SOME GENERALIZATIONS OF THE CAUCHY-SCHWARZ AND THE CAUCHY-BUNYAKOVSKY INEQUALITIES INVOLVING FOUR FREE PARAMETERS AND THEIR APPLICATIONS

MOHAMMAD MASJED-JAMEI, SEVER S. DRAGOMIR, AND H. M. SRIVASTAVA[1]

Abstract. Some generalizations of the well-known Cauchy-Schwarz inequality and the analogous Cauchy-Bunyakovsky inequality involving four free parameters are given for both discrete and continuous cases. Several particular cases of interest are also analyzed. Some of the applications of our main results include (for example) the Wagner inequality.

1. Introduction

Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be two sequences of real numbers. It is well known that the *discrete* version of the Cauchy-Schwarz inequality [3] can be stated as follows:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leqq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right). \tag{1.1}$$

The equality holds true in (1.1) if and only if the sequences are proportional, that is, if and only if

$$a_k = r b_k \qquad (k \in \{1, \ldots, n\}), \tag{1.2}$$

where $r \in \mathbb{R}$ is a constant of proportionality.

To date, a large number of generalizations and refinements of the Cauchy-Schwarz inequality (1.1) have been investigated in the literature (see, for example, the survey paper [4], the book [9] and the numerous references cited therein; see also several closely-related works such as [7], [8] and [10]).

In this paper, we present further generalizations of the Cauchy-Schwarz inequality (1.1) and the analogous Cauchy-Bunyakovsky inequality (4.1) below in terms of four free parameters and study its several particular cases of interest. One of the applications of our main results includes (for example) the Wagner inequality.

2. A Generalization of the Cauchy-Schwarz Inequality

Our first result is contained in Theorem 1 below.

Theorem 1. *If $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are two sequences of real numbers and*

$$p, q, r, s \in \mathbb{R},$$

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[1]Corresponding Author.

then

$$\left[\sum_{k=1}^{n} a_k b_k + A_1\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + B_1\left(\sum_{k=1}^{n} a_k\right)^2 + C_1\left(\sum_{k=1}^{n} b_k\right)^2\right]^2 \tag{2.1}$$
$$\leqq \left[\sum_{k=1}^{n} a_k^2 + A_2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + B_2\left(\sum_{k=1}^{n} a_k\right)^2 + C_2\left(\sum_{k=1}^{n} b_k\right)^2\right]$$
$$\cdot \left[\sum_{k=1}^{n} b_k^2 + A_3\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + B_3\left(\sum_{k=1}^{n} a_k\right)^2 + C_3\left(\sum_{k=1}^{n} b_k\right)^2\right],$$

in which the coefficients involved are defined by the following matrix equation:

$$M = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} p+s+ps+qr & r(1+p) & q(1+s) \\ 2q(1+p) & p(p+2) & q^2 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}.$$

Moreover, the inequality (2.1) *is equivalent to*

$$\left[\sum_{k=1}^{n} a_k b_k + \frac{p+s+ps+qr}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right. \tag{2.2}$$
$$\left. + \frac{r(1+p)}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{q(1+s)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right]^2$$
$$\leqq \left(\sum_{k=1}^{n} a_k^2 + \frac{1}{n}\sum_{k=1}^{n}(pa_k + qb_k)\sum_{k=1}^{n}[(p+2)a_k + qb_k]\right)$$
$$\cdot \left(\sum_{k=1}^{n} b_k^2 + \frac{1}{n}\sum_{k=1}^{n}(ra_k + sb_k)\sum_{k=1}^{n}[ra_k + (s+2)b_k]\right),$$

and is a generalization of the Cauchy-Schwarz inequality (1.1), *which corresponds to the special case of* (2.1) *when*

$$A_j = B_j = C_j = 0 \qquad (j = 1, 2, 3).$$

The equality holds true in (2.1) *if*

$$a_k = b_k \qquad (k \in \{1, \ldots, n\}) \quad \text{and} \quad A_j = B_j = C_j \qquad (j = 1, 2, 3).$$

