# SCHUR-CONVEXITY AND SCHUR-GEOMETRICALLY CONCAVITY OF SEIFFERT'S MEAN

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ABSTRACT. The Schur-convexity and the Schur-geometrically convexity with variables $(a, b) \in \mathbb{R}^2_{++}$ of Seiffert's mean are discussed. Some new inequalities are obtained.

## 1. Introduction

Throughout the paper we assume that the set of the real number, the nonnegative real number and the positive real number by $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

Let$(a, b) \in \mathbb{R}^2_{++}$. In 1993, H.-J. Seiffert [1] introduced the mean

$$P = P(a,b) = \begin{cases} \frac{a-b}{4\arctan\sqrt{a/b}-\pi}, & a \neq b, \\ a, & a = b. \end{cases}$$

In recent years, some further generalizations and applications about Seifferts mean have been obtained in [2],[3],[4] and [5] and the references therein.

In this paper, the Schur-convexity and the Schur-geometrically convexity with variables $(a, b) \in \mathbb{R}^2_{++}$ of Seiffert's mean are discussed. Besides, some new inequalities are obtained.

## 2. Definitions and Lemmas

We need the following definitions and lemmas.

**Definition 1** ([6, 7] )**.** Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$. let $\Omega \subset \mathbb{R}^n$, $\varphi$: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. $\varphi$ is said to be decreasing if and only if $-\varphi$ is increasing.

(*iii*) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where $\alpha$ and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(*iv*) let $\Omega \subset \mathbb{R}^n$, $\varphi$: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on $\Omega$ if $\boldsymbol{x} \prec \boldsymbol{y}$ on $\Omega$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. $\varphi$ is said to be a Schur-concave function on $\Omega$ if and only if $-\varphi$ is Schur-convex function.

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**Definition 2** ([8, 9])**.** Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$.

(*i*) $\Omega \subset \mathbb{R}^n_{++}$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where $\alpha$ and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(*ii*) let $\Omega \subset \mathbb{R}^n_{++}$, $\varphi$: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on $\Omega$ if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on $\Omega$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. $\varphi$ is said to be a Schur-geometrically concave function on $\Omega$ if and only if $-\varphi$ is Schur-geometrically convex function.

**Lemma 1** ([6, 7] )**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior set. $\Omega^0$ is the interior of $\Omega$. $\varphi : \Omega \to \mathbb{R}$ is continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\varphi$ is the $Schur-convex(Schur-concave) function$, if and only if $\varphi$ is symmetric on $\Omega$ and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0)$$

*holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.*

**Lemma 2** ([8] )**.** *Let $\Omega \subset \mathbb{R}^n_{++}$ is a symmetric and has a nonempty interior geometrically convex set. $\Omega^0$ is the interior of $\Omega$. $\varphi : \Omega \to \mathbb{R}_+$ is continuous on $\Omega$ and differentiable in $\Omega^0$. If $\varphi$ is symmetric on $\Omega$ and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0)$$

*holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then $\varphi$ is the Schur-geometrically convex ( Schur-geometrically concave) function.*

**Lemma 3.** *Let $a \leq b, u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$. If $1/2 \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq 1$, then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{1}$$

*Proof.* Case 1. When $1/2 \leq t_2 \leq t_1 \leq 1$, it is easy to see that $u(t_1) \geq v(t_1), u(t_2) \geq v(t_2), u(t_1) \geq u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = a + b$, this is (1) holds.

Case 2. When $0 \leq t_1 \leq t_2 \leq 1$, then $1/2 \leq 1 - t_2 \leq 1 - t_1 \leq 1$, by the Case 1, it follows

$$(u(1-t_2), v(1-t_2)) \prec (u(1-t_1), v(1-t_1)),$$

i.e. $(u(t_2), v(t_2)) \prec (u(t_1), v(t_1))$. □

**Lemma 4** ([10])**.** *Let $0 \leq a \leq b, c \geq 0$. Then*

$$\left(\frac{a+c}{a+b+2c}, \frac{b+c}{a+b+2c}\right) \prec \left(\frac{a}{a+b}, \frac{b}{a+b}\right). \tag{2}$$

## 3. Main results and their proofs

In the following, we are in a position to state our main results and give proofs of them.

