

A FUNCTIONAL ASSOCIATED WITH TWO BOUNDED LINEAR OPERATORS IN HILBERT SPACES AND RELATED INEQUALITIES

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ABSTRACT. In this paper several inequalities for the functional $\mu(A, B) := \sup_{\|x\|=1} \{\|Ax\| \|Bx\|\}$ under various assumptions for the operators involved, including operators satisfying the uniform (α, β) -property and operators for which the transform $C_{\alpha, \beta}(\cdot, \cdot)$ is accretive, are given.

1. INTRODUCTION

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers \mathbb{C} given by [9, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, x \in H, \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [9, p. 8]:

$$(1.1) \quad w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \rightarrow H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [9, p. 9]:

$$(1.2) \quad w(T) \leq \|T\| \leq 2w(T),$$

for any $T \in B(H)$

For other results on numerical radii, see [10], Chapter 11. For some recent and interesting results concerning inequalities for the numerical radius, see [11] and [12].

If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then

$$(1.3) \quad w(AB) \leq 4w(A)w(B).$$

In the case that $AB = BA$, then

$$(1.4) \quad w(AB) \leq 2w(A)w(B).$$

The following results are also well known [9, p. 38]:

If A is a unitary operator that commutes with another operator B , then

$$(1.5) \quad w(AB) \leq w(B).$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. If the operators A and B double commute, then [9, p. 38]

$$(1.6) \quad w(AB) \leq w(B) \|A\|.$$

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As a consequence of the above, we have [9, p. 39]:

Let A be a normal operator commuting with B , then

$$(1.7) \quad w(AB) \leq w(A)w(B).$$

For other results and historical comments on the above see [9, p. 39–41].

For two bounded linear operators A, B in the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ we define the functional

$$(1.8) \quad \mu(A, B) := \sup_{\|x\|=1} \{\|Ax\| \|Bx\|\} (\geq 0).$$

It is obvious that μ is symmetric and sub-additive in each variable, $\mu(A, A) = \|A\|^2$, $\mu(A, I) = \|A\|$, where I is the identity operator, $\mu(\alpha A, \beta B) = |\alpha\beta| \mu(A, B)$ and $\mu(A, B) \leq \|A\| \|B\|$. We also have the following inequalities

$$(1.9) \quad \mu(A, B) \geq w(B^*A)$$

and

$$(1.10) \quad \mu(A, B) \|A\| \|B\| \geq \mu(AB, BA).$$

The inequality (1.9) follows by the Schwarz inequality $\|Ax\| \|Bx\| \geq |\langle Ax, Bx \rangle|$, $x \in H$, while (1.10) can be obtained by multiplying the inequalities $\|ABx\| \leq \|A\| \|Bx\|$ and $\|BAx\| \leq \|B\| \|Ax\|$.

From (1.9) we also get

$$(1.11) \quad \|A\|^2 \geq \mu(A, A^*) \geq w(A^2)$$

for any A .

Motivated by the above results we establish in this paper several inequalities for the functional $\mu(\cdot, \cdot)$ under various assumptions for the operators involved, including operators satisfying the uniform (α, β) -property and operators for which the transform $C_{\alpha, \beta}(\cdot, \cdot)$ is accretive.

2. GENERAL INEQUALITIES

The following result concerning some general power operator inequalities may be stated:

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$ we have the inequality*

$$(2.1) \quad \mu^r(A, B) \leq \frac{1}{2} \|(A^*A)^r + (B^*B)^r\|.$$

The constant $\frac{1}{2}$ is best possible.

Proof. Utilising the arithmetic mean - geometric mean inequality and the convexity of the function $f(t) = t^r$ for $r \geq 1$ we have successively

$$(2.2) \quad \begin{aligned} \|Ax\| \|Bx\| &\leq \frac{1}{2} [\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle] \\ &\leq \left[\frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right]^{\frac{1}{r}}, \end{aligned}$$

for any $x \in H$.

