A NEW PROOF OF COMPLETE MONOTONICITY OF A FUNCTION INVOLVING PSI FUNCTION

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Abstract. A new proof for the conclusion that the function $x^{\alpha}[\ln x-\psi(x)]$ is completely monotonic in $(0,\infty)$ if and only if $\alpha\le 1$ is provided.

1. Introduction

Recall [16] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x)\ge 0 \tag{1}$$

for all $x\in I$ and $n\in\mathbb{N}\cup\{0\}$. The well-known Bernstein's Theorem in [16, p. 160, Theorem 12a] states that a function f on $[0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\alpha(t) \tag{2}$$

converges for $x\in[0,\infty)$.

Recall also [4, 5, 8, 12, 14] that a positive function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n[\ln f(x)]^{(n)}\ge 0 \tag{3}$$

for all $x\in I$ and $n\in\mathbb{N}$.

It was proved explicitly in [5, 12, 14] that a logarithmically completely monotonic function must be completely monotonic. For more information on the logarithmically completely monotonic functions, please refer to [5, 15] and the references therein.

It is well-known that Euler's gamma function is defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t \tag{4}$$

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for $\Re z > 0$. The logarithmic derivative of $\Gamma(z)$, denoted by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{5}$$

is called the psi or digamma function, and $\psi^{(k)}$ for $k \in \mathbb{N}$ are called the polygamma functions.

In [3], the function

$$\theta(x) = x[\ln x - \psi(x)] \tag{6}$$

was proved to be decreasing and convex in $(0, \infty)$ and two limits

$$\lim_{x\to 0^+} \theta(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} \theta(x) = \frac{1}{2} \tag{7}$$

were presented complicatedly.

In [2, p. 374], it was pointed out that the limits in (7) can follow immediately from the representations

$$\theta(x) = x\ln x - x\psi(x+1) + 1 \quad \text{and} \quad \theta(x) = \frac{1}{2} + \frac{1}{12x} - \frac{\tau}{120x^3}$$

for $x > 0$ and $\tau \in (0, 1)$.

From (7) and the decreasing monotonicity of $\theta(x)$, the inequality

$$\frac{1}{2x} < \ln x - \psi(x) < \frac{1}{x} \tag{8}$$

for $x > 0$ is concluded. This extends a result in [10] which says that inequality (8) is valid for $x > 1$. Refinements and generalizations of (8) were given in [7, 11, 13] and related references therein.

In [9], by employing the monotonicity of $\theta(x)$, it was recovered simply that the double inequality

$$\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-1/2}}{e^{x-1}} \tag{9}$$

holds for $x > 1$, the constants γ and $\frac{1}{2}$ are the best possible, the left-hand side inequality in (9) holds also for $0 < x < 1$, but the right-hand side inequality in (9) reverses, where γ is Euler-Mascheroni's constant. Furthermore, by virtue of the decreasingly monotonic property and convexity of $\theta(x)$, it was showed in [9] that the function

$$h(x) = \frac{e^x\Gamma(x)}{x^{x-\theta(x)}} \tag{10}$$

in $(0, \infty)$ has a unique maximum e at $x = 1$, with two limits

$$\lim_{x\to 0^+} h(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} h(x) = \sqrt{2\pi}\,. \tag{11}$$

Consequently, three sharp inequalities

$$\frac{x^{x-\theta(x)}}{e^x} < \Gamma(x) \le \frac{x^{x-\theta(x)}}{e^{x-1}} \tag{12}$$

in $(0, 1]$,

$$\frac{\sqrt{2\pi}\,x^{x-\theta(x)}}{e^x} < \Gamma(x) \le \frac{x^{x-\theta(x)}}{e^{x-1}} \tag{13}$$

in $[1, \infty)$, and

$$I(x,y) < \left\{\frac{x^{\theta(x)}\Gamma(x)}{y^{\theta(y)}\Gamma(y)}\right\}^{1/(x-y)} \tag{14}$$

for $x \ge 1$ and $y \ge 1$ with $x \ne y$, where

$$I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \tag{15}$$

for $a>0$ and $b>0$ with $a\ne b$ is called the identric or exponential mean, are deduced directly. If $0<x\le 1$ and $0<y\le 1$ with $x\ne y$, inequality (14) is reversed.