Proof. Let us define the positive quadratic polynomial

$$Q : \mathbb{R} \to \mathbb{R}$$

as follows:

$$Q(x; p, q, r, s) = \sum_{k=1}^{n}\left[\left(a_k + \frac{p}{n}\sum_{k=1}^{n} a_k + \frac{q}{n}\sum_{k=1}^{n} b_k\right)x\right. \tag{2.3}$$
$$\left. + \left(b_k + \frac{r}{n}\sum_{k=1}^{n} a_k + \frac{s}{n}\sum_{k=1}^{n} b_k\right)\right]^2,$$

in which

$$p, q, r, s \in \mathbb{R},$$

and $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are real numbers. Since a simple calculation reveals that

$$\begin{aligned}(2.4)\quad Q(x;p,q,r,s) &= \sum_{k=1}^n \left(a_k + \frac{p}{n}\sum_{k=1}^n a_k + \frac{q}{n}\sum_{k=1}^n b_k\right)^2 x^2 \\ &+ 2\sum_{k=1}^n \left(a_k + \frac{p}{n}\sum_{k=1}^n a_k + \frac{q}{n}\sum_{k=1}^n b_k\right)\left(b_k + \frac{r}{n}\sum_{k=1}^n a_k + \frac{s}{n}\sum_{k=1}^n b_k\right) x \\ &+ \sum_{k=1}^n \left(b_k + \frac{r}{n}\sum_{k=1}^n a_k + \frac{s}{n}\sum_{k=1}^n b_k\right)^2 \geqq 0 \qquad (x \in \mathbb{R}),\end{aligned}$$

the discriminant Δ of Q must be negative, that is,

$$\begin{aligned}(2.5)\quad \frac{1}{4}\Delta &= \left[\sum_{k=1}^n \left(a_k + \frac{p}{n}\sum_{k=1}^n a_k + \frac{q}{n}\sum_{k=1}^n b_k\right)\left(b_k + \frac{r}{n}\sum_{k=1}^n a_k + \frac{s}{n}\sum_{k=1}^n b_k\right)\right]^2 \\ &- \sum_{k=1}^n \left(a_k + \frac{p}{n}\sum_{k=1}^n a_k + \frac{q}{n}\sum_{k=1}^n b_k\right)^2 \\ &\cdot \left[\sum_{k=1}^n \left(b_k + \frac{r}{n}\sum_{k=1}^n a_k + \frac{s}{n}\sum_{k=1}^n b_k\right)^2\right] \leqq 0.\end{aligned}$$

On the other hand, the elements of $\Delta/4$ can be simplified as follows:

$$\begin{aligned}(2.6\text{a})\quad &\sum_{k=1}^n \left(a_k + \frac{p}{n}\sum_{k=1}^n a_k + \frac{q}{n}\sum_{k=1}^n b_k\right)\left(b_k + \frac{r}{n}\sum_{k=1}^n a_k + \frac{s}{n}\sum_{k=1}^n b_k\right) \\ &= \sum_{k=1}^n a_k b_k + \frac{p+s+ps+qr}{n}\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) \\ &+ \frac{r(1+p)}{n}\left(\sum_{k=1}^n a_k\right)^2 + \frac{q(1+s)}{n}\left(\sum_{k=1}^n b_k\right)^2,\end{aligned}$$

$$\begin{aligned}(2.6\text{b})\quad &\sum_{k=1}^n \left(a_k + \frac{p}{n}\sum_{k=1}^n a_k + \frac{q}{n}\sum_{k=1}^n b_k\right)^2 \\ &= \sum_{k=1}^n a_k^2 + \frac{2q(1+p)}{n}\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) \\ &+ \frac{p(p+2)}{n}\left(\sum_{k=1}^n a_k\right)^2 + \frac{q^2}{n}\left(\sum_{k=1}^n b_k\right)^2\end{aligned}$$

and

$$\sum_{k=1}^{n}\left(b_k+\frac{r}{n}\sum_{k=1}^{n}a_k+\frac{s}{n}\sum_{k=1}^{n}b_k\right)^2 = \sum_{k=1}^{n}b_k^2+\frac{2r(1+s)}{n}\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) + \frac{r^2}{n}\left(\sum_{k=1}^{n}a_k\right)^2+\frac{s(s+2)}{n}\left(\sum_{k=1}^{n}b_k\right)^2. \tag{2.6c}$$

So, upon substituting the results from (2.6) into the inequality (2.5), the first part of Theorem 1 is proved.

