MONOTONICITIES AND SCHUR-CONVEXITY FOR A TYPE OF SYMMETRIC MEAN VALUES

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Abstract. In the present paper, we discuss the basic properties of a type of symmetric mean values. Such results as monotonicities, Schur-convexity for some special cases of this type of mean values are established.

1. Introduction

Throughout this paper, we assume that the set of $(n+1)$-dimensional row vector on real number field by $\mathbb{R}^{n+1}$, and

$$\mathbb{R}_{+}^{n+1} := \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geqq 0 \text{ for } 0 \leqq k \leqq n\},$$
$$\mathbb{R}_{++}^{n+1} := \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i > 0 \text{ for } 0 \leqq k \leqq n\}.$$

The weighted arithmetic mean $A(\boldsymbol{x}, \boldsymbol{t})$ and the power mean $M_r(\boldsymbol{x}, \boldsymbol{u})$ of order r with respect to the positive numbers $x_0, x_1, \ldots, x_n$ and the positive weights $t_0, t_1, \ldots, t_n$ for

$$\sum_{i=0}^{n} t_i = 1$$

are defined, respectively, as follows:

$$A(\boldsymbol{x}, \boldsymbol{t}) = \sum_{i=0}^{n} t_i x_i,$$

and

$$M_\alpha(\boldsymbol{x}, \boldsymbol{t}) = \left(\sum_{i=0}^{n} t_i x_i^\alpha\right)^{\frac{1}{\alpha}}$$

for $\alpha \neq 0$, and

$$M_0(\boldsymbol{x}, \boldsymbol{t}) = \prod_{i=0}^{n} x_i^{t_i}.$$

Let $\boldsymbol{x} \in \mathbb{R}_{+}^{n+1}$ and $r \in \mathbb{N} := \{1, 2, \ldots\}$. Then the well-known r-th complete symmetric function is

$$E_n^{[r]} = E_n^{[r]}(\boldsymbol{x}) := \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ (i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}} \prod_{k=0}^{n} x_k^{i_k} \qquad (\mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{1.1}$$

with $E_n^{[0]} = E_n^{[0]}(\boldsymbol{x}) = 1$ for $n \geqq 1$. Correspondingly, the r-th generalized symmetric mean is

$$\sum_n^{[r]} = \sum_n^{[r]}(\boldsymbol{x}) := \frac{E_n^{[r]}(\boldsymbol{x})}{\binom{n+r}{r}},$$

where

$$\binom{n+r}{r} = \frac{(n+r)!}{r!n!}.$$

In 1934, I. Schur [3, p. 164] obtained the following identity:

$$\sum_n^{[r]}(\boldsymbol{x}) = n! \int \cdots \int (A(\boldsymbol{x}, \boldsymbol{t}))^r \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \mathrm{d}t_n$$

2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 26D20.
Key words and phrases. Monotonicity, Schur-convexity, convex function, symmetric function.

with

$$t_0 = 1 - \sum_{i=1}^{n} t_i,$$

for the integral is taken over

$$t_k \geqq 0 \quad \text{for} \quad 0 \leqq k \leqq n,$$

and proved that

$$\sum_n^{[r-1]}(\boldsymbol{x}) \cdot \sum_n^{[r+1]}(\boldsymbol{x}) \geqq \left[\sum_n^{[r]}(\boldsymbol{x})\right]^2,$$

and

$$\left[\sum_n^{[r]}(\boldsymbol{x})\right]^{1/r} \leqq \left[\sum_n^{[r]}(\boldsymbol{x})\right]^{1/(r+1)}.$$

These are to say that $\sum_n^{[r]}(\boldsymbol{x})/\sum_n^{[r-1]}(\boldsymbol{x})$ and $\left[\sum_n^{[r]}(\boldsymbol{x})\right]^{1/r}$ both are the monotone increasing functions with respect to $r \in \mathbb{N}$.

