Some Subclasses of Analytic Functions with Respect to $2k$-Symmetric Conjugate Points

Hong-Cai Luo[1] **and Zhi-Gang Wang**[2]

[1]Department of Mathematics
Xiangnan University
Chenzhou 423000, Hunan, People's Republic of China
E-Mail: xnxylhc@163.com

[2]Department of Mathematics and Computing Science
Changsha University of Science and Technology
Changsha 410076, Hunan, People's Republic of China
E-Mail: zhigwang@163.com

Abstract

In the present paper, we introduce two new subclasses $\mathcal{P}_{sc}^{(k)}(\lambda,\alpha)$ and $\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$ of analytic functions with respect to $2k$-symmetric conjugate points. Such results as integral representations, convolution conditions and coefficient inequalities for these classes are provided.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are starlike and convex of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$, respectively.

Let $\mathcal{S}_{sc}^{(k)}(\alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the following inequality:

$$\Re\left(\frac{zf'(z)}{f_{2k}(z)}\right) > \alpha \qquad (z \in \mathbb{U}),$$

where $0 \leqq \alpha < 1,\ k \geqq 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality:

$$f_{2k}(z) = \frac{1}{2k}\sum_{\nu=0}^{k-1}\left(\varepsilon^{-\nu} f(\varepsilon^{\nu} z) + \varepsilon^{\nu}\overline{f(\varepsilon^{\nu}\overline{z})}\right) \qquad \left(\varepsilon = \exp\left(\frac{2\pi i}{k}\right);\ z \in \mathbb{U}\right). \tag{1.3}$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_{sc}^{(k)}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}_{sc}^{(k)}(\alpha)$. The class $\mathcal{S}_{sc}^{(k)}(0)$ of functions starlike with respect to $2k$-symmetric conjugate points was introduced and investigated by Al-Amiri *et al.* [1].

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of all functions which are of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0).$$

Let $\mathcal{T}(\lambda, \alpha)$ be the subclass of $\mathcal{T}$ consisting of functions $f(z)$ which satisfy the inequality:

$$\Re\left(\frac{\frac{zf'(z)}{f(z)}}{\lambda\frac{zf'(z)}{f(z)} + (1-\lambda)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < 1)$ and λ $(0 \leqq \lambda < 1)$. And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}(\lambda, \alpha)$ if and only if $zf'(z) \in \mathcal{T}(\lambda, \alpha)$. These classes were first introduced and investigated by Altintas and Owa [2], then were studied by Aouf *et al.* [3].

We note that the functions $f(z)$ with missing real coefficients belonging to $\mathcal{S}^*(\alpha)$ $(\mathcal{C}(\alpha))$ are in the class $\mathcal{S}_{sc}^{(k)}(\alpha)$ $(\mathcal{C}_{sc}^{(k)}(\alpha))$. Motivated by the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$, we now introduce the following subclasses of $\mathcal{A}$ with respect to $2k$-symmetric conjugate points, and obtain some interesting results.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{P}_{sc}^{(k)}\lambda, \alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)} + (1-\lambda)}\right) > \alpha \qquad (z \in \mathbb{U}), \tag{1.4}$$

where $0 \leqq \alpha < 1,\ 0 \leqq \lambda < 1$ and $f_{2k}(z)$ is defined the equality (1.3). And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$ if and only if $zf'(z) \in \mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$. For simplicity, we write $\mathcal{P}_{sc}^{(k)}(\lambda, \alpha) \cap \mathcal{T}$ simple as $\mathcal{T}\mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha) \cap \mathcal{T}$ simple as $\mathcal{T}\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$.

In the present paper, we shall provide some integral representations and convolution conditions for the classes $\mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$, we shall also provide some coefficient inequalities for functions belonging to these classes and their subclasses with negative coefficients.

2. Integral Representations

We first give some integral representations of functions in the classes $\mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$.