To prove the second part (i.e., the equality condition), let us assume that

$$b_k = va_k \qquad (k\in\{1,\dots,n\})$$

and substitute it into (2.1) to get

$$\left[v\sum_{k=1}^{n}a_k^2+(C_1v^2+A_1v+B_1)\left(\sum_{k=1}^{n}a_k\right)^2\right]^2 = \left[\sum_{k=1}^{n}a_k^2+(C_2v^2+A_2v+B_2)\left(\sum_{k=1}^{n}a_k\right)^2\right] \cdot\left[v^2\sum_{k=1}^{n}a_k^2+(C_3v^2+A_3v+B_3)\left(\sum_{k=1}^{n}a_k\right)^2\right]. \tag{2.7}$$

After some computations, the above equality leads us to the following nonlinear system:

$$\begin{cases} (C_1v^2+A_1v+B_1)^2=(C_2v^2+A_2v+B_2)(C_3v^2+A_3v+B_3), \\ 2v(C_1v^2+A_1v+B_1)=v^2(C_2v^2+A_2v+B_2)+(C_3v^2+A_3v+B_3). \end{cases} \tag{2.8}$$

Obviously, one of the solutions of the equation (2.8) is given by

$$A_j=B_j=C_j \qquad (j=1,2,3) \quad \text{and} \quad v=1.$$

□

Remark 1. *There exist various sub-cases of the inequality* (2.1). *However, due to page limitations, here we only consider a particular case of* (2.1) *and investigate its sub-cases. Naturally, therefore, other special cases can be separately studied. The details involved are being left as an exercise for the interested reader.*

3. The Particular Case When $B_1 = C_1 = 0$

A total of four cases can occur for the inequality (2.1) when

$$B_1 = C_1 = 0.$$

They are given, respectively, as follows:

(i) $(r,q)=(0,0)$;
(ii) $(r,s)=(0,-1)$;

(iii) $(p, q) = (-1, 0)$;
(iv) $(p, s) = (-1, -1)$.

3.1. **The Case When $q = r = 0$ and $p, s \in \mathbb{R}$ in (2.1).** In this case, we have

$$B_1 = C_1 = A_2 = C_2 = A_3 = B_3 = 0$$

and the inequality (2.1) is reduced to the following form:

$$\left[\sum_{k=1}^{n} a_k b_k + \frac{p+s+ps}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2 \leqq \left[\sum_{k=1}^{n} a_k^2 + \frac{p(p+2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 + \frac{s(s+2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right]. \tag{3.1}$$

This inequality has some interesting sub-cases as detailed below.

3.1.1. ***Sub-Case*** 1. $p = s \in \mathbb{R} \setminus (-2, 0)$: (***A Generalization of the Wagner Inequality***). The following inequality for sequences of real numbers is known in the literature as the Wagner inequality [12] (see also [6]):

Let $\{a_k\}_{k=1}^n$ *and* $\{b_k\}_{k=1}^n$ *be two sequences of real numbers. If* $w \geqq 0$, *then*

$$\left[\sum_{k=1}^{n} a_k b_k + w\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2 \leqq \left[\sum_{k=1}^{n} a_k^2 + w\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 + w\left(\sum_{k=1}^{n} b_k\right)^2\right]. \tag{3.2}$$

