MONOTONICITY PROPERTIES OF THE FUNCTIONS INVOLVING THE PSI FUNCTION

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Abstract. The monotonicity properties of the functions involving the psi function are considered. An inequality for the Euler's constant is obtained as consequence

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in I \quad \text{and} \quad n = 0, 1, 2, \ldots. \tag{1}$$

If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} \, \mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$, see [7, p.161]. A detailed collection of the most important properties of completely monotonic functions can be found in [7, Chapter IV].

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t \quad (x > 0)$$

is one of the most important functions in analysis and its applications. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [1, pp. 259-260] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, \mathrm{d}t, \tag{2}$$

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} \, \mathrm{d}t \tag{3}$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n \to \infty} D_n, \quad \text{where} \quad D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

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In 1991, R. M. Young [6] presented an elegant geometrical proof for the double inequality

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \quad n = 1, 2, \ldots. \tag{4}$$

In [2, 3, 4, 5], other bounds for $D_n - \gamma$ were established. Since

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k},$$

the inequality (4) can be written as

$$\frac{1}{2(n+1)} < \psi(n+1) - \log n < \frac{1}{2n}, \quad n = 1, 2, \ldots. \tag{5}$$

Define for $x > 0$,

$$f(x) = (x+1)\left[\psi(x+1) - \log x\right] \quad \text{and} \quad g(x) = x\left[\psi(x+1) - \log x\right].$$

In this paper, we prove that f is strictly completely monotonic on $(0, \infty)$, and g is a so-called Bernstein function on $(0, \infty)$. That is, $g > 0$ and g' is strictly completely monotonic on $(0, \infty)$.

From the representations [1, pp. 258-259]

$$\psi(x) = \log x - \frac{1}{2x} + O(x^{-2}),$$
$$\psi(x+1) = \psi(x) + \frac{1}{x},$$

we conclude

$$\lim_{x\to\infty} f(x) = \frac{1}{2} \quad \text{and} \quad \lim_{x\to\infty} g(x) = \frac{1}{2},$$

the inequalities (4) are immediate consequences of the fact that f is strictly decreasing on $(0, \infty)$ and g is strictly increasing on $(0, \infty)$.

Theorem 1. *The function $f(x) = (x+1)\left[\psi(x+1) - \log x\right]$ is strictly completely monotonic on $(0, \infty)$, and $g(x) = x\left[\psi(x+1) - \log x\right]$ is a Bernstein function on $(0, \infty)$.*

Proof. Using the representations [1, p. 259]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt \tag{6}$$

and

$$\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt, \tag{7}$$

we imply

$$f(x) = (x+1)\int_0^\infty \delta(t) e^{-(x+1)t} dt,$$

where

$$\delta(t) = \frac{1}{t} - \frac{1}{e^t - 1}, t > 0.$$

Easy computations reveal that the function δ is strictly decreasing on $(0, \infty)$ with $\lim_{x\to 0} \delta(t) = \frac{1}{2}$ and $\lim_{x\to 0} \delta(t) = 0$.

For $x>0, n=0,1,2,\ldots$, we have

$$\begin{aligned}
(-1)^n f^{(n)}(x) &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (x+1)^{(n-k)} \left(\int_0^\infty \delta(t) e^{-(x+1)t} dt\right)^{(k)} \\
&= (x+1)\int_0^\infty \delta(t) e^{-(x+1)t} t^n dt - n\int_0^\infty \delta(t) e^{-(x+1)t} t^{n-1} dt \\
&= \int_0^{n/(x+1)} \delta(t) e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt \\
&\qquad + \int_{n/(x+1)}^\infty \delta(t) e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt \qquad (8) \\
&> \delta(n/(x+1)) \int_0^{n/(x+1)} e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt \\
&\qquad + \delta(n/(x+1)) \int_{n/(x+1)}^\infty e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt \\
&= \delta(n/(x+1)) \int_0^\infty e^{-(x+1)t} t^{n-1} [(x+1)t-n] dt.
\end{aligned}$$

Since

$$\frac{m!}{(x+s)^{m+1}} = \int_0^\infty t^m e^{-(x+s)t}\,\mathrm{d}t \quad (x>0; s\ge 0, m=0,1,2,\ldots), \tag{9}$$

we conclude

$$\int_0^\infty e^{-(x+1)t} t^{n-1} [(x+1)t-n]\,\mathrm{d}t = 0,$$

so that (8) implies

$$(-1)^n f^{(n)}(x) > 0 \ (x>0, n=0,1,2,\ldots).$$

Hence, the function f is strictly completely monotonic on $(0,\infty)$

Using the representations (6) and (7), we imply

$$g(x) = x\int_0^\infty \omega(t) e^{-xt} dt, \tag{10}$$

where

$$\omega(t) = \frac{e^t}{t} - \frac{1}{e^t-1} - 1, t>0. \tag{11}$$

Easy computations reveal that

$$\frac{t^2(e^t-1)^2}{e^t}\omega'(t) = \sum_{n=4}^\infty \left[(n-2)2^{n-1} - 2(n-1)\right] \frac{t^n}{n!} > 0, t>0, \tag{12}$$

hence, the function ω is strictly increasing on $(0,\infty)$, and $\omega(t) > \lim_{t\to 0}\omega(t) = \frac{1}{2}$, and then $g(x) > 0$. For $x > 0, n = 1, 2, \ldots$, we have

$$\begin{aligned}
(-1)^n g^{(n)}(x) &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} \left(\int_0^\infty \omega(t) e^{-xt} dt\right)^{(k)} \\
&= x \int_0^\infty \omega(t) e^{-xt} t^n dt - n \int_0^\infty \omega(t) e^{-xt} t^{n-1} dt \\
&= \int_0^{n/x} \omega(t) e^{-xt} t^{n-1} (xt - n) dt \\
&\qquad + \int_{n/x}^\infty \omega(t) e^{-xt} t^{n-1} (xt - n) dt \qquad (13) \\
&< \omega(n/x) \int_0^{n/x} e^{-xt} t^{n-1} (xt - n) dt \\
&\qquad + \omega(n/x) \int_{n/x}^\infty e^{-xt} t^{n-1} (xt - n) dt \\
&= \omega(n/x) \int_0^\infty e^{-xt} t^{n-1} (xt - n) dt = 0.
\end{aligned}$$

Hence, g is a Bernstein function on $(0,\infty)$. The proof of Theorem 1 is complete. □

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