

GENERALIZATIONS OF AN INTEGRAL INEQUALITY

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ABSTRACT. In this short paper, an integral inequality posed in the 11th International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004 is generalized.

1. INTRODUCTION

The Problem 2 of the 11th International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004 (see [1]) reads as follows.

Proposition 1 ([1]). *Let $f, g : [a, b] \rightarrow [0, \infty)$ be two continuous and non-decreasing functions such that*

$$\int_a^x \sqrt{f(t)} \, dt \leq \int_a^x \sqrt{g(t)} \, dt \quad (1)$$

for $x \in [a, b]$ and

$$\int_a^b \sqrt{f(t)} \, dt = \int_a^b \sqrt{g(t)} \, dt. \quad (2)$$

Then

$$\int_a^b \sqrt{1 + f(t)} \, dt \geq \int_a^b \sqrt{1 + g(t)} \, dt. \quad (3)$$

It is clear that, considering (2), inequality (1) can be rewritten as

$$\int_x^b \sqrt{f(t)} \, dt \geq \int_x^b \sqrt{g(t)} \, dt. \quad (4)$$

If replacing $f(x)$ by $\sqrt{f(x)}$ and $g(x)$ by $\sqrt{g(x)}$, then Proposition 1 can be simplified into the following Proposition 2.

Proposition 2. *Let $f, g : [a, b] \rightarrow [0, \infty)$ be two continuous and non-decreasing functions such that*

$$\int_x^b f(t) \, dt \geq \int_x^b g(t) \, dt \quad (5)$$

for $x \in [a, b]$ and

$$\int_a^b f(t) \, dt = \int_a^b g(t) \, dt. \quad (6)$$

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Then

$$\int_a^b \sqrt{1+f^2(t)} dt \geq \int_a^b \sqrt{1+g^2(t)} dt. \quad (7)$$

If denoting

$$F(x) = \int_a^x f(t) dt \quad \text{and} \quad G(x) = \int_a^x g(t) dt, \quad (8)$$

then $F(x) \leq G(x)$ for $x \in [a, b]$ and $G(x)$ is a convex function on $[a, b]$. On the other hand, since $F(a) = G(a)$ and $F(b) = G(b)$, then inequality (7) is valid apparently, because the length of the curve $y = F(x)$ is not less than that of the curve $y = G(x)$. This explains the geometric meaning of Proposition 2 and gives a solution to Proposition 1.

The main aim of this paper is to generalize Proposition 1 and Proposition 2 above.

Our main results are included in a couple of theorems below.

Theorem 1. *Let $f : [a, b] \rightarrow [0, \infty)$ be a continuous function and $g : [a, b] \rightarrow [0, \infty)$ a continuous and non-decreasing function satisfying (5) and (6). Then*

$$\int_a^b h(f(t)) dt \geq \int_a^b h(g(t)) dt \quad (9)$$

validates for every convex function h on $[0, \infty)$.

Theorem 2. *Let $f : [a, b] \rightarrow [0, \infty)$ be a continuous function and $g : [a, b] \rightarrow [0, \infty)$ a continuous and non-increasing function satisfying (6) and the reverse of (5). Then inequality (9) holds true for every convex function h on $[0, \infty)$.*

Remark 1. It is easy to see that the function $\sqrt{1+x^2}$ is convex on $[a, b]$. Hence, inequality (7) can be deduced easily from (9).

2. PROOFS OF THEOREM 1 AND THEOREM 2

In order to prove our theorems, the well known second mean value theorem for integrals will be available.

Lemma 1 ([2, p. 35]). *Let $f(x)$ be bounded and monotonic and $g(x)$ integrable on $[a, b]$. Then there exists some $\xi \in [a, b]$ such that*

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx + f(b) \int_\xi^b g(x) dx. \quad (10)$$

Proof of Theorem 1. Denote

$$\phi(x) = - \int_x^b f(t) dt \quad \text{and} \quad \varphi(x) = - \int_x^b g(t) dt. \quad (11)$$

Since h is convex, then

$$h(t) \geq h(s) + (t-s)h'(s)$$

for $a \leq s, t \leq b$, therefore,

$$h(\phi'(t)) \geq h(\varphi'(t)) + [\phi'(t) - \varphi'(t)]h'(\varphi'(t)) \quad (12)$$

which gives

$$\int_a^b h(\phi'(t)) dt \geq \int_a^b h(\varphi'(t)) dt + \int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t)) dt. \quad (13)$$

Now it is sufficient to prove that

$$\int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t)) dt \geq 0. \quad (14)$$

Since $h(t)$ is convex, the function $h'(t)$ is non-decreasing; Since $g(t)$ is non-decreasing, the function $\varphi'(t)$ is also non-decreasing. Thus, the composite function $h'(\varphi'(t))$ is non-decreasing with respect to t . Using Lemma 1 yields

$$\begin{aligned} & \int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t)) dt \\ &= h'(\varphi'(a)) \int_a^\xi [\phi'(t) - \varphi'(t)] dt + h'(\varphi'(b)) \int_\xi^b [\phi'(t) - \varphi'(t)] dt \\ &= h'(g(a))[\phi(\xi) - \phi(a) - \varphi(\xi) + \varphi(a)] + h'(g(b))[\phi(b) - \phi(\xi) - \varphi(b) + \varphi(\xi)] \\ &= [\phi(\xi) - \varphi(\xi)][h'(g(a)) - h'(g(b))] \\ &\geq 0, \end{aligned}$$

where $\xi \in [a, b]$, since $\phi(\xi) \leq \varphi(\xi)$ and $h'(g(a)) \leq h'(g(b))$. The proof of Theorem 1 is complete. \square

Proof of Theorem 2. Denote

$$\psi(x) = \int_a^x f(t) dt \quad \text{and} \quad \theta(x) = \int_a^x g(t) dt \quad (15)$$

for $a \leq x \leq b$. Along with the proof of Theorem 1, it is obtained that, in order to show Theorem 2, it suffices to prove

$$\int_a^b [\psi'(t) - \theta'(t)][-h'(\theta'(t))] dt \leq 0. \quad (16)$$

Since h is convex, then $-h'$ is non-increasing; Since g is non-increasing, then θ' is also non-increasing. Consequently, the composite function $-h'(\theta'(t))$ is non-increasing with respect to t . Utilizing Lemma 1 leads to

$$\begin{aligned} & \int_a^b [\psi'(t) - \theta'(t)]h'(\theta'(t)) dt \\ &= h'(\theta'(a)) \int_a^\xi [\psi'(t) - \theta'(t)] dt + h'(\theta'(b)) \int_\xi^b [\psi'(t) - \theta'(t)] dt \\ &= [\psi(\xi) - \theta(\xi)][h'(g(a)) - h'(g(b))] \\ &\geq 0, \end{aligned}$$

where $\xi \in [a, b]$, since $\psi(\xi) \geq \theta(\xi)$ and $h'(g(a)) \geq h'(g(b))$. The proof of Theorem 2 is complete. \square

REFERENCES

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