In order to deduce (3.2) from our results, it is sufficient to assume in (3.1) that

$$\frac{p+s+ps}{n} = \frac{p(p+2)}{n} = \frac{s(s+2)}{n} \geqq 0,$$

which obviously holds true for

$$p = s \in \mathbb{R} \setminus (-2, 0)$$

and readily yields the Wagner inequality (3.2) for

$$w = \frac{p(p+2)}{n} \geqq 0.$$

We note that, if in (3.1) we let

$$p(p+2) \leqq 0 \qquad \text{and} \qquad s(s+2) \leqq 0,$$

then

$$\begin{aligned} &\left[\sum_{k=1}^{n} a_k b_k + \frac{p+s+ps}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2 \\ &\leqq \left[\sum_{k=1}^{n} a_k^2 + \frac{p(p+2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 + \frac{s(s+2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \\ &\leqq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \Longleftrightarrow p \in [-2, 0] \qquad \text{and} \qquad s \in [-2, 0]. \end{aligned} \tag{3.3}$$

3.1.2. ***Sub-Case*** 2. $p = s \in [-2, 0]$: (***A Refinement for the Cauchy-Schwarz Inequality***). Suppose in (3.1) that

$$p = s \in [-2, 0] \qquad \text{and} \qquad p(p+2) = u.$$

Consequently, we have $u \in [-1, 0]$. By noting these assumptions, we can obtain a refinement for the Cauchy-Schwarz inequality (1.1). For this purpose, we first consider the following inequality, which is directly provable *via* some algebraic computations:

$$\begin{aligned} (3.4) \quad & \left[\sum_{k=1}^{n} a_k^2 + \frac{u}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 + \frac{u}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \\ & \leqq \left[\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} + \frac{u}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2, \end{aligned}$$

which leads us eventually to

$$(3.5) \qquad \frac{u}{n}\left[\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\sum_{k=1}^{n} b_k - \left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}}\sum_{k=1}^{n} a_k\right]^2 \leqq 0 \qquad (u \in [-1, 0]).$$

Hence, by referring to the inequalities (3.1) and (3.4), we can at last arrive at the following corollary.

Corollary 1. *Let* $\{a_k\}_{k=1}^n$ *and* $\{b_k\}_{k=1}^n$ *be two positive sequences of real numbers and* $\alpha \in [0, 1]$. *Then*

$$\begin{aligned} (3.6) \qquad & \left[\sum_{k=1}^{n} a_k b_k - \frac{\alpha}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2 \\ & \leqq \left[\sum_{k=1}^{n} a_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \\ & \leqq \left[\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} - \frac{\alpha}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2, \end{aligned}$$

which is equivalent to the following inequality:

$$\begin{aligned} (3.7) \quad & \left(\sum_{k=1}^{n} a_k b_k\right)^2 \\ & \leqq \left[\frac{\alpha}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + \sqrt{\sum_{k=1}^{n} a_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^{n} a_k\right)^2}\sqrt{\sum_{k=1}^{n} b_k^2 - \frac{\alpha}{n}\left(\sum_{k=1}^{n} b_k\right)^2}\right]^2 \\ & \leqq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right). \end{aligned}$$

The equality holds true in (3.7) *when*

$$b_k = \lambda a_k \qquad (k \in \{1, \ldots, n\}),$$

where λ is a constant.

For other refinements of the Cauchy-Schwarz inequality (1.1), we refer the reader to the earlier works [1] and [13].

3.1.3. ***Sub-Case*** 3. It may be interesting to add that, if

$$\frac{1}{p}+\frac{1}{s}=-1 \qquad (p,s\in\mathbb{R}\setminus\{0\}),$$

then (3.1) is reduced to the following form:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leqq \left[\sum_{k=1}^{n} a_k^2+\frac{p(p+2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right] \cdot \left[\sum_{k=1}^{n} b_k^2+\frac{s(s+2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right], \tag{3.8}$$

which, for $p=s=-2$, yields the Cauchy-Schwarz inequality (1.1).