In [2, pp. 374–375, Theorem 1], by using the well-known Binet's formula and complicated calculating techniques for integrals, the monotonicity and convexity of $\theta(x)$ was extended to the complete monotonicity: For real number α, the function

$$\theta_\alpha(x)=x^\alpha[\ln x-\psi(x)] \tag{16}$$

is completely monotonic in $(0,\infty)$ if and only if $\alpha\le 1$.

The aim of this paper is to give a new proof for the complete monotonicity of the function $\theta_\alpha(x)$, which can be restated as the following Theorem 1, since this function $\theta_\alpha(x)$ has so many meaningful applications as stated above.

Theorem 1. *For real number α, the function $\theta_\alpha(x)$ defined by* (16) *is completely monotonic in* $(0,\infty)$ *if and only if $\alpha\le 1$, with two limits*

$$\lim_{x\to0^+}\theta_1(x)=1,\quad \lim_{x\to\infty}\theta_1(x)=\frac12 \tag{17}$$

and, for $\alpha<1$,

$$\lim_{x\to0^+}\theta_\alpha(x)=\infty,\quad \lim_{x\to\infty}\theta_\alpha(x)=0. \tag{18}$$

2. Lemmas

In order to prove Theorem 1, the following lemmas are necessary.

Lemma 1 ([1])**.** *For $i\in\mathbb{N}$, $x>0$, $a>0$ and $b>0$,*

$$\psi^{(i-1)}(x+1)=\psi^{(i-1)}(x)+\frac{(-1)^{i-1}(i-1)!}{x^i}, \tag{19}$$

$$\ln\frac{b}{a}=\int_0^\infty\frac{e^{-at}-e^{-bt}}{t}\,\mathrm{d}t, \tag{20}$$

$$\psi^{(i)}(x)=(-1)^{i+1}\int_0^\infty\frac{t^ie^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{21}$$

$$\psi(x)-\ln x+\frac1x=\int_0^\infty\left(\frac1t-\frac{1}{e^t-1}\right)e^{-xt}\,\mathrm{d}t. \tag{22}$$

Lemma 2 ([13])**.** *For $x>0$,*

$$\frac{1}{2x}-\frac{1}{12x^2}<\psi(x+1)-\ln x<\frac{1}{2x}, \tag{23}$$

$$\frac{1}{2x^2}-\frac{1}{6x^3}<\frac1x-\psi'(x+1)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}. \tag{24}$$

Lemma 3. *Inequalities*

$$\ln x-\frac1x\le\psi(x)\le\ln x-\frac{1}{2x} \tag{25}$$

and

$$\frac{(k-1)!}{x^k}+\frac{k!}{2x^{k+1}}\le(-1)^{k+1}\psi^{(k)}(x)\le\frac{(k-1)!}{x^k}+\frac{k!}{x^{k+1}} \tag{26}$$

hold in $(0,\infty)$ *for* $k\in\mathbb{N}$.

Proof. In [11], the function $\psi(x)-\ln x+\frac{\alpha}{x}$ was proved to be completely monotonic in $(0,\infty)$ if and only if $\alpha\ge1$ and so was its negative if and only if $\alpha\le\frac12$. In [6], the function $\frac{e^x\Gamma(x)}{x^{x-\alpha}}$ was proved to be logarithmically completely monotonic in $(0,\infty)$ if and only if $\alpha\ge1$ and so was its reciprocal if and only if $\alpha\le\frac12$. From these, considering (1) and (3), inequalities in (26) are derived straightforwardly. $\square$

Lemma 4. *If* $f(x)$ *is a function defined in an infinite interval* I *such that* $f(x)-f(x+\varepsilon)>0$ *and* $\lim_{x\to\infty}f(x)=\delta$ *for* $x\in I$ *and some* $\varepsilon>0$, *then* $f(x)>\delta$ *in* I.