**Theorem 1.** *$P(a, b)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_{++}$.*

*Proof.* Firstly, it is easy to check that $P(a, b) = P(b, a)$. In fact, for $(a, b) \in \mathbb{R}^2_{++}$ with $a \neq b$, we have

$$P(a, b) = \frac{a - b}{4\int_1^{\sqrt{a/b}} \frac{1}{1+t^2}\, dt},$$

by substitution $t = 1/s$, it follows

$$P(a,b) = \frac{b-a}{4\int_1^{\sqrt{b/a}} \frac{1}{1+s^2}\, ds} = P(b,a).$$

Secondly, let $u = u(a,b) = \sqrt{a/b}$ and $\varphi = \varphi(a,b) = 4\arctan u - \pi$, then $P(a,b) = (a-b)/\varphi$, and

$$\begin{aligned}
\Delta :&= (b-a)\left(\frac{\partial P(a,b)}{\partial a} - \frac{\partial P(a,b)}{\partial b}\right) \\
&= \frac{a-b}{\varphi^2}\left[2\varphi + (a-b)\left(\frac{\partial \varphi}{\partial b} - \frac{\partial \varphi}{\partial a}\right)\right] \\
&= \frac{a-b}{\varphi^2}\left[2\varphi + (a-b)\left(\frac{4}{1+u^2}\frac{\partial u}{\partial b} - \frac{4}{1+u^2}\frac{\partial u}{\partial a}\right)\right] \\
&= \frac{a-b}{\varphi^2}\left[2\varphi + \frac{4(a-b)}{1+u^2}\left(-\frac{a}{2ub^2} - \frac{1}{2ub}\right)\right] \\
&= \frac{a-b}{\varphi^2}\left[2\varphi - \frac{2(a-b)}{u(1+u^2)}\left(\frac{a}{b^2} + \frac{1}{b}\right)\right] \\
&= \frac{2(a-b)}{\varphi^2}\left(\varphi + \frac{1}{u} - u\right) = \frac{2(a-b)}{\varphi^2} f(u),
\end{aligned}$$

where

$$f(u) = \varphi + \frac{1}{u} - u = 4\arctan u - \pi + \frac{1}{u} - u.$$

Since

$$f^{'}(u) = \frac{4}{1+u^2} - \frac{1}{u^2} - 1 = \frac{-u^4 + 2u^2 - 1}{u^2(1+u^2)} = -\frac{(u^2-1)^2}{u^2(1+u^2)} \le 0,$$

$f(u)$ is decreasing with $u$ on $\mathbb{R}_{++}$. And then if $a < b$, i.e. $u < 1$, then $f(u) \ge f(1) = 0$, further $\Delta \le 0$; if $a > b$, i.e. $u > 1$, then $f(u) \le f(1) = 0$, further $\Delta \le 0$. According to Lemma 1, it follows that $P(a,b)$ is Schur-concave with $(a,b) \in \mathbb{R}^2_{++}$.

The proof is complete. □

**Theorem 2.** *$P(a,b)$ is the Schur-geometrically convex with $(a,b) \in \mathbb{R}^2_{++}$.*

*Proof.* Let $u = u(a,b) = \sqrt{a/b}$ and $\varphi = \varphi(a,b) = 4\arctan u - \pi$, then $P(a,b) = (a-b)/\varphi$, and

$$\begin{aligned}
\Lambda :&= (\ln b - \ln a)\left(a\frac{\partial P(a,b)}{\partial a} - b\frac{\partial P(a,b)}{\partial b}\right)\\
&= (\ln b - \ln a)\left[\frac{a}{\varphi^2}\left(\varphi - (a-b)\frac{\partial \varphi}{\partial a}\right) - \frac{b}{\varphi^2}\left(\varphi - (a-b)\frac{\partial \varphi}{\partial b}\right)\right]\\
&= \frac{\ln a - \ln b}{\varphi^2}\left[(a+b)\varphi - (a-b)\left(b\frac{\partial \varphi}{\partial b} - a\frac{\partial \varphi}{\partial a}\right)\right]\\
&= \frac{\ln a - \ln b}{\varphi^2}\left[(a+b)\varphi + (a-b)\left(\frac{4b}{1+u^2}\frac{\partial u}{\partial b} - \frac{4a}{1+u^2}\frac{\partial u}{\partial a}\right)\right]\\
&= \frac{\ln a - \ln b}{\varphi^2}\left[(a+b)\varphi + (a-b)\frac{4}{1+u^2}\left(b\frac{\partial u}{\partial b} - a\frac{\partial u}{\partial a}\right)\right]\\
&= \frac{\ln a - \ln b}{\varphi^2}\left[(a+b)\varphi + (a-b)\frac{4u}{1+u^2}\right]\\
&= b\frac{\ln a - \ln b}{\varphi^2}\left[(u^2+1)\varphi - (u^2-1)\frac{4u}{1+u^2}\right]\\
&= b\frac{\ln a - \ln b}{\varphi^2}(u^2+1)\left[\varphi - \frac{4u(u^2-1)}{(1+u^2)^2}\right]\\
&= \frac{b(\ln a - \ln b)(u^2+1)}{\varphi^2}\left[\varphi - \frac{4u^3}{(1+u^2)^2} + \frac{4u}{(1+u^2)^2}\right]\\
&= \frac{b(a-b)(u^2+1)}{\varphi^2}\cdot\frac{\ln a - \ln b}{a-b}\cdot g(u),
\end{aligned}$$