It is well known that, if P is a positive operator, then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [13])

$$(2.3) \quad \langle Px, x \rangle^r \leq \langle P^r x, x \rangle.$$

Applying this inequality to the positive operators A^*A and B^*B we deduce that

$$(2.4) \quad \left[\frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right]^{\frac{1}{r}} \leq \left\langle \frac{[(A^*A)^r + (B^*B)^r]x}{2}, x \right\rangle^{\frac{1}{r}},$$

for any $x \in H$ with $\|x\| = 1$.

Now, on making use of the inequalities (2.2) and (2.4) we get

$$(2.5) \quad \|Ax\| \|Bx\| \leq \left\langle \frac{[(A^*A)^r + (B^*B)^r]x}{2}, x \right\rangle^{\frac{1}{r}},$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we obtain the desired result (2.1).

For $r = 1$ and $B = A$ we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). ■

Corollary 1. For any $A \in B(H)$ and $r \geq 1$ we have the inequality

$$(2.6) \quad \mu^r(A, A^*) \leq \frac{1}{2} \|(A^*A)^r + (AA^*)^r\|$$

and the inequality

$$(2.7) \quad \|A\|^r \leq \frac{1}{2} \|(A^*A)^r + I\|,$$

respectively.

The following similar result for powers of operators can be stated as well:

Theorem 2. For any $A, B \in B(H)$, any $\alpha \in (0, 1)$ and $r \geq 1$ we have the inequality

$$(2.8) \quad \mu^{2r}(A, B) \leq \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (B^*B)^{r/(1-\alpha)} \right\|.$$

The inequality is sharp.

Proof. Observe that, for any $\alpha \in (0, 1)$ we have

$$(2.9) \quad \|Ax\|^2 \|Bx\|^2 = \langle (A^*A)x, x \rangle \langle (B^*B)x, x \rangle \\ = \left\langle [(A^*A)^{1/\alpha}]^\alpha x, x \right\rangle \left\langle [(B^*B)^{1/(1-\alpha)}]^{1-\alpha} x, x \right\rangle,$$

where $x \in H$.

It is well known that (see for instance [13]), if P is a positive operator and $q \in (0, 1)$, then

$$(2.10) \quad \langle P^q x, x \rangle \leq \langle Px, x \rangle^q.$$

Applying this property to the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$, where $\alpha \in (0, 1)$, we have

$$(2.11) \quad \left\langle [(A^*A)^{1/\alpha}]^\alpha x, x \right\rangle \left\langle [(B^*B)^{1/(1-\alpha)}]^{1-\alpha} x, x \right\rangle \\ \leq \left\langle (A^*A)^{1/\alpha} x, x \right\rangle^\alpha \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^{1-\alpha},$$

for any $x \in H$ with $\|x\| = 1$.

Now, on utilising the weighted arithmetic mean-geometric mean inequality, i.e.,

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b, \quad \text{where } \alpha \in (0, 1) \text{ and } a, b \geq 0;$$

we get

$$(2.12) \quad \left\langle (A^*A)^{1/\alpha} x, x \right\rangle^\alpha \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^{1-\alpha} \\ \leq \alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle,$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, by the elementary inequality

$$\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{1/r}, \quad \text{where } \alpha \in (0, 1) \text{ and } a, b \geq 0;$$

we have successively

$$(2.13) \quad \alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle \\ \leq \left[\alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle^r + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^r \right]^{\frac{1}{r}} \\ \leq \left[\alpha \cdot \left\langle (A^*A)^{r/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{r/(1-\alpha)} x, x \right\rangle \right]^{\frac{1}{r}},$$

for any $x \in H$ with $\|x\| = 1$, where for the last inequality we have used the property (2.3) for the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$.

Now, on making use of the identity (2.9) and the inequalities (2.11)-(2.13) we get

$$\|Ax\|^2 \|Bx\|^2 \leq \left[\left\langle \left[\alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (B^*B)^{r/(1-\alpha)} \right] x, x \right\rangle \right]^{\frac{1}{r}},$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we deduce the desired result (2.8).