When $1 \leqq r \leqq n+1$, the fact that $E_n^{[r]}(\boldsymbol{x})$ or $\sum_n^{[r]}(\boldsymbol{x})$ is Schur-convex in $\mathbb{R}_+^{n+1}$ has generalized by Baston (see [9, p. 82]). Guan [2] proved that the functions $E_n^{[r]}(\boldsymbol{x})$ or $\sum_n^{[r]}(\boldsymbol{x})$ and $E_n^{[r]}(\boldsymbol{x})/E_n^{[r-1]}(\boldsymbol{x})$ both are Schur-convex in $\mathbb{R}_+^{n+1}$ for all $r \in \mathbb{N}$.

In 2007, Zheng *et al.* [20] found the Schur-convexity of the following increasing Pittenger means $L_\alpha(\boldsymbol{x})$ [12] and Pearce means $F_\alpha(\boldsymbol{x})$ [11] with respect to $\alpha \in \mathbb{R}$:

$$L_\alpha(\boldsymbol{x}) = \begin{cases} \left(n! \int \cdots \int (A(\boldsymbol{x}, \boldsymbol{t}))^\alpha \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \mathrm{d}t_n\right)^{\frac{1}{\alpha}} & (\alpha \neq 0), \\ \exp\left(n! \int \cdots \int \ln A(\boldsymbol{x}, \boldsymbol{t}) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \mathrm{d}t_n\right) & (\alpha = 0), \end{cases} \tag{1.2}$$

and

$$F_\alpha(\boldsymbol{x}) = n! \int \cdots \int M_\alpha(\boldsymbol{x}, \boldsymbol{t}) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \mathrm{d}t_n, \tag{1.3}$$

where

$$t_0 = 1 - \sum_{i=1}^{n} t_i,$$

for the integral is taken over

$$t_k \geqq 0 \quad \text{for} \quad 0 \leqq k \leqq n.$$

For the similarities of (1.1), (1.2) and (1.3), we will consider two classes of sum-type symmetric mean values as follows:

$$F_n^{[r]}(\boldsymbol{x}; \alpha, \beta) = \left[\frac{1}{\binom{n+r}{n}} \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ (i_0,i_1,\ldots,i_n \in \mathbb{N}_0)}} \left(\frac{1}{r}\sum_{k=0}^{n} i_k x_k^\alpha\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}, \tag{1.4}$$

$$F_n^{[r]}(\boldsymbol{x}; 0, \beta) = \left(\frac{1}{\binom{n+r}{n}} \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ (i_0,i_1,\ldots,i_n \in \mathbb{N}_0)}} \prod_{k=0}^{n} x_k^{\frac{i_k \beta}{r}}\right)^{\frac{1}{\beta}}, \tag{1.5}$$

$$F_n^{[r]}(\boldsymbol{x}; \alpha, 0) = \prod_{\substack{i_0+i_1+\cdots+i_n=r\\ (i_0,i_1,\ldots,i_n \in \mathbb{N}_0)}} \left(\frac{1}{r}\sum_{k=0}^{n} i_k x_k^\alpha\right)^{\frac{1}{\binom{n+r}{n}\alpha}}, \tag{1.6}$$

$$F_n^{[r]}(\boldsymbol{x}; 0, 0) = \prod_{i=0}^{n} x_i^{\frac{1}{n+1}}; \tag{1.7}$$

and

$$T_n^{[r]}(\boldsymbol{x};\alpha,\beta)=\left[\frac{1}{\binom{n+r-1}{n}}\sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N})}}\left(\frac{1}{n+r}\sum_{k=0}^{n}i_kx_k^{\alpha}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}, \tag{1.8}$$

$$T_n^{[r]}(\boldsymbol{x};0,\beta)=\left(\frac{1}{\binom{n+r-1}{n}}\sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N})}}\prod_{k=0}^{n}x_k^{\frac{i_k\beta}{n+r}}\right)^{\frac{1}{\beta}}, \tag{1.9}$$

$$T_n^{[r]}(\boldsymbol{x};\alpha,0)=\prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\(i_0,i_1,\ldots,i_n\in\mathbb{N})}}\left(\frac{1}{r}\sum_{k=0}^{n}i_kx_k^{\alpha}\right)^{\frac{1}{\binom{n+r-1}{n}\alpha}}, \tag{1.10}$$

$$T_n^{[r]}(\boldsymbol{x};0,0)=F_n^{[r]}(\boldsymbol{x};0,0)=\prod_{i=0}^{n}x_i^{\frac{1}{n+1}}. \tag{1.11}$$

