\infty)$.

In particular, (a) for $s = p$, $r = q$, $t = 0$, $0 < \lambda \leq \min\{p,q\}$, $F(x) = x^{(1-\lambda)/q} f(x)$, $G(y) = y^{(1-\lambda)/p} g(y)$, $k = \frac{1}{\lambda^2} B^2\left(\frac{1}{p}, \frac{1}{q}\right)$, we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)dx\right\}^p dy \leq k^p ||F||_p^p, \tag{3.9}$$

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)g(y)dxdy \leq k||F||_p||G||_q\{1 - R^2(F,G)\}^{m(p)/2}. \tag{3.10}$$

(b) for $s = q$, $r = p$, $t = 1$, $2 - \min\{p,q\} < \lambda \leq 2$, $F(x) = x^{(1-\lambda)/p} f(x)$, $G(y) = y^{(1-\lambda)/q} g(y)$, $k = \frac{1}{\lambda^2} B^2\left(\frac{1}{\lambda}\left(\frac{\lambda}{p} + \frac{2}{q} - 1\right), \frac{1}{\lambda}\left(\frac{\lambda}{q} + \frac{2}{p} - 1\right)\right)$, we have

$$\int_0^\infty y^{(\lambda-1)p/q}\left\{\int_0^\infty \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)dx\right\}^p dy \leq k^p ||F||_p^p, \tag{3.11}$$

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y)}{x^\lambda - y^\lambda} f(x)g(y)dxdy \leq k||F||_p||G||_q\{1 - R^2(F,G)\}^{m(p)/2}. \tag{3.12}$$

(3) Setting

$$K(x,y) = \frac{1}{(\max\{x,y\})^\lambda} \quad ((2 - \min\{r,s\})t < \lambda \leq (2 - \min\{r,s\})t + \min\{r,s\}).$$

It is clear that $K(x,y) \geq 0$ for $x, y \in (0,\infty)$, $K^{1/\lambda}(x,y)$ is homogeneous -1 expression.

By Theorem 2.1, we have

Corollary 3.3 Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $t \in [0,1]$ and $(2 - \min\{r,s\})t < \lambda \leq (2 - \min\{r,s\})t + \min\{r,s\}$. $f(x), g(x) \geq 0$, $F(x) = x^{[(1-t)+(2t-\lambda)/r]-1/p} f(x) \in L^p(0,\infty)$, $G(y) = y^{[(1-t)+(2t-\lambda)/s]} \cdot y^{-1/q} g(y) \in L^q(0,\infty)$, $k = \frac{sr\lambda}{[st-(2t-\lambda)][rt-(2t-\lambda)]}$, then

$$\int_0^\infty y^{p[t-(2t-\lambda)/s]-1}\left\{\int_0^\infty \frac{f(x)}{(\max\{x,y\})^\lambda} dx\right\}^p dy \leq k^p ||F||_p^p, \tag{3.13}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\max\{x,y\})^\lambda} dxdy k||F||_p||G||_q\{1 - R^2(F,G)\}^{m(p)/2}, \tag{3.14}$$

where $m(p) = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $R(F,G) = \frac{(G^q,e)}{||G||_q^q} - \frac{(F^p,E)}{||F||_p^p}$, $E(x) = \int_0^\infty \frac{1}{\max(\omega,1)^\lambda} e(\frac{x}{\omega})\omega^{-[(1-t)r+2t-\lambda]/r} d\omega$, $1 - e(x) + e(y) \geq 0$, for x, $y \in (0,\infty)$.

In particular, (a) for $s = p$, $r = q$, $t = 0$, $0 < \lambda \leq \min\{p,q\}$, $F(x) = x^{(1-\lambda)/q} f(x)$, $G(y) = y^{(1-\lambda)/p} g(y)$, $k = \frac{pq}{\lambda}$, we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{f(x)}{(\max\{x,y\})^\lambda} dx\right\}^p dy \leq k^p ||F||_p^p, \tag{3.15}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\max\{x,y\})^\lambda} dxdy \leq k||F||_p||G||_q\{1 - R^2(F,G)\}^{m(p)/2}. \tag{3.16}$$

(b) for $s = q$, $r = p$, $t = 1$, $2 - \min\{p,q\} < \lambda \leq 2$, $F(x) = x^{(1-\lambda)/p} f(x)$, $G(y) = y^{(1-\lambda)/q} g(y)$, $k = \frac{pq\lambda}{[p-2+\lambda][q-2+\lambda]}$, we have

$$\int_0^\infty y^{(\lambda-1)p/q}\left\{\int_0^\infty \frac{f(x)}{(\max\{x,y\})^\lambda} dx\right\}^p dy \leq k^p ||F||_p^p, \tag{3.17}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\max\{x,y\})^\lambda} dxdy \leq k||F||_p||G||_q\{1 - R^2(F,G)\}^{m(p)/2}. \tag{3.18}$$