Notice that the inequality is sharp since for $r = 1$ and $B = A$ we get in both sides of (2.8) the same quantity $\|A\|^4$. ■

Corollary 2. For any $A \in B(H)$, any $\alpha \in (0, 1)$ and $r \geq 1$, we have the inequalities

$$\mu^{2r}(A, A^*) \leq \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (AA^*)^{r/(1-\alpha)} \right\|, \\ \|A\|^{2r} \leq \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot I \right\|$$

and

$$\|A\|^{4r} \leq \left\| \alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (A^*A)^{r/(1-\alpha)} \right\|,$$

respectively.

The following reverse of the inequality (1.9) maybe stated as well:

Theorem 3. For any $A, B \in B(H)$ we have the inequality

$$(2.14) \quad (0 \leq) \mu(A, B) - w(B^*A) \leq \frac{1}{2} \|A - B\|^2$$

and the inequality

$$(2.15) \quad \mu\left(\frac{A+B}{2}, \frac{A-B}{2}\right) \leq \frac{1}{2} w(B^*A) + \frac{1}{2} \|A - B\|^2,$$

respectively.

Proof. We have

$$(2.16) \quad \begin{aligned} \|Ax - Bx\|^2 &= \|Ax\|^2 + \|Bx\|^2 - 2 \operatorname{Re} \langle B^* Ax, x \rangle \\ &\geq 2 \|Ax\| \|Bx\| - 2 |\langle B^* Ax, x \rangle|, \end{aligned}$$

for any $x \in H, \|x\| = 1$, which gives the inequality

$$\|Ax\| \|Bx\| \leq |\langle B^* Ax, x \rangle| + \frac{1}{2} \|Ax - Bx\|^2,$$

for any $x \in H, \|x\| = 1$.

Taking the supremum over $\|x\| = 1$ we deduce the desired result (2.14).

By the parallelogram identity in the Hilbert space H we also have

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \frac{1}{2} (\|Ax + Bx\|^2 + \|Ax - Bx\|^2) \\ &\geq \|Ax + Bx\| \|Ax - Bx\|, \end{aligned}$$

for any $x \in H$.

Combining this inequality with the first part of (2.16) we get

$$\|Ax + Bx\| \|Ax - Bx\| \leq \|Ax - Bx\|^2 + 2 |\langle B^* Ax, x \rangle|,$$

for any $x \in H$. Taking the supremum in this inequality over $\|x\| = 1$ we deduce the desired result (2.15). ■

Corollary 3. *Let $A \in B(H)$. If $\operatorname{Re}(A) := \frac{A+A^*}{2}$ and $\operatorname{Im}(A) := \frac{A-A^*}{2i}$ are the real and imaginary parts of A , then we have the inequality*

$$(0 \leq) \mu(A, A^*) - w(A^2) \leq 2 \cdot \|\operatorname{Im}(A)\|^2$$

and

$$\mu(\operatorname{Re}(A), \operatorname{Im}(A)) \leq \frac{1}{2} w(A^2) + 2 \cdot \|\operatorname{Im}(A)\|^2,$$

respectively.

Moreover, we have

$$(0 \leq) \mu(\operatorname{Re}(A), \operatorname{Im}(A)) - w(\operatorname{Re}(A) \operatorname{Im}(A)) \leq \frac{1}{2} \|A\|^2.$$

Corollary 4. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ we have the inequality (see also [6])*

$$(2.17) \quad (0 \leq) \|A\| - w(A) \leq \frac{1}{2|\lambda|} \|A - \lambda I\|^2.$$

For a bounded linear operator T consider the quantity $\ell(T) := \inf_{\|x\|=1} \|Tx\|$. We can state the following result as well.

Theorem 4. *For any $A, B \in B(H)$ with $A \neq B$ and such that $\ell(B) \geq \|A - B\|$ we have*

$$(2.18) \quad (0 \leq) \mu^2(A, B) - w^2(B^* A) \leq \|A\|^2 \|A - B\|^2.$$

Proof. Denote $r := \|A - B\| > 0$. Then for any $x \in H$ with $\|x\| = 1$ we have $\|Bx\| \geq r$ and by the first part of (2.16) we can write that

$$(2.19) \quad \|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2} \right)^2 \leq 2 |\langle B^* Ax, x \rangle|,$$

for any $x \in H$ with $\|x\| = 1$.