The main object of this paper is to discuss the basic properties of two symmetric mean values above. Such results as monotonicities and Schur-convexity are proved.

2. Preliminary Results

The convex functions are well-known and the foundations of its theory are due to J. L. Jensen [5]. In order to prove our main results, we shall require the following lemmas.

Lemma 1. *If $f(x)$ is a convex function defined on an interval $\mathbb{I}\subseteq\mathbb{R}$, then*

$$f\left(\frac{\sum_{i=0}^{n}u_ix_i}{\sum_{i=0}^{n}u_i}\right)\leqq\frac{\sum_{i=0}^{n}u_if(x_i)}{\sum_{i=0}^{n}u_i},$$

where $x_i\in\mathbb{I}$ and $u_i\in\mathbb{R}_+$ with $0\leqq i\leqq n$.

Lemma 2. *The function $f(x)=x^{\alpha}$ is convex if $\alpha(\alpha-1)>0$, and concave if $\alpha(\alpha-1)<0$ defined on an interval $\mathbb{I}\subseteq\mathbb{R}_+$.*

The Schur-convex functions were introduced by Schur [9] in 1923, and it has many important applications in analytic inequalities. Hardy *et al.* [4] were also interested in some inequalities that are related to Schur-convex functions, the following definitions can be found in many references such as [7, 9, 10, 13, 15, 17].

Lemma 3. *A function $\varphi(\boldsymbol{x})$ is increasing if and only if $\nabla\varphi(\boldsymbol{x})\geqq 0$ for $\boldsymbol{x}\in\Omega$, where $\Omega\subset\mathbb{R}^{n+1}$ is an open set, $\varphi:\Omega\to R$ is differentiable, and*

$$\nabla\varphi(\boldsymbol{x})=\left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_0},\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1},\ldots,\frac{\partial\varphi(\boldsymbol{x})}{\partial x_n}\right)\in\mathbb{R}^{n+1}.$$

Every Schur-convex function is a symmetric function [18]. But it is not hard to see that not every symmetric function can be a Schur-convex function [13, p. 258]. However, we have the following so-called Schur's condition.

Lemma 4. *Let $\Omega\subset\mathbb{R}^{n+1}$ be symmetric to have a nonempty interior convex set, Ω^0 the interior of Ω and $\varphi:\Omega\to\mathbb{R}$ continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex$ $(Schur-concave)$ $function$, if and only if φ is symmetric on Ω and*

$$(x_1-x_2)\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{\partial x_2}\right)\geqq(\leqq)0$$

holds for any $\boldsymbol{x}\in\Omega^0$.

3. Monotonicities

By using the well-known mean inequality, it is easy to obtain the following theorem.

Theorem 1. *Given $r \in \mathbb{N}$, $F_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ and $T_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ both are the monotone increasing functions with respect to α and β.*

Theorem 2. *Let $r \in \mathbb{N}$. Then $F_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ is a monotone decreasing function and $T_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ is a monotone increasing function with respect to r, i.e. the following two inequalities*

$$F_n^{[r]}(\boldsymbol{x};\alpha,\beta) \geqq F_n^{[r+1]}(\boldsymbol{x};\alpha,\beta), \tag{3.1}$$

$$T_n^{[r]}(\boldsymbol{x};\alpha,\beta) \leqq T_n^{[r+1]}(\boldsymbol{x};\alpha,\beta) \tag{3.2}$$

hold if $\alpha < \beta$, and inequalities (3.1) and (3.2) inverse if $\alpha > \beta$. The equalities in (3.1) and (3.2) are valid if and only if $x_0 = x_1 = \cdots = x_n$.