Theorem 1. *Let $f(z) \in \mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$ with $k \geqq 2$. Then*

$$f_{2k}(z) = z \cdot \exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)}\right.\right.$$

$$\left.\left. + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta\right), \tag{2.1}$$

where $f_{2k}(z)$ is defined by equality (1.3), $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$, we know that the condition (1.4) can be written as follows:

$$\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)} + (1-\lambda)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

where " $\prec$ " stands for the subordination. It follows that

$$\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)}+(1-\lambda)}=\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)},$$

where $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0)=0$, $|\omega(z)|<1$. This yields

$$\frac{zf'(z)}{f_{2k}(z)}=\frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}. \tag{2.2}$$

Substituting z by $\varepsilon^\mu z$ ($\mu=0,1,2,\ldots,k-1$) in (2.2), respectively, we get

$$\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)}{f_{2k}(\varepsilon^\mu z)}=\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}. \tag{2.3}$$

From (2.3), we have

$$\frac{\overline{\varepsilon^\mu \overline{z}}\ \overline{f'(\varepsilon^\mu \overline{z})}}{\overline{f_{2k}(\varepsilon^\mu \overline{z})}}=\frac{(1-\lambda)\left[1+(1-2\alpha)\overline{\omega(\varepsilon^\mu \overline{z})}\ \right]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu \overline{z})}}. \tag{2.4}$$

Note that $f_{2k}(\varepsilon^\mu z)=\varepsilon^\mu f_{2k}(z)$ and $\overline{f_{2k}(\varepsilon^\mu \overline{z})}=\varepsilon^{-\mu}f_{2k}(z)$, summing equalities (2.3) and (2.4), we can obtain

$$\frac{z(f'(\varepsilon^\mu z)+\overline{f'(\varepsilon^\mu \overline{z})})}{f_{2k}(z)}=\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}+\frac{(1-\lambda)\left[1+(1-2\alpha)\overline{\omega(\varepsilon^\mu \overline{z})}\ \right]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu \overline{z})}}. \tag{2.5}$$

Let $\mu=0,1,2,\ldots,k-1$ in (2.5), respectively, and summing them we can get

$$\frac{zf'_{2k}(z)}{f_{2k}(z)}=\frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}+\frac{(1-\lambda)\left[1+(1-2\alpha)\overline{\omega(\varepsilon^\mu \overline{z})}\ \right]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu \overline{z})}}. \tag{2.6}$$

From (2.6), we can get

$$\frac{f'_{2k}(z)}{f_{2k}(z)}-\frac{1}{z}=\frac{1}{2k}\sum_{\mu=0}^{k-1}\left(\frac{(1-\lambda)[1+(1-2\alpha)\omega(\varepsilon^\mu z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu z)}+\frac{(1-\lambda)\left[1+(1-2\alpha)\overline{\omega(\varepsilon^\mu \overline{z})}\ \right]}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu \overline{z})}}-2\right). \tag{2.7}$$

Integrating (2.6), we have

$$\log\left(\frac{f_{2k}(z)}{z}\right)=\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)}\right.$$
$$\left.+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta. \tag{2.8}$$

From (2.8), we can get (2.1) easily. Hence the proof of Theorem 1 is complete.

Theorem 2. *Let $f(z)\in\mathcal{P}_{sc}^{(k)}(\lambda,\alpha)$ with $k\geqq 2$. Then*

$$f(z)=\int_0^z\exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi\frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)}\right.\right.$$
$$\left.\left.+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta\right)\cdot\frac{(1-\lambda)[1+(1-2\alpha)\omega(\xi)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\xi)}d\xi, \tag{2.9}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0)=0$, $|\omega(z)|<1$.

Proof. Suppose that $f(z)\in\mathcal{P}_{sc}^{(k)}(\lambda,\alpha)$, from equalities (2.1) and (2.2), we can get

$$\begin{aligned}f'(z)=&\frac{f_{2k}(z)}{z}\cdot\frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}\\=&\exp\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z\frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)}\right.\right.\\&\left.\left.+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta\right)\cdot\frac{(1-\lambda)[1+(1-2\alpha)\omega(z)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(z)}.\end{aligned}$$

Integrating the above equality, we can easily get (2.6).

Similarly, for the class $\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$, we have

Corollary 1. *Let $f(z)\in\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$ with $k\geqq 2$. Then*

$$\begin{aligned}f_k(z)=\int_0^z\exp&\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi\frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)}\right.\right.\\&\left.\left.+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta\right)d\xi,\end{aligned}$$

where $f_{2k}(z)$ is defined by equality (1.3), $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0)=0$, $|\omega(z)|<1$.

Corollary 2. *Let $f(z)\in\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$ with $k\geqq 2$. Then*

$$\begin{aligned}f(z)=\int_0^z\frac{1}{t}\int_0^t\exp&\left(\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi\frac{2(1-\alpha)}{\zeta}\left(\frac{\omega(\varepsilon^\mu\zeta)}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\varepsilon^\mu\zeta)}\right.\right.\\&\left.\left.+\frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-\lambda-(1+\lambda-2\alpha\lambda)\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right)d\zeta\right)\cdot\frac{(1-\lambda)[1+(1-2\alpha)\omega(\xi)]}{1-\lambda-(1+\lambda-2\alpha\lambda)\omega(\xi)}d\xi dt,\end{aligned}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ and $\omega(0)=0$, $|\omega(z)|<1$.