On the other hand we have

$$(2.20) \quad \|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2} \right)^2 \geq 2 \cdot \|Ax\| \sqrt{\|Bx\|^2 - r^2},$$

for any $x \in H$ with $\|x\| = 1$.

Combining (2.19) with (2.20) we deduce

$$\|Ax\| \sqrt{\|Bx\|^2 - r^2} \leq |\langle B^* Ax, x \rangle|$$

which is clearly equivalent with

$$(2.21) \quad \|Ax\|^2 \|Bx\|^2 \leq |\langle B^* Ax, x \rangle|^2 + \|Ax\|^2 \|A - B\|^2,$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum in (2.21) over $x \in H$ with $\|x\| = 1$, we deduce the desired inequality (2.18). ■

Corollary 5. *For any $A \in B(H)$ a non self adjoint operator and such that $\ell(A^*) \geq 2 \cdot \|\text{Im}(A)\|$ we have*

$$(2.22) \quad (0 \leq) \mu^2(A, A^*) - w^2(A^2) \leq 4 \cdot \|A\|^2 \|\text{Im}(A)\|^2.$$

Corollary 6. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and $|\lambda| \geq \|A - \lambda I\|$ we have the inequality (see also [6])*

$$(0 \leq) \|A\|^2 - w^2(A) \leq \frac{1}{|\lambda|^2} \cdot \|A\|^2 \|A - \lambda I\|^2$$

or, equivalently,

$$(0 \leq) \sqrt{1 - \frac{\|A - \lambda I\|^2}{|\lambda|^2}} \leq \frac{w(A)}{\|A\|} (\leq 1).$$

3. INEQUALITIES FOR OPERATORS SATISFYING THE UNIFORM (α, β) -PROPERTY

The following result that may be of interest in itself, holds:

Lemma 1. *Let $T \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. The following statements are equivalent:*

(i) *We have*

$$(3.1) \quad \text{Re} \langle \beta y - Tx, Tx - \alpha y \rangle \geq 0,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$;

(ii) *We have*

$$(3.2) \quad \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\alpha - \beta|,$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. Follows by the following identity

$$\text{Re} \langle \beta y - Tx, Tx - \alpha y \rangle = \frac{1}{4} |\alpha - \beta|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\|^2,$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$. ■

Remark 1. For any operator $T \in B(H)$ if we choose $\alpha = a \|T\| (1 + 2i)$ and $\beta = a \|T\| (1 - 2i)$ with $a \geq 1$, then

$$\frac{\alpha + \beta}{2} = a \|T\| \quad \text{and} \quad \frac{|\alpha - \beta|}{2} = 2a \|T\|$$

showing that

$$\begin{aligned} \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| &\leq \|Tx\| + \left| \frac{\alpha + \beta}{2} \right| \leq \|T\| + a \|T\| \\ &\leq 2a \|T\| = \frac{1}{2} \cdot |\alpha - \beta|, \end{aligned}$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$, i.e., T satisfies the condition (3.1) with the scalars α and β given above.

Definition 1. For given $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ and $y \in H$ with $\|y\| = 1$, we say that the operator $T \in B(H)$ has the (α, β, y) -property if either (3.1) or, equivalently, (3.2) holds true for any $x \in H$ with $\|x\| = 1$. Moreover, if T has the (α, β, y) -property for any $y \in H$ with $\|y\| = 1$, then we say that this operator has the uniform (α, β) -property.

Remark 2. The above Remark 1 shows that any bounded linear operator has the uniform (α, β) -property for infinitely many (α, β) appropriately chosen. For a given operator satisfying an (α, β) -property, it is an open problem to find the possibly nonzero lower bound for the quantity $|\alpha - \beta|$.