Proof. For $\alpha = 0$ or $\beta = 0$, the proofs of inequalities (3.1) and (3.2) are obtained in [8, 16, 19].

We will only prove that $T_n^{[r]} := T_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ is monotone increasing function with $r \in \mathbb{N}$. The proof of the results of $F_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ is similar.

Indeed, from the well-known weighted Jensen's inequality (Lemma 1) and Lemma 2, if

$$\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}-1\right) > 0,$$

we then have

$$\begin{aligned}
&\binom{n+r-1}{n}\left(T_n^{[r+1]}\right)^{\beta} \\
&= \frac{r}{n+r}\sum_{\substack{i_0+i_1+\cdots+i_n=n+r+1,\\(i_0,i_1,\ldots,i_n\in\mathbb{N})}}\left(\frac{1}{n+r+1}\sum_{k=0}^{n} i_k x_k^{\alpha}\right)^{\frac{\beta}{\alpha}} \\
&= \sum_{\substack{i_0+i_1+\cdots+i_n=n+r+1,\\(i_0,i_1,\ldots,i_n\in\mathbb{N})}}\frac{\sum\limits_{k=0}^{n}(i_k-1)}{n+r}\cdot\left(\frac{\sum_{k=0}^{n} i_k x_k^{\alpha}}{n+r+1}\right)^{\frac{\beta}{\alpha}} \\
&= \sum_{\substack{j_0+j_1+\cdots+j_n=n+r,\\(j_0,j_1,\ldots,j_n\in\mathbb{N})}}\sum_{k=0}^{n}\left[\frac{j_k}{n+r}\cdot\left(\frac{\sum_{v=0}^{n} j_v x_v^{\alpha}+x_k^{\alpha}}{n+r+1}\right)^{\frac{\beta}{\alpha}}\right] \\
&\geqq \sum_{\substack{j_0+j_1+\cdots+j_n=n+r,\\(j_0,j_1,\ldots,j_n\in\mathbb{N})}}\left(\sum_{k=0}^{n}\frac{j_k}{n+r}\cdot\frac{\sum_{v=0}^{n} j_v x_v^{\alpha}+x_k^{\alpha}}{n+r+1}\right)^{\frac{\beta}{\alpha}} \\
&= \sum_{\substack{j_0+j_1+\cdots+j_n=n+r,\\(j_0,j_1,\ldots,j_n\in\mathbb{N})}}\left(\frac{\sum_{k=0}^{n} j_k x_k^{\alpha}}{n+r}\right)^{\frac{\beta}{\alpha}} \\
&= \binom{n+r-1}{n}\left(T_n^{[r]}\right)^{\beta},
\end{aligned}$$

or

$$\left[T_n^{[r]}(\boldsymbol{x};\alpha,\beta)\right]^{\beta} \leqq \left[T_n^{[r+1]}(\boldsymbol{x};\alpha,\beta)\right]^{\beta}, \tag{3.3}$$

and inequalities (3.3) inverses if

$$\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}-1\right) < 0.$$

The equalities above are valid if and only if

$$\sum_{k=0}^{n} i_k x_k^\alpha + x_0^\alpha = \sum_{k=0}^{n} i_k x_k^\alpha + x_1^\alpha = \cdots = \sum_{k=0}^{n} i_k x_k^\alpha + x_n^\alpha$$

which is equivalent to $x_0 = x_1 = \cdots = x_n$.

If $(\alpha/\beta)(\alpha/\beta - 1) > 0$, that is $\alpha/\beta < 0$ or $\alpha/\beta > 1$, then $\beta > \alpha$ and $\beta > 0$, we immediately find inequality (3.2) from (3.3). If $(\alpha/\beta)(\alpha/\beta - 1) < 0$, then $\alpha < \beta < 0$, and we also obtain inequality (3.2) from inverses (3.3). It is to say that inequality (3.2) holds if $\alpha > \beta$. Similarly, we have that inequality (3.2) inverses if $\alpha < \beta$.