3.2. **The Case When $r=0$, $s=-1$ and $p,q\in\mathbb{R}$ in (2.1).** In this case, we have

$$B_1=C_1=A_3=B_3=0$$

and the inequality (2.1) is reduced to the following form:

$$\left[\sum_{k=1}^{n} a_k b_k-\frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2 \leqq \left[\sum_{k=1}^{n} b_k^2-\frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \cdot \left[\sum_{k=1}^{n} a_k^2-\frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2+\frac{1}{n}\left(\sum_{k=1}^{n}[(p+1)a_k+qb_k]\right)^2\right]. \tag{3.9}$$

However, since

$$\sum_{k=1}^{n} a_k^2-\frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 \geqq 0, \tag{3.10}$$

the best option for p and q in (3.9) is when $p=-1$ and $q=0$. Furthermore, we note that the above-mentioned third case, that is,

$$p=-1,\ \ q=0 \quad \text{and} \quad r,s\in\mathbb{R},$$

gives us the same result as in (3.9).

3.3. **The Case When $p = s = -1$ and $q, r \in \mathbb{R}$ in (2.1).** In this case, we have

$$B_1 = C_1 = A_2 = A_3 = 0$$

and the inequality (2.1) is reduced to the following form:

$$\left[\sum_{k=1}^{n} a_k b_k + \frac{qr-1}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2 \leqq \left[\sum_{k=1}^{n} a_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2 + \frac{q^2}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right] \cdot \left[\sum_{k=1}^{n} b_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)^2 + \frac{r^2}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]. \tag{3.11}$$

An interesting case in (3.11) occurs when $q = r = 1$. We are thus led to the following inequality:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leqq \left[\sum_{k=1}^{n} a_k^2 + \frac{1}{n}\left\{\left(\sum_{k=1}^{n} b_k\right)^2 - \left(\sum_{k=1}^{n} a_k\right)^2\right\}\right] \cdot \left[\sum_{k=1}^{n} b_k^2 + \frac{1}{n}\left\{\left(\sum_{k=1}^{n} a_k\right)^2 - \left(\sum_{k=1}^{n} b_k\right)^2\right\}\right]. \tag{3.12}$$

4. A Generalization of the Cauchy-Bunyakovsky Inequality

In a similar manner, the integral version of the Cauchy-Schwarz inequality (1.1), which is known in the literature as the Cauchy-Bunyakovsky inequality [2] and has the following form:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leqq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx, \tag{4.1}$$

can also be generalized as follows.

Theorem 2. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions on $[a,b]$ and*

$$p, q, r, s \in \mathbb{R}.$$

Then the following inequality holds true:

$$\left[\int_a^b f(x)g(x)dx + A_1^* \int_a^b f(x)dx \int_a^b g(x)dx + B_1^* \left(\int_a^b f(x)dx\right)^2 + C_1^* \left(\int_a^b g(x)dx\right)^2\right]^2 \tag{4.2}$$

$$\leqq \left[\int_a^b [f(x)]^2 dx + A_2^* \int_a^b f(x)dx \int_a^b g(x)dx\right.$$

$$+B_2^*\left(\int_a^b f(x)dx\right)^2+C_2^*\left(\int_a^b g(x)dx\right)^2\Bigg]$$

$$\cdot\left[\int_a^b [g(x)]^2dx+A_3^*\int_a^b f(x)dx\int_a^b g(x)dx\right.$$

$$\left.+B_3^*\left(\int_a^b f(x)dx\right)^2+C_3^*\left(\int_a^b g(x)dx\right)^2\right],$$

in which

$$M^*=\begin{pmatrix} A_1^* & B_1^* & C_1^* \\ A_2^* & B_2^* & C_2^* \\ A_3^* & B_3^* & C_3^* \end{pmatrix}=\frac{1}{b-a}\begin{pmatrix} p+s+ps+qr & r(1+p) & q(1+s) \\ 2q(1+p) & p(p+2) & q^2 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}.$$