(4) Setting

$$K(x,y)=\frac{x^{b+\frac{2\lambda}{r}-\lambda}y^{c+\frac{2\lambda}{s}-\lambda}}{(x+y)^{b+c+\lambda}}((2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}).$$

It is clear that $K(x,y)\ge 0$ for $x,y\in(0,\infty)$, $K^{1/\lambda}(x,y)$ is homogeneous -1 expression.
By Theorem 2.1, we have

Corollary 3.4 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$ and $(2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}$, $b>\lambda-t-\frac{3\lambda-2t}{r}$, $c>\lambda-t-\frac{3\lambda-2t}{s}$. $f(x)$, $g(x)\ge 0$, $F(x)=x^{[(1-t)+(2t-\lambda)/r]-1/p}f(x)\in L^p(0,\infty)$, $G(y)=y^{[(1-t)+(2t-\lambda)/s]-1/q}g(y)\in L^q(0,\infty)$, $k=B\left(b+t-\lambda+\frac{3\lambda-2t}{r},c+t-\lambda+\frac{3\lambda-2t}{s}\right)$, then

$$\int_0^\infty y^{p[t-(2t-\lambda)/s]-1}\left\{\int_0^\infty \frac{x^{b+\frac{2\lambda}{r}-\lambda}y^{c+\frac{2\lambda}{s}-\lambda}}{(x+y)^{b+c+\lambda}}f(x)dx\right\}^p dy\le k^p||F||_p^p, \tag{3.19}$$

$$\int_0^\infty\int_0^\infty \frac{x^{b+\frac{2\lambda}{r}-\lambda}y^{c+\frac{2\lambda}{s}-\lambda}}{(x+y)^{b+c+\lambda}}f(x)g(y)dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}, \tag{3.20}$$

where $m(p)=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $R(F,G)=\frac{(G^q,e)}{||G||_q^q}-\frac{(F^p,E)}{||F||_p^p}$, $E(x)=\int_0^\infty \frac{\omega^{b+\frac{2\lambda}{r}-\lambda}}{(\omega+1)^{b+c+\lambda}}e(\frac{x}{\omega})\omega^{-[(1-t)r+2t-\lambda]/r}d\omega$, $1-e(x)+e(y)\ge 0$, for x, $y\in(0,\infty)$.

In particular, (a) for $s=p$, $r=q$, $t=0$, $0<\lambda\le\min\{p,q\}$, $F(x)=x^{(1-\lambda)/q}f(x)$, $G(y)=y^{(1-\lambda)/p}g(y)$, $k=B\left(b-\lambda+\frac{3\lambda}{q},c-\lambda+\frac{3\lambda}{p}\right)$, we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{x^{b+\frac{2\lambda}{q}-\lambda}y^{c+\frac{2\lambda}{p}-\lambda}}{(x+y)^{b+c+\lambda}}f(x)dx\right\}^p dy\le k^p||F||_p^p, \tag{3.21}$$

$$\int_0^\infty\int_0^\infty \frac{x^{b+\frac{2\lambda}{q}-\lambda}y^{c+\frac{2\lambda}{p}-\lambda}}{(x+y)^{b+c+\lambda}}f(x)g(y)dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}. \tag{3.22}$$

(b) for $s=q$, $r=p$, $t=1$, $2-\min\{p,q\}<\lambda\le 2$, $F(x)=x^{(1-\lambda)/p}f(x)$, $G(y)=y^{(1-\lambda)/q}g(y)$, $k=B\left(b+1-\lambda+\frac{3\lambda-2}{p},c+1-\lambda+\frac{3\lambda-2}{q}\right)$, we have

$$\int_0^\infty y^{(\lambda-1)p/q}\left\{\int_0^\infty \frac{x^{b+\frac{2\lambda}{p}-\lambda}y^{c+\frac{2\lambda}{q}-\lambda}}{(x+y)^{b+c+\lambda}}f(x)dx\right\}^p dy\le k^p||F||_p^p, \tag{3.23}$$

$$\int_0^\infty\int_0^\infty \frac{x^{b+\frac{2\lambda}{p}-\lambda}y^{c+\frac{2\lambda}{q}-\lambda}}{(x+y)^{b+c+\lambda}}f(x)g(y)dxdy\le k||F||_p||G||_q\{1-R^2(F,G)\}^{m(p)/2}. \tag{3.24}$$

(5) Setting

$$K(x,y)=\begin{cases} y^{(\alpha-1)\lambda}x^{-\alpha\lambda} & , \quad x\le y,\\ 0 & , \quad x>y,\end{cases}\quad ((2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}).$$