The proof of Theorem 2 is completed. □

Theorem 3. *If $r \in \mathbb{N}$, then $F_n^{[r]}(\boldsymbol{x}; \alpha, \beta)$ and $T_n^{[r]}(\boldsymbol{x}; \alpha, \beta)$ both are the monotone increasing functions with $\boldsymbol{x} \in \mathbb{I}^{n+1}$.*

Proof. The required results are obtained from Lemma 3, and

$$\frac{\partial F_n^{[r]}(\boldsymbol{x}; \alpha, \beta)}{\partial x_i} = \frac{1}{\binom{n+r}{n}} \left[F_n^{[r]}(\boldsymbol{x}; \alpha, \beta)\right]^{1-\beta} \cdot \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ (i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}} \frac{i_i x_k^{\alpha-1}}{r} \left(\frac{1}{r}\sum_{k=0}^{n} i_k x_k^\alpha\right)^{\frac{\beta}{\alpha}-1} \geqq 0,$$

and other similar ones. □

4. Schur-Convexity

For any $\alpha, \beta \in \mathbb{R}$, it is difficult to know the Schur-convexity of $F_n^{[r]}(\boldsymbol{x}; \alpha, \beta)$ and $T_n^{[r]}(\boldsymbol{x}; \alpha, \beta)$. But in the some special cases of α and β, we can establish their Schur-convexity. By similarly applying the method of proof of Theorems 3.1 and 3.2 obtained by N.-G. Zheng *et al.* [20], we can get the following results. Here we only give the proof of Theorem 4.

Theorem 4. *Given $r \in \mathbb{N}$, $F_n^{[r]}(\boldsymbol{x}; \alpha, 1)$ and $T_n^{[r]}(\boldsymbol{x}; \alpha, 1)$ both are the Schur-convex functions if $\alpha \geqq 1$, and the Schur-concave functions $\alpha \leqq 1$.*

Proof. We only give the proof of the result of $F_n^{[r]}(\boldsymbol{x}; \alpha, \beta)$, and other is similar.

Denote

$$M_\alpha(\boldsymbol{x}; i_0, i_1) = \left(\frac{1}{r}\sum_{k=0}^{n} i_k x_k^\alpha\right)^{\frac{1}{\alpha}}.$$

For $\alpha \neq 0$, we thus have

$$\frac{\partial M_\alpha(\boldsymbol{x}; i_0, i_1)}{\partial x_0} = \frac{i_0}{r} x_0^{\alpha-1} \left(M_\alpha(\boldsymbol{x}; i_0, i_1)\right)^{1-\alpha} = \frac{i_0}{r}\left[\frac{M_\alpha(\boldsymbol{x}; i_0, i_1)}{x_0}\right]^{1-\alpha}, \tag{4.1}$$

$$\frac{\partial M_\alpha(\boldsymbol{x}; i_1, i_0)}{\partial x_1} = \frac{i_0}{r} x_1^{\alpha-1} \left(M_\alpha(\boldsymbol{x}; i_1, i_0)\right)^{1-\alpha} = \frac{i_0}{r}\left[\frac{M_\alpha(\boldsymbol{x}; i_1, i_0)}{x_1}\right]^{1-\alpha}. \tag{4.2}$$

Combination of (4.1) to (4.2), yields

$$\frac{\partial M_\alpha(\boldsymbol{x}; i_0, i_1)}{\partial x_0} - \frac{\partial M_\alpha(\boldsymbol{x}; i_1, i_0)}{\partial x_1} = \frac{i_0}{r}\left\{\left[\frac{M_\alpha(\boldsymbol{x}; i_0, i_1)}{x_0}\right]^{1-\alpha} - \left[\frac{M_\alpha(\boldsymbol{x}; i_1, i_0)}{x_1}\right]^{1-\alpha}\right\}. \tag{4.3}$$