3. Convolution Conditions

In this section, we provide some convolution conditions for the classes $\mathcal{P}_{sc}^{(k)}(\lambda,\alpha)$ and $\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$. Let $f,\ g\in\mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z)=z+\sum_{n=2}^{\infty}c_nz^n.$$

Then the Hadamard product (or convolution) $f*g$ is defined (as usual) by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nc_nz^n=(g*f)(z).$$

Theorem 3. *A function $f(z)\in\mathcal{P}_{sc}^{(k)}(\lambda,\alpha)$ if and only if*

$$\frac{1}{z}\left\{f*\left\{\frac{z}{(1-z)^2}\left\{(1-e^{i\theta})-\lambda[1+(1-2\alpha)e^{i\theta}]\right\}-\frac{(1-\lambda)[1+(1-2\alpha)e^{i\theta}]}{2}h\right\}(z)\right.$$

$$-\overline{f * \left\{\frac{(1-\lambda)[1+(1-2\alpha)e^{i\theta}]}{2}h\right\}(\overline{z})}\Bigg\} \neq 0 \tag{3.1}$$

for all $z \in \mathbb{U}$ and $0 \leqq \theta < 2\pi$, where $h(z)$ is given by (3.6).

Proof. Suppose that $f(z) \in \mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$, since the condition (1.4) is equivalent to

$$\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)} + (1-\lambda)} \neq \frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}} \tag{3.2}$$

for all $z \in \mathbb{U}$ and $0 \leqq \theta < 2\pi$. And the condition (3.2) can be written as follows:

$$\frac{1}{z}\left\{zf'(z)(1-e^{i\theta}) - [\lambda zf'(z) + (1-\lambda)f_{2k}(z)][1+(1-2\alpha)e^{i\theta}]\right\} \neq 0. \tag{3.3}$$

On the other hand, it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{3.4}$$

And from the definition of $f_{2k}(z)$, we know

$$f_{2k}(z) = z + \sum_{n=2}^{\infty} \frac{a_n + \overline{a_n}}{2} c_n z^n = \frac{1}{2}\left((f*h)(z) + \overline{(f*h)(\overline{z})}\right), \tag{3.5}$$

where

$$h(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{z}{1-\varepsilon^{\upsilon} z}. \tag{3.6}$$

Substituting (3.4) and (3.5) into (3.3), we can easily get (3.1) This completes the proof of Theorem 3.

Similarly, for the class $\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$, we have

Corollary 3. *A function $f(z) \in \mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$ if and only if*

$$\frac{1}{z}\Bigg\{f * \left\{z\left\{\frac{z}{(1-z)^2}\left\{(1-e^{i\theta}) - \lambda[1+(1-2\alpha)e^{i\theta}]\right\} - \frac{(1-\lambda)[1+(1-2\alpha)e^{i\theta}]}{2}h\right\}'\right\}(z)$$

$$-\overline{f * \left\{z\left\{\frac{(1-\lambda)[1+(1-2\alpha)e^{i\theta}]}{2}h\right\}'\right\}(\overline{z})}\Bigg\} \neq 0 \tag{3.7}$$

for all $z \in \mathbb{U}$ and $0 \leqq \theta < 2\pi$, where $h(z)$ is given by (3.6).

4. Coefficient Inequalities

In this section, we provide the sufficient conditions for functions belonging to the classes $\mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$ and $\mathcal{Q}_{sc}^{(k)}(\lambda, \alpha)$.

Theorem 4. *Let $0 \leqq \alpha < 1$ and $0 \leqq \lambda < 1$. If*

$$\sum_{n=1}^{\infty}[(1-\lambda\alpha)(nk+1)\left|a_{nk+1}\right| - \alpha(1-\lambda)\left|\Re(a_{nk+1})\right|] + \sum_{\substack{n=2 \\ n\neq lk+1}}^{\infty}(1-\lambda\alpha)n\left|a_n\right| \leqq 1-\alpha. \tag{4.1}$$

Then $f(z) \in \mathcal{P}_{sc}^{(k)}(\lambda, \alpha)$.