The following results may be stated:

Theorem 5. Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y) -property while B^* has the (γ, δ, y) -property, then

$$(3.3) \quad \left| \|Ay\| \|By\| - \|BA^*\| \right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta|.$$

Moreover, if A^* has the uniform (α, β) -property and B^* has the uniform (γ, δ) -property, then

$$(3.4) \quad |\mu(A, B) - \|BA^*\|| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta|.$$

Proof. Since A^* has the (α, β, y) -property while B^* has the (γ, δ, y) -property, then on making use of Lemma 1 we have that

$$\left\| A^*x - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\beta - \alpha|$$

and

$$\left\| B^*z - \frac{\gamma + \delta}{2} \cdot y \right\| \leq \frac{1}{2} |\gamma - \delta|$$

for any $x, z \in H$, with $\|x\| = \|z\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [1] (see also [2] or [7, p. 43]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} , $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$(3.5) \quad \operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \quad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0$$

or, equivalently,

$$(3.6) \quad \left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha|, \quad \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma|,$$

then

$$(3.7) \quad |\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|.$$

Applying (3.7) for $u = A^*x$, $v = B^*z$ and $e = y$ we deduce

$$(3.8) \quad |\langle BA^*x, z \rangle - \langle x, Ay \rangle \langle z, By \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$, which is an inequality of interest in itself.

Observing that

$$\| |\langle BA^*x, z \rangle| - |\langle x, Ay \rangle \langle z, By \rangle| \| \leq |\langle BA^*x, z \rangle - \langle x, Ay \rangle \langle z, By \rangle|,$$

then by (3.7) we deduce the inequality

$$\| |\langle BA^*x, z \rangle| - |\langle x, Ay \rangle \langle z, By \rangle| \| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$. This is equivalent with the following two inequalities

$$(3.9) \quad |\langle BA^*x, z \rangle| \leq |\langle x, Ay \rangle \langle z, By \rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$(3.10) \quad |\langle x, Ay \rangle \langle z, By \rangle| \leq |\langle BA^*x, z \rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$.

Taking the supremum over $x, z \in H$, $\|x\| = \|z\| = 1$ in (3.9) and (3.10) we get the inequalities

$$(3.11) \quad \|BA^*\| \leq \|Ay\| \|By\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$(3.12) \quad \|Ay\| \|By\| \leq \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

which are clearly equivalent with (3.3).

Now, if A has the uniform (α, β) -property and B has the uniform (γ, δ) -property, then the inequalities (3.11) and (3.12) hold for any $y \in H$ with $\|y\| = 1$. Taking the supremum over $y \in H$ with $\|y\| = 1$ in these inequalities we deduce

$$\|BA^*\| \leq \mu(A, B) + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$\mu(A, B) \leq \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

which are equivalent with (3.4). ■

Corollary 7. *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A has the (α, β, y) -property while A^* has the (γ, δ, y) -property, then*

$$\left| \|A^*y\| \|Ay\| - \|A^2\| \right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta|.$$

Moreover, if A has the uniform (α, β) -property and A^* has the uniform (γ, δ) -property, then

$$\left| \mu(A, A^*) - \|A^2\| \right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta|.$$

The following results may be stated as well:

Theorem 6. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y) -property while B^* has the (γ, δ, y) -property, then*

$$(3.13) \quad \begin{aligned} & \left| \|Ay\| \|By\| - \|BA^*\| \right| \\ & \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|A\| + \|Ay\|)(\|B\| + \|By\|)}. \end{aligned}$$

Moreover, if A^* has the uniform (α, β) -property and B^* has the uniform (γ, δ) -property, then

$$(3.14) \quad \left| \mu(A, B) - \|BA^*\| \right| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{\|A\| \|B\|}.$$

Proof. We make use of the following inequality obtained by the author in [5] (see also [7, p. 65]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} , $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$ and such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \quad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0$$

or, equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha|, \quad \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma|,$$

then

$$(3.15) \quad \begin{aligned} & |\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \\ & \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|u\| + |\langle u, e \rangle|)(\|v\| + |\langle v, e \rangle|)}. \end{aligned}$$

Applying (3.15) for $u = A^*x$, $v = B^*z$ and $e = y$ we deduce

$$\begin{aligned} & |\langle BA^*x, z \rangle - \langle x, Ay \rangle \langle z, By \rangle| \\ & \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|A^*x\| + |\langle x, Ay \rangle|)(\|B^*z\| + |\langle z, By \rangle|)}, \end{aligned}$$

for any $x, y, z \in H$, $\|x\| = \|y\| = \|z\| = 1$.