By using the mean value theorem, we find

$$\begin{aligned}
&\left[\frac{M_\alpha(\boldsymbol{x};i_0,i_1)}{x_0}\right]^{1-\alpha}-\left[\frac{M_\alpha(\boldsymbol{x};i_1,i_0)}{x_1}\right]^{1-\alpha}\\
&=\left(\frac{1}{rx_0^\alpha}\sum_{k=0}^n i_kx_k^\alpha\right)^{\frac{1-\alpha}{\alpha}}-\left[\frac{1}{rx_1^\alpha}\left(i_0x_1^\alpha+i_1x_0^\alpha+\sum_{k=2}^n i_kx_k^\alpha\right)\right]^{\frac{1-\alpha}{\alpha}}\\
&=\frac{1-\alpha}{\alpha}\left(\frac{i_1x_1^\alpha+\sum_{k=2}^n i_kx_i^\alpha}{rx_0^\alpha}-\frac{i_1x_0^\alpha+\sum_{k=2}^n i_kx_k^\alpha}{rx_1^\alpha}\right)(i_0+\theta_1)^{\frac{1-2\alpha}{\alpha}}\\
&=\frac{1-\alpha}{\alpha}\cdot\frac{i_1x_1^{2\alpha}+x_1^\alpha\sum_{k=2}^n i_kx_k^\alpha-i_1x_0^{2\alpha}-x_0^\alpha\sum_{k=2}^n i_kx_k^\alpha}{rx_0^\alpha x_1^\alpha}\cdot(i_0+\theta_1)^{\frac{1-2\alpha}{\alpha}}\\
&=(1-\alpha)(x_1-x_0)(i_0+\theta_1)^{\frac{1-2\alpha}{\alpha}}\mathfrak{M}(\boldsymbol{x},\theta_2),
\end{aligned}\tag{4.4}$$

where θ_1 is between to

$$\frac{i_1x_1^\alpha+\sum_{k=2}^n i_kx_k^\alpha}{x_0^\alpha}\quad\text{and}\quad\frac{i_1x_0^\alpha+\sum_{k=2}^n i_kx_k^\alpha}{x_1^\alpha},$$

θ_2 is between to x_0 and x_1, and

$$\mathfrak{M}(\boldsymbol{x},\theta_2)=\frac{2i_1\theta_2^{2\alpha-1}+\theta_2^{\alpha-1}\sum_{k=2}^n i_kx_k^\alpha}{x_0^\alpha x_1^\alpha}\geqq 0.$$

Owing to the symmetry of $F_n^{[r]}(\boldsymbol{x};\alpha,1)$ with respect to $x_1,x_2,\ldots,x_n$, we have

$$\begin{aligned}
F_n^{[r]}(\boldsymbol{x};\alpha,1)&=\frac{1}{\binom{n+r}{n}}\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}}M_\alpha(\boldsymbol{x};i_0,i_1)\\
&=\frac{1}{\binom{n+r}{n}}\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}}M_\alpha(\boldsymbol{x};i_1,i_0).
\end{aligned}\tag{4.5}$$

From (4.4) and (4.5), we get

$$\begin{aligned}
&(x_0-x_1)\left[\frac{\partial F_n^{[r]}(\boldsymbol{x};\alpha,1)}{\partial x_0}-\frac{\partial F_n^{[r]}(\boldsymbol{x};\alpha,1)}{\partial x_1}\right]\\
&=\frac{1}{\binom{n+r}{n}}\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}}(x_0-x_1)\left[\frac{\partial M_0(\boldsymbol{x};i_0,i_1)}{\partial x_0}-\frac{\partial M_0(\boldsymbol{x};i_1,i_0)}{\partial x_1}\right]\\
&=\frac{1}{\binom{n+r}{n}}\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}}\frac{i_0}{r}(\alpha-1)(x_0-x_1)^2(i_0+\theta_1)^{\frac{1-2\alpha}{\alpha}}\mathfrak{M}(\boldsymbol{x},\theta_2).
\end{aligned}$$

It follows that $F_n^{[r]}(\boldsymbol{x};\alpha,1)$ is Schur-convex for $\alpha\geqq 1$ and Schur-concave for $\alpha\leqq 1$ and $\alpha\neq 0$ by Lemma 4.