It is clear that $K(x,y)\ge 0$ for $x,y\in(0,\infty)$, $K^{1/\lambda}(x,y)$ is homogeneous -1 expression.
By Theorem 2.1, we have

Corollary 3.5 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $t\in[0,1]$ and $(2-\min\{r,s\})t<\lambda\le(2-\min\{r,s\})t+\min\{r,s\}$. $f(x)$, $g(x)\ge 0$, $F(x)=x^{[(1-t)+(2t-\lambda)/r]-1/p}f(x)\in L^p(0,\infty)$, $ts-(\alpha-1)s\lambda+(\lambda-2t)>0$, $k=\frac{s}{ts-(\alpha-1)s\lambda+(\lambda-2t)}$, then

$$\int_0^\infty y^{p[t-(2t-\lambda)/s]-1}\left\{\int_0^y y^{(\alpha-1)\lambda}x^{-\alpha\lambda}f(x)dx\right\}^p dy\le k^p||F||_p^p. \tag{3.25}$$

In particular, (a) for $s=p$, $r=q$, $t=0$, $0<\lambda\le\min\{p,q\}$, $F(x)=x^{(1-\lambda)/q}f(x)$, $k=\frac{p}{p(1-\alpha)\lambda+\lambda}$, we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^y y^{(\alpha-1)\lambda}x^{-\alpha\lambda}f(x)dx\right\}^p dy\le k^p||F||_p^p. \tag{3.26}$$

(b) for $s = q$, $r = p$, $t = 1$, $2 - \min\{p, q\} < \lambda \le 2$, $F(x) = x^{(1-\lambda)/p} f(x)$, $k = \frac{q}{q - q(\alpha-1)\lambda + \lambda - 2}$, we have

$$\int_0^\infty y^{(\lambda-1)p/q} \left\{ \int_0^y y^{(\alpha-1)\lambda} x^{-\alpha\lambda} f(x) dx \right\}^p dy \le k^p ||F||_p^p. \tag{3.27}$$

Remark 4 Inequalities (3.3)-(3.6) and (3.27) are obtained by Hu[2].

Remark 5 By Corollary 3.4, one obtains

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $t \in [0, 1]$ and $(2 - \min\{r, s\})t < \lambda \le (2 - \min\{r, s\})t + \min\{r, s\}$, $b > \lambda - t - \frac{3\lambda - 2t}{r}$, $c > \lambda - t - \frac{3\lambda - 2t}{s}$. $f(x)$, $g(x) \ge 0$, $F(x) = x^{[\lambda - b + (1-t) + (2t-\lambda)/r] - 1/p} f(x) \in L^p(0, \infty)$, $G(y) = y^{[\lambda - c + (1-t) + (2t-\lambda)/s] - 1/q} g(y) \in L^q(0, \infty)$, $k = B\left(b + t - \lambda + \frac{3\lambda - 2t}{r}, c + t - \lambda + \frac{3\lambda - 2t}{s}\right)$, then

$$\int_0^\infty y^{p[c+t-\lambda+(3\lambda-2t)/s]-1} \left\{ \int_0^\infty \frac{f(x)}{(x+y)^{b+c+\lambda}} dx \right\}^p dy \le k^p ||F||_p^p, \tag{3.28}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{b+c+\lambda}} dx dy \le k ||F||_p ||G||_q \{1 - R^2(F, G)\}^{m(p)/2}, \tag{3.29}$$

where $m(p) = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $R(F, G) = \frac{(G^q, e)}{||G||_q^q} - \frac{(F^p, E)}{||F||_p^p}$, $E(x) = \int_0^\infty \frac{1}{(\omega+1)^{b+c+\lambda}} e(\frac{x}{\omega}) \omega^{-[(1-t)r+2t-\lambda]/r} d\omega$, $1 - e(x) + e(y) \ge 0$, for x, $y \in (0, \infty)$.

In particular, $r = p$, $s = q$, $\lambda = t = e(x) = 1$, $F(x) = x^{1-b-\frac{2}{p}} f(x)$, $G(y) = y^{1-c-\frac{2}{q}} g(y)$, $k = B\left(b + \frac{1}{p}, c + \frac{1}{q}\right)$, then

$$\int_0^\infty y^{p+pc-2} \left\{ \int_0^\infty \frac{f(x)}{(x+y)^{b+c+1}} dx \right\}^p dy \le k^p ||F||_p^p, \tag{3.30}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{b+c+1}} dx dy \le k ||F||_p ||G||_q. \tag{3.31}$$

where inequality (3.31) is given in [1] and [5].