Moreover, the inequality (4.2) *is equivalent to the following integral inequality*:

$$\left[\int_a^b f(x)g(x)dx+\frac{p+s+ps+qr}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx\right.$$
$$\left.+\frac{r(1+p)}{b-a}\left(\int_a^b f(x)dx\right)^2+\frac{q(1+s)}{b-a}\left(\int_a^b g(x)dx\right)^2\right]^2$$
$$\leqq\left(\int_a^b [f(x)]^2dx+\frac{1}{b-a}\int_a^b [pf(x)+qg(x)]dx\int_a^b [(p+2)f(x)+qg(x)]dx\right)$$
$$\cdot\left(\int_a^b [g(x)]^2dx+\frac{1}{b-a}\int_a^b [rf(x)+sg(x)]dx\int_a^b [rf(x)+(s+2)g(x)]dx\right),\tag{4.3}$$

and would reduce to the Cauchy-Bunyakovsky inequality (4.1) *in its special case when*

$$A_j^*=B_j^*=C_j^*=0\qquad(j=1,2,3).$$

The equality holds true in (4.2) *if*

$$f(x)=g(x)\qquad\text{and}\qquad A_j^*=B_j^*=C_j^*\quad(j=1,2,3).$$

Although the proof is similar to the proof of Theorem 1, by defining the positive quadratic polynomial $R(x;p,q,r,s)$ by

$$R(x;p,q,r,s)=\int_a^b\left[\left(f(t)+\frac{p}{b-a}\int_a^b f(x)dx+\frac{q}{b-a}\int_a^b g(x)dx\right)x\right.$$
$$\left.+\left(g(t)+\frac{r}{b-a}\int_a^b f(x)dx+\frac{s}{b-a}\int_a^b g(x)dx\right)\right]^2dt\geqq 0,\tag{4.4}$$

we note that the following relations are required to be applied in our proof of Theorem 2:

$$\int_a^b\left(f(x)+\frac{p}{b-a}\int_a^b f(x)dx+\frac{q}{b-a}\int_a^b g(x)dx\right)$$
$$\cdot\left(g(x)+\frac{r}{b-a}\int_a^b f(x)dx+\frac{s}{b-a}\int_a^b g(x)dx\right)dx\tag{4.5}$$

$$= \int_a^b f(x)g(x)dx + \frac{p+s+ps+qr}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx + \frac{r(1+p)}{b-a}\left(\int_a^b f(x)dx\right)^2 + \frac{q(1+s)}{b-a}\left(\int_a^b g(x)dx\right)^2,$$

$$\int_a^b \left(f(x) + \frac{p}{b-a}\int_a^b f(x)dx + \frac{q}{b-a}\int_a^b g(x)dx\right)^2 dx = \int_a^b [f(x)]^2 dx + \frac{2q(1+p)}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx + \frac{p(p+2)}{b-a}\left(\int_a^b f(x)dx\right)^2 + \frac{q^2}{b-a}\left(\int_a^b g(x)dx\right)^2, \tag{4.6}$$

and

$$\int_a^b \left(g(x) + \frac{r}{b-a}\int_a^b f(x)dx + \frac{s}{b-a}\int_a^b g(x)dx\right)^2 dx = \int_a^b [g(x)]^2 dx + \frac{2r(1+s)}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx + \frac{r^2}{b-a}\left(\int_a^b f(x)dx\right)^2 + \frac{s(s+2)}{b-a}\left(\int_a^b g(x)dx\right)^2, \tag{4.7}$$

successively. Furthermore, we note that all of the above-mentioned sub-cases for the inequality (2.1) can similarly be considered for the continuous (integral) case given by (4.2). For the sake of completeness, we can state the following corollary as one of the results derived in this manner.