where

$$g(u) = \arctan u - \pi - \frac{4u^3}{(1+u^2)^2} + \frac{4u}{(1+u^2)^2}.$$

Since

$$\begin{aligned}
g^{'}(u) &= \frac{4}{1+u^2} - 4\cdot\frac{3u^2(1+u^2)^2 - 2u^3(1+u^2)2u}{(1+u^2)^4} + 4\cdot\frac{(1+u^2)^2 - 4u^2(1+u^2)}{(1+u^2)^4}\\
&= 4\left[\frac{(1+u^2)^2}{(1+u^2)^3} - \frac{3u^2+3u^4-4u^4}{(1+u^2)^3} + \frac{1+u^2-4u^2}{(1+u^2)^3}\right]\\
&= \frac{4}{(1+u^2)^3}\left(u^4+2u^2+1-3u^2+u^4+1+u^2-4u^2\right)\\
&= \frac{4}{(1+u^2)^3}\left(2u^4-4u^2+2\right)\\
&= \frac{8(u^4-2u^2+1)}{(1+u^2)^3}\\
&= \frac{8(u^2-1)^2}{(1+u^2)^3} \geq 0,
\end{aligned}$$

$g(u)$ is increasing with $u$ on $\mathbb{R}_{++}$. And then if $a < b$, i.e. $u < 1$, then $g(u) \leq g(1) = 0$, further $\Lambda \leq 0$; if $a > b$, i.e. $u > 1$, then $g(u) \geq g(1) = 0$, further $\Lambda \geq 0$. According to Lemma 2, it follows that $P(a,b)$ is Schur-geometrically convex with $(a,b) \in \mathbb{R}^2_{++}$.

The proof is complete. □

## 4. Applications

**Theorem 3.** *Let* $a \le b, u(t) = tb+(1-t)a, v(t) = ta+(1-t)b$*. If* $1/2 \le t_2 \le t_1 \le 1$ *or* $0 \le t_1 \le t_2 \le 1$*, then we have*

$$\begin{aligned} G(a,b) &\le P\left(a^{u(t_1)}b^{v(t_1)}, a^{v(t_1)}b^{u(t_1)}\right) \le P\left(a^{u(t_2)}b^{v(t_2)}, a^{v(t_2)}b^{u(t_2)}\right) \\ &\le P(a,b) \le P\left(u(t_2), v(t_2)\right) \le P\left(u(t_1), v(t_1)\right) \le A(a,b). \end{aligned} \tag{3}$$

*Proof.* Combining Lemma 3 with Theorem 1, we have

$$P(a,b) \le P\left(u(t_2), v(t_2)\right) \le P\left(u(t_1), v(t_1)\right) \le A(a,b).$$

On the other hand, since

$$\begin{aligned} \left(\ln\sqrt{ab}, \ln\sqrt{ab}\right) &\prec \left(\ln a^{u(t_1)}b^{v(t_1)}, \ln a^{v(t_1)}b^{u(t_1)}\right) \\ &\prec \left(\ln a^{u(t_2)}b^{v(t_2)}, \ln a^{v(t_2)}b^{u(t_2)}\right) \prec (\ln a, \ln b), \end{aligned}$$

we have

$$\begin{aligned} G(a,b) = P\left(\ln\sqrt{ab}, \ln\sqrt{ab}\right) &\le P\left(a^{u(t_1)}b^{v(t_1)}, a^{v(t_1)}b^{u(t_1)}\right) \\ &\le P\left(a^{u(t_2)}b^{v(t_2)}, a^{v(t_2)}b^{u(t_2)}\right) \le P(a,b). \end{aligned}$$

The proof is complete. □

**Theorem 4.** *Let* $0 \le a \le b, c \ge 0$*. Then*

$$(a+b+2c)\left(\arctan\sqrt{\frac{a+c}{b+c}} - (a+b)\arctan\sqrt{\frac{a}{b}}\right) \ge \frac{c\pi}{2}. \tag{4}$$

*Proof.* By Lemma 4 and Theorem 1, it follows that

$$P\left(\frac{a+c}{a+b+2c}, \frac{b+c}{a+b+2c}\right) \ge P\left(\frac{a}{a+b}, \frac{b}{a+b}\right).$$

i.e.

$$\frac{\frac{a-b}{a+b+2c}}{4\arctan\sqrt{\frac{a+c}{b+c}} - \pi} \ge \frac{\frac{a-b}{a+b}}{4\arctan\sqrt{\frac{a}{b}} - \pi},$$

making a little deformation,it deduce to (4).

The proof is complete. □

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