Now, on making use of a similar argument to the one from the proof of Theorem 5, we deduce the desired results (3.13) and (3.14). The details are omitted. ■

Corollary 8. *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A has (α, β, y) -property while A^* has the (γ, δ, y) -property, then*

$$\| \|A^*y\| \|Ay\| - \|A^2\| \|y\| \|y\| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|A\| + \|A^*y\|)(\|A\| + \|Ay\|)}.$$

Moreover, if A has the uniform (α, β) -property and A^* has the uniform (γ, δ) -property, then

$$|\mu(A, A^*) - \|A^2\|| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \|A\|.$$

4. THE TRANSFORM $C_{\alpha, \beta}(\cdot, \cdot)$ AND OTHER INEQUALITIES

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$C_{\alpha, \beta}(T, U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform $C_{\alpha, \beta}(T) := (T^* - \bar{\alpha}I)(\beta I - T) = C_{\alpha, \beta}(T, I)$, where I is the identity operator, which has been introduced in [8] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re} \langle Ty, y \rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$(4.1) \quad \begin{aligned} \operatorname{Re} \langle C_{\alpha, \beta}(T, U)x, x \rangle &= \operatorname{Re} \langle C_{\beta, \alpha}(T, U)x, x \rangle \\ &= \frac{1}{4} |\beta - \alpha|^2 \|Ux\|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot Ux \right\|^2, \end{aligned}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 2. *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

- (i) *The transform $C_{\alpha, \beta}(T, U)$ (or, equivalently, $C_{\beta, \alpha}(T, U)$) is accretive;*
- (ii) *We have the norm inequality*

$$(4.2) \quad \left\| Tx - \frac{\alpha + \beta}{2} \cdot Ux \right\| \leq \frac{1}{2} |\beta - \alpha| \|Ux\|,$$

for any $x \in H$.

As a consequence of the above lemma we can state

Corollary 9. *Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $C_{\alpha, \beta}(T, U)$ is accretive, then*

$$(4.3) \quad \left\| T - \frac{\alpha + \beta}{2} \cdot U \right\| \leq \frac{1}{2} |\beta - \alpha| \|U\|.$$

Remark 3. *In order to give examples of operators $T, U \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha, \beta}(T, U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w ($w \neq 0$) with the property that $\|Sx - zVx\| \leq |w| \|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we observe that T and U satisfy (4.2), i.e., $C_{\alpha, \beta}(T, U)$ is accretive.*

We are able now to give the following result concerning other reverse inequalities for the case when the involved operators satisfy the accretivity property described above.

Theorem 7. *Let $\alpha, \beta \in \mathbb{C}$ and $A, B \in B(H)$. If $C_{\alpha, \beta}(A, B)$ is accretive, then*

$$(4.4) \quad (0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|B\|^4.$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(4.5) \quad (0 \leq) \mu(A, B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|B\|^2.$$

In addition, if $\operatorname{Re}(\alpha\bar{\beta}) > 0$, then also

$$(4.6) \quad (1 \leq) \frac{\mu(A, B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}(\alpha\bar{\beta})}}$$

and

$$(4.7) \quad (0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}(\alpha\bar{\beta})} \right) w(B^*A) \|B\|^2,$$

respectively.

Proof. By Lemma 2, since $C_{\alpha, \beta}(A, B)$ is accretive, then

$$(4.8) \quad \left\| Ax - \frac{\alpha + \beta}{2} \cdot Bx \right\| \leq \frac{1}{2} |\beta - \alpha| \|Bx\|,$$

for any $x \in H$.