Proof. By induction, for any $x\in I$,

$$f(x)>f(x+\varepsilon)>f(x+2\varepsilon)>\cdots>f(x+k\varepsilon)\to\delta$$

as $k\to\infty$. The proof of Lemma 4 is complete. $\square$

3. A new proof of Theorem 1

Straightforward computation gives

$$\begin{aligned}\theta_1(x+1)-\theta_1(x)&=(x+1)\ln(x+1)-x\ln x+x[\psi(x)-\psi(x+1)]-\psi(x+1)\\&=(x+1)\ln(x+1)-x\ln x-\psi(x+1)-1\end{aligned}$$

and

$$\begin{aligned}[\theta_1(x+1)-\theta_1(x)]'&=\ln(x+1)-\ln x-\psi'(x+1)\\&=\int_0^\infty\left[\frac{1-e^{-t}}{t}-\frac{te^{-t}}{1-e^{-t}}\right]e^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\frac{e^{-t}+e^t-t^2-2}{t(e^t-1)}e^{-xt}\,\mathrm{d}t\\&>0\end{aligned}$$

by using formulas (19), (20) and (21). Hence,

$$(-1)^i[\theta_1(x+1)-\theta_1(x)]^{(i)}=\left[(-1)^i\theta_1^{(i)}(x+1)\right]-\left[(-1)^i\theta_1^{(i)}(x)\right]<0$$

in $(0,\infty)$ for $i\in\mathbb{N}$.

Using inequality (23) yields

$$\begin{aligned}(x+1)\ln\left(1+\frac1x\right)-\frac1{2x}-1<\theta_1(x+1)-\theta_1(x)\\<(x+1)\ln\left(1+\frac1x\right)-\frac1{2x}+\frac1{12x^2}-1,\end{aligned}$$

this implies that $\lim_{x\to\infty}[\theta_1(x+1)-\theta_1(x)]=0$. Since the function $\theta_1(x+1)-\theta_1(x)$ is increasing in $(0,\infty)$, it is obtained that $\theta_1(x+1)-\theta_1(x)<0$ in $(0,\infty)$.

Utilizing (19) and (23) leads easily to $\lim_{x\to\infty}\theta_1(x)=\frac12$.

Direct calculation gives $\theta_1'(x)=\ln x-\psi(x)-x\psi'(x)+1$ and

$$\theta_1^{(i)}(x)=\frac{(-1)^i(i-2)!}{x^{i-1}}-i\psi^{(i-1)}(x)-x\psi^{(i)}(x)$$

for $i\ge2$. Combination of (19), (23) and (24) yields that $\lim_{x\to\infty}\theta_1'(x)=0$. Inequality (26) means that $\lim_{x\to\infty}\theta_1^{(i)}(x)=0$ for $i\ge2$.

By above argument and Lemma 4, it is concluded that $(-1)^k\theta_1^{(k)}(x) \ge 0$ in $(0,\infty)$ for $k \ge 0$, which says that the function $\theta_1(x)$ is completely monotonic in $(0,\infty)$ with $\lim_{x\to\infty}\theta_1(x) = \frac{1}{2}$.

The validity of the limit $\lim_{x\to 0^+}\theta_1(x) = 1$ follows from formula (22).

It is clear that $\theta_\alpha(x) = x^{\alpha-1}\theta_1(x)$ and $x^{\alpha-1}$ is also completely monotonic in $(0,\infty)$ for $\alpha < 1$. Since the product of any finite completely monotonic functions on an interval I is also completely monotonic on I, the function $\theta_\alpha(x)$ is completely monotonic in $(0,\infty)$ for $\alpha < 1$.

Conversely, if the function $\theta_\alpha(x)$ is completely monotonic in $(0,\infty)$, then $\theta_\alpha(x)$ is decreasing and positive in $(0,\infty)$. From formula (19) and inequality (23), it follows that

$$\frac{1}{2x} + \frac{1}{12x^2} > \ln x - \psi(x) > \frac{1}{2x} \tag{27}$$

and

$$\frac{1}{2x^{1-\alpha}} + \frac{1}{12x^{2-\alpha}} > x^\alpha[\ln x - \psi(x)] > \frac{1}{2x^{1-\alpha}} \tag{28}$$

for $x > 0$, which means that $x^\alpha[\ln x - \psi(x)]$ tends to ∞ as $x \to \infty$ if $\alpha > 1$. This contradicts with the decreasingly monotonic property of $\theta_\alpha(x)$ in $(0,\infty)$. Hence, the necessary condition $\alpha \le 1$ follows.

It is obvious that inequality (28) implies the two limits in (18). The proof of Theorem 1 is complete.

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