For $\alpha=0$, from

$$\frac{i_0-i_1}{r}-1\leqq 0,$$

and

$$(x_0-x_1)\left(x_0^{\frac{i_0-i_1}{r}-1}-x_1^{\frac{i_0-i_1}{r}-1}\right)\leqq 0,$$

we similarly obtain

$$\begin{aligned}
&(x_0-x_1)\left[\frac{\partial F_n^{[r]}(\boldsymbol{x};0,1)}{\partial x_0}-\frac{\partial F_n^{[r]}(\boldsymbol{x};0,1)}{\partial x_1}\right]\\
&=\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\(i_0,i_1,\ldots,i_n\in\mathbb{N}_0)}}\frac{i_0(x_0-x_1)}{r\binom{n+r}{n}}\left(x_0^{\frac{i_0-i_1}{r}-1}-x_1^{\frac{i_0-i_1}{r}-1}\right)x_0^{\frac{i_1}{r}}x_1^{\frac{i_1}{r}}\prod_{k=2}^n x_k^{\frac{i_k}{r}}\leqq 0.
\end{aligned}$$

It is clear to see that $F_n^{[r]}(\boldsymbol{x};0,1)$ is a Schur-concave function. □

Theorem 5. *Given $r \in \mathbb{N}$, $F_n^{[r]}(\boldsymbol{x};1,\beta)$ and $T_n^{[r]}(\boldsymbol{x};1,\beta)$ both are the Schur-convex functions if $\beta \geqq 1$, and the Schur-concave functions if $\beta \leqq 1$.*

5. Some Remarks

Remark 1. If $r \in \mathbb{N}$, then $\lim_{r\to\infty} F_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ and $\lim_{r\to\infty} T_n^{[r]}(\boldsymbol{x};\alpha,\beta)$ exist, and

$$\begin{aligned}\lim_{r\to\infty} F_n^{[r]}(\boldsymbol{x};\alpha,\beta) &= \lim_{r\to\infty} T_n^{[r]}(\boldsymbol{x};\alpha,\beta)\\ &= F_n(\boldsymbol{x};\alpha,\beta) = \left(n!\int\cdots\int\left(\sum_{i=0}^{n} x_i^{\alpha}t_i\right)^{\frac{\beta}{\alpha}} dt_1dt_2\cdots dt_n\right)^{\frac{1}{\beta}},\end{aligned}$$

where $t_0 = 1 - \sum_{i=1}^{n} t_i$ and the integral is taken over $t_k \geqq 0$ for $0 \leqq k \leqq n$.

Remark 2. From the monotonicities of the generalized symmetric mean values above, we can easily find some inequalities. For example:
Let $r \in \mathbb{N}$ and $\alpha < \beta$. Then

$$\begin{aligned}\left(\frac{1}{n+1}\sum_{i=0}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha}} &= T_n^{[1]}(\boldsymbol{x};\alpha,\beta) \leqq T_n^{[r]}(\boldsymbol{x};\alpha,\beta) \leqq F_n(\boldsymbol{x};\alpha,\beta)\\ &\leqq F_n^{[r]}(\boldsymbol{x};\alpha,\beta) \leqq F_n^{[1]}(\boldsymbol{x};\alpha,\beta) = \left(\frac{1}{n+1}\sum_{i=0}^{n} x_i^{\beta}\right)^{\frac{1}{\beta}}.\end{aligned} \tag{5.1}$$

Obviously, inequality (5.1) is a refinement of power mean inequality.

Acknowledgements. The present investigation was supported by the *Scientific Research Fund of the Educational Department* under Grant 07C718 and the *"Eleventh-Five-Year" Special Foundation of Construction for Key Subject* of Hunan Province, People's Republic of China.

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