Proof. It suffices to show that

$$\left|\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)}+(1-\lambda)}-1\right|<1-\alpha.$$

Note that for $|z|=r<1$, we have

$$\begin{aligned}\left|\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)}+(1-\lambda)}-1\right| &= \left|\frac{\sum_{n=2}^{\infty}(1-\lambda)(na_n-\Re(a_n)b_n)z^{n-1}}{1+\sum_{n=2}^{\infty}[\lambda na_n+(1-\lambda)\Re(a_n)b_n]z^{n-1}}\right| \\ &\leqq \frac{\sum_{n=2}^{\infty}(1-\lambda)(na_n-\Re(a_n)b_n)\,|z|^{n-1}}{1-\sum_{n=2}^{\infty}[\lambda na_n+(1-\lambda)\Re(a_n)b_n]\,|z|^{n-1}} \\ &\leqq \frac{\sum_{n=2}^{\infty}(1-\lambda)(na_n-\Re(a_n)b_n)}{1-\sum_{n=2}^{\infty}[\lambda na_n+(1-\lambda)\Re(a_n)b_n]},\end{aligned}$$

where

$$b_n=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{(n-1)\nu}=\begin{cases}1, & n=lk+1,\\ 0, & n\neq lk+1.\end{cases} \tag{4.2}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}[(1-\lambda\alpha)n\,|a_n|-\alpha(1-\lambda)\,|\Re(a_n)|\,b_n]\leqq 1-\alpha. \tag{4.3}$$

Since inequality (4.3) can be written as inequality (4.1), hence $f(z)$ satisfies the condition (1.4). This completes the proof of Theorem 4.

Similarly, for the class $\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$, we have

Corollary 4. *Let* $0\leqq\alpha<1$ *and* $0\leqq\lambda<1$. *If*

$$\sum_{n=1}^{\infty}(nk+1)[(1-\lambda\alpha)(nk+1)\,|a_{nk+1}|-\alpha(1-\lambda)\,|\Re(a_{nk+1})|]+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}(1-\lambda\alpha)n^2\,|a_n|\leqq 1-\alpha.$$

Then $f(z)\in\mathcal{Q}_{sc}^{(k)}(\lambda,\alpha)$.

We now provide the necessary and sufficient coefficient conditions for functions belonging to the classes $\mathcal{TP}_{sc}^{(k)}(\lambda,\alpha)$, $\mathcal{TQ}_{sc}^{(k)}(\lambda,\alpha)$.

Theorem 5. *Let* $0\leqq\alpha<1,\ 0\leqq\lambda<1$ *and* $f(z)\in\mathcal{T}$. *Then* $f(z)\in\mathcal{TP}^{(k)}(\lambda,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}[(1-\lambda\alpha)(nk+1)-\alpha(1-\lambda)]a_{nk+1}+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}(1-\lambda\alpha)na_n\leqq 1-\alpha. \tag{4.4}$$

Proof. In view of Theorem 2, we need only to prove the necessity. Suppose that $f(z)\in\mathcal{TP}_{sc}^{(k)}(\lambda,\alpha)$, then from (1.4), we can get

$$\Re\left(\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f_{2k}(z)}\right)>\alpha,$$

that is,

$$\Re\left(\frac{1-\sum_{n=2}^{\infty}na_nz^{n-1}}{1-\sum_{n=2}^{\infty}[\lambda n+(1-\lambda)b_n]a_nz^{n-1}}\right)>\alpha, \tag{4.5}$$

where b_n is given by (4.2). By letting $z \to 1^-$ through real values in (4.5), we can get

$$\frac{1-\sum_{n=2}^{\infty} na_n}{1-\sum_{n=2}^{\infty}[\lambda n+(1-\lambda)b_n]a_n} \geqq \alpha,$$

or equivalently,

$$\sum_{n=2}^{\infty}[(1-\lambda\alpha)n-\alpha(1-\lambda)b_n]a_n \leqq 1-\alpha. \tag{4.6}$$

Substituting (4.2) into inequality (4.6), we can get inequality (4.4) easily. This completes the proof of Theorem 5.

Similarly, for the class $\mathcal{TQ}_{sc}^{(k)}(\lambda,\alpha)$, we have

Corollary 5. *Let* $0 \leqq \alpha < 1,\ 0 \leqq \lambda < 1$ *and* $f(z) \in \mathcal{T}$. *Then* $f(z) \in \mathcal{TQ}_{sc}^{(k)}(\lambda,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(nk+1)[(1-\lambda\alpha)(nk+1)-\alpha(1-\lambda)]a_{nk+1}+\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}(1-\lambda\alpha)n^2a_n \leqq 1-\alpha.$$

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