We utilize the following reverse of the Schwarz inequality in inner product spaces obtained by the author in [3] (see also [7, p. 4]):

If $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $u, v \in H$ are such that

$$(4.9) \quad \operatorname{Re} \langle \Gamma v - u, u - \gamma v \rangle \geq 0$$

or, equivalently,

$$(4.10) \quad \left\| u - \frac{\gamma + \Gamma}{2} \cdot v \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|v\|,$$

then

$$(4.11) \quad 0 \leq \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|v\|^4.$$

Now, on making use of (4.11) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ we can write the inequality

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^*Ax, x \rangle|^2 + \frac{1}{4} |\beta - \alpha|^2 \|Bx\|^4,$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ in this inequality produces the desired result (4.4).

Now, by utilizing the result from [5] (see also [7, p. 29]), namely:

If $\gamma, \Gamma \in \mathbb{K}$ with $\gamma + \Gamma \neq 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.9) holds true, then

$$(4.12) \quad 0 \leq \|u\| \|v\| - |\langle u, v \rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|v\|^2.$$

Now, on making use of (4.12) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha$, $\Gamma = \beta$ and using the same procedure outlined above, we deduce the second inequality (4.5).

The inequality (4.6) follows from the result presented below obtained in [4] (see also [7, p. 21]):

If $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.9) holds true, then

$$(4.13) \quad \|u\| \|v\| \leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle u, v \rangle|,$$

by choosing $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha$, $\Gamma = \beta$ and taking the supremum over $\|x\| = 1$.

Finally, on making use of the inequality (see [6])

$$(4.14) \quad \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right) |\langle u, v \rangle| \|v\|^2$$

that is valid provided $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.9) holds true, we obtain the last inequality (4.7). The details are omitted. ■

Remark 4. Let $M > m > 0$ and $A, B \in B(H)$. If $C_{m,M}(A, B)$ is accretive, then

$$(0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \frac{1}{4} \cdot (M - m)^2 \|B\|^4,$$

$$(0 \leq) \mu(A, B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M} \|B\|^2,$$

$$(1 \leq) \frac{\mu(A, B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{m + M}{\sqrt{mM}}$$

and

$$(0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \left(\sqrt{M} - \sqrt{m} \right)^2 w(B^*A) \|B\|^2,$$

respectively.

Corollary 10. Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$. If $C_{\alpha,\beta}(A, A^*)$ is accretive, then

$$(0 \leq) \mu^2(A, A^*) - w^2(A^2) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|A\|^4.$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(0 \leq) \mu(A, A^*) - w(A^2) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|A\|^2.$$

In addition, if $\operatorname{Re}(\alpha\bar{\beta}) > 0$, then also

$$(1 \leq) \frac{\mu(A, A^*)}{w(A^2)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}(\alpha\bar{\beta})}}$$

and

$$(0 \leq) \mu^2(A, A^*) - w^2(A^2) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}(\alpha\bar{\beta})} \right) w(A^2) \|A\|^2,$$

respectively.

Remark 5. In a similar manner, if $N > n > 0$, $A \in B(H)$ and $C_{n,N}(A, A^*)$ is accretive, then

$$\begin{aligned} (0 \leq) \mu^2(A, A^*) - w^2(A^2) &\leq \frac{1}{4} \cdot (N - n)^2 \|A\|^4, \\ (0 \leq) \mu(A, A^*) - w(A^2) &\leq \frac{1}{4} \cdot \frac{(N - n)^2}{n + N} \|A\|^2, \\ (1 \leq) \frac{\mu(A, A^*)}{w(A^2)} &\leq \frac{1}{2} \cdot \frac{n + N}{\sqrt{nN}} \end{aligned}$$

and

$$(0 \leq) \mu^2(A, A^*) - w^2(A^2) \leq (\sqrt{N} - \sqrt{n})^2 w(A^2) \|A\|^2,$$

respectively.

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