Corollary 2 (A Refinement of the Cauchy-Bunyakovsky Inequality)**.**

Let

$$f, g : [a,b] \to \mathbb{R}$$

be two positive integrable functions on the interval $[a,b]$ *and* $\alpha \in [0,1]$. *Then*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leqq \left[\frac{\alpha}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx + \sqrt{\int_a^b [f(x)]^2 dx - \frac{\alpha}{b-a}\left(\int_a^b f(x)dx\right)^2} \cdot \sqrt{\int_a^b [g(x)]^2 dx - \frac{\alpha}{b-a}\left(\int_a^b g(x)dx\right)^2}\,\right]^2 \leqq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx. \tag{4.8}$$

5. A Unified Approach for the Classification of (2.1) and (4.2)

As we observed in the preceding sections, there are, respectively, two special matrices M and M^* for the inequalities (2.1) and (4.2) having 9 elements. Consequently, each sub-case of these two inequalities can be characterized by means the matrices M or M^* *directly*. For instance, the following discrete inequality:

$$\left[\sum_{k=1}^n a_k b_k - \frac{s+2}{n}\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) - \frac{r}{n}\left(\sum_{k=1}^n a_k\right)^2\right]^2 \leqq \sum_{k=1}^n a_k^2 \left(\sum_{k=1}^n b_k^2 + \frac{1}{n}\sum_{k=1}^n (ra_k + sb_k)\sum_{k=1}^n [ra_k + (s+2)b_k]\right), \tag{5.1}$$

which yields the Cauchy-Schwarz inequality (1.1) in its special case when

$$r = 0 \qquad \text{and} \qquad s = -2,$$

has the characteristic matrix given by

$$M[\text{Ineq. (5.1)}] = \frac{1}{n}\begin{pmatrix} -s-2 & -r & 0 \\ 0 & 0 & 0 \\ 2r(1+s) & r^2 & s(s+2) \end{pmatrix}, \tag{5.2}$$

while the following continuous inequality:

$$\left[\int_a^b f(x)g(x)dx + \frac{p}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx + \frac{q}{b-a}\left(\int_a^b g(x)dx\right)^2\right]^2 \leqq \left(\int_a^b [f(x)]^2 dx + \frac{1}{b-a}\int_a^b [pf(x)+qg(x)]dx \cdot \int_a^b [(p+2)f(x)+qg(x)]dx\right)\left(\int_a^b [g(x)]^2 dx\right), \tag{5.3}$$

corresponds to the matrix given by

$$M^*[\text{Ineq. (5.3)}] = \frac{1}{b-a}\begin{pmatrix} p & 0 & q \\ 2q(1+p) & p(p+2) & q^2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.4}$$

Similarly, for the inequalities (3.2), (3.9), (3.11) and (3.12), we have

$$M[\text{Ineq. (3.2)}] = \frac{p(p+2)}{n}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.5}$$

$$M[\text{Ineq. (3.9)}] = \frac{1}{n}\begin{pmatrix} -1 & 0 & 0 \\ 2q(1+p) & p(p+2) & q^2 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.6}$$

$$M[\text{Ineq. (3.11)}] = \frac{1}{n}\begin{pmatrix} qr-1 & 0 & 0 \\ 0 & -1 & q^2 \\ 0 & r^2 & -1 \end{pmatrix} \tag{5.7}$$

and

$$M[\text{Ineq. } (3.12)] = \frac{1}{n}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \tag{5.8}$$

respectively.

Finally, we mention that the usual Cauchy-Schwarz and Cauchy-Bunyakovsky inequalities correspond, respectively, to

$$M = 0 \qquad \text{and} \qquad M^* = 0,$$

which can be obtained for $p = q = r = s = 0$.

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Department of Applied Mathematics, K. N. Toosi University of Technology, P. O. Box 1618, Tehran 16315–1618, Iran
E-mail address: mmjamei@aut.ac.ir

School of Computer Science and Mathematics, Victoria University, P. O. Box 14428, Melbourne City MC, Victoria 8001, Australia
E-mail address: sever.dragomir@vu.edu.au
URL: http://www.staff.vu.edu.au/rgmia/dragomir/

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada
E-mail address: harimsri@math.uvic.ca
URL: http://www.math.uvic.ca/faculty/harimsri/