4 A application in Knopp's inequality

Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $r > 0$, $f(x) \ge 0$, $f \in L^p(0, \infty)$, set

$$f_r(y) = \frac{1}{\Gamma(r)} \int_0^y \frac{(y-x)^{r-1}}{y^r} f(x) dx.$$

Then

$$\int_0^\infty f_r^p(y) dy \le \left(\frac{\Gamma(\frac{1}{q})}{\Gamma(r + \frac{1}{q})} \right)^p \int_0^\infty f^p(x) dx. \tag{4.1}$$

This well-known inequality (4.1) is many times referred to as Knopp's inequality, with the reference to paper [4].

Now we give a corollary of an application in Knopp's inequality (4.1).

Corollary 4.1 let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $t \in [0, 1]$ and $(2 - \min\{r, s\})t < \lambda$. $f(x)$, $g(x) \ge 0$, $c, n > 0$, $c + \lambda(n-1) > 0$, $x^{[(1-t)+(2t-\lambda)/r]-1/p} f(x) \in L^p(0, \infty)$. Set

$$f_n(y) = \frac{c y^{[(2t-\lambda)/s - t] + 1/p}}{\Gamma(1 + \frac{\lambda(n-1)}{c})} \int_0^y \frac{(y^c - x^c)^{\lambda(n-1)/c}}{y^{\lambda n}} f(x) dx.$$

Then

$$\int_0^\infty f_n^p(y) dy \le \frac{\Gamma^p\left(\frac{rt-(2t-\lambda)}{cr}\right)}{\Gamma^p\left(1 + \frac{\lambda(n-1)}{c} + \frac{rt-(2t-\lambda)}{cr}\right)} \int_0^\infty x^{p[(1-t)+(2t-\lambda)/r]-1} f^p(x) dx. \tag{4.2}$$

In particular, (a) for $s=p$, $r=q$, $t=0$, $0<\lambda$, thus

$$f_n(y)=\frac{cy^{(1-\lambda)/p}}{\Gamma(1+\frac{\lambda(n-1)}{c})}\int_0^y\frac{(y^c-x^c)^{\lambda(n-1)/c}}{y^{\lambda n}}f(x)dx.$$

Then

$$\int_0^\infty f_n^p(y)dy\le\frac{\Gamma^p\left(\frac{\lambda}{cp}\right)}{\Gamma^p\left(1+\frac{\lambda(n-1)}{c}+\frac{\lambda}{cp}\right)}\int_0^\infty x^{(p-1)(1-\lambda)}f^p(x)dx. \tag{4.3}$$

(b) for $s=q$, $r=p$, $t=1$, $2-\min\{p,q\}<\lambda$, thus

$$f_n(y)=\frac{cy^{(1-\lambda)/q}}{\Gamma(1+\frac{\lambda(n-1)}{c})}\int_0^y\frac{(y^c-x^c)^{\lambda(n-1)/c}}{y^{\lambda n}}f(x)dx.$$

Then

$$\int_0^\infty f_n^p(y)dy\le\frac{\Gamma^p\left(\frac{\lambda-2+p}{cp}\right)}{\Gamma^p\left(1+\frac{\lambda(n-1)}{c}+\frac{\lambda-2+p}{cp}\right)}\int_0^\infty x^{1-\lambda}f^p(x)dx. \tag{4.4}$$

Proof Set

$$K(x,y)=\begin{cases}\frac{c(y^c-x^c)^{\lambda(n-1)/c}}{y^{\lambda n}\Gamma(1+\frac{\lambda(n-1)}{c})} & , \quad 0\le x\le y,\\ 0 & , \quad 0<y<x.\end{cases}$$

It is clear that $K(x,y)\ge 0$ and $K(x,y)=K(y,x)$ for $x,y\in(0,\infty)$, $K^{1/\lambda}(x,y)$ is homogeneous -1 expression.

$$\begin{aligned}k &= \int_0^\infty K(x,1)x^{-[(1-t)r+2t-\lambda]/r}dx\\ &= \int_0^\infty \frac{c(1-x^c)^{\lambda(n-1)/c}}{\Gamma(1+\frac{\lambda(n-1)}{c})}\cdot x^{-[(1-t)r+2t-\lambda]/r}dx\\ &= \int_0^\infty (1-u)^{\lambda(n-1)/c}u^{\frac{rt-(2t-\lambda)}{cr}-1}du\\ &= B\left(1+\frac{\lambda(n-1)}{c},\frac{rt-(2t-\lambda)}{cr}\right).\end{aligned}$$

Due to (2.2) in Theorem 2.1, we obtain Corollary 2.1.

Remark 6 Inequalities (4.3) and (4.4) are obtained by Hu[2].

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