ASYMPTOTIC REPRESENTATIONS FOR STOLARSKY, GINI AND THE GENERALIZED MUIRHEAD MEANS

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Abstract. We establish the asymptotic representations relating to Stolarsky, Gini and the generalized Muirhead means, and we prove the monotonicity results for the generalized logarithmic mean and the power mean, and we consider Schur-convexity of the generalized Muirhead mean $\sum_{r,s}(a,b)$ with respect to (r,s). Finally, we pose three conjectures.

1. Introduction

A mean in two variables is any function $M:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ (where $\mathbb{R}_+$ denotes the set of positive reals), satisfying for all $a,b>0$ the following conditions:

(i) $M(a,b)=M(b,a)$ (symmetry property);

(ii) $\min(a,b)\le M(a,b)\le \max(a,b)$ (mean valus property).

Sometimes, condition (ii) is replaced by the weaker requirement (e.g. [1]).

(iii) $M(a,a)=a$ (reflexivity property).

Clearly, (ii) imples (iii), but the converse is not always true.

The following two-parameter families of bivariabte means have evoked the interest of many mathematicians in the last three decades.

The first family is that of Stolarsky means (sometimes called difference means). Let $r,s\in\mathbb{R}$ and let $a,b>0$, the Stolarsky mean $E_{r,s}(a,b)$ of order (r,s) of a and b with $a\ne b$ are defined as

$$E_{r,s}(a,b)=\begin{cases}\left(\dfrac{r}{s}\cdot\dfrac{b^s-a^s}{b^r-a^r}\right)^{1/(s-r)}, & rs(r-s)\ne 0,\\ \exp\left(-\dfrac{1}{r}+\dfrac{a^r\ln a-b^r\ln b}{a^r-b^r}\right), & r=s\ne 0,\\ \left(\dfrac{1}{r}\cdot\dfrac{b^r-a^r}{\ln b-\ln a}\right)^{1/r}, & r\ne 0, s=0,\\ \sqrt{ab}, & r=s=0,\end{cases}\tag{1}$$

with $E_{r,s}(a,a)=a$ (see [2, 3]).

The mean $E_{r,s}(a,b)$ satisfies both (i) and (ii). Moreover, $E_{r,s}(a,b)$ are continuous on the domain

$$\{(r,s,a,b)|r,s\in\mathbb{R},a,b\in\mathbb{R}_+\}.\tag{2}$$

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In fact, the function

$$(r,s,a,b) \mapsto E_{r,s}(a,b)$$

is infinitely many times differentiable on the domain (2), see [4].

The second family is that of Gini means (sometimes called sum means). Let $r,s \in \mathbb{R}$ and let $a,b>0$, the Gini mean $G_{r,s}(a,b)$ of order (r,s) of a and b are defined in [5] as

$$G_{r,s}(a,b) = \begin{cases} \left(\dfrac{a^s+b^s}{a^r+b^r}\right)^{1/(s-r)}, & r \neq s, \\ \exp\left(\dfrac{a^r \ln a + b^r \ln b}{a^r+b^r}\right), & r = s \neq 0, \\ \sqrt{ab}, & r = s = 0. \end{cases} \tag{3}$$

The mean $G_{r,s}(a,b)$ satisfies both (i) and (ii). Moreover, $G_{r,s}(a,b)$ are continuous on the domain (2).

K. B. Stolarsky [2], Leach and Sholander [6] showed that $E_{r,s}(a,b)$ are for $a \neq b$ strictly increasing with both r and s. For $a \neq b$, $G_{r,s}(a,b)$ are also strictly increasing with both r and s, see [7, 8]. Leach and Sholander [9] and Páles [10] solved the problem of comparison of Stolarsky mean. The problem of comparison of Gini mean was completely solved by Páles [11] (see also the paper by P. Czinder and Zs. Ples [12]). A problem of comparability of Gini and Stolarsky means was addressed by Neuman and Páles [13]. The log-convexity of Gini and Stolarsky means was presented in [6, 7, 14, 2]. The Schur-convexity of Stolarsky and Gini means with respect to (r,s) were considered in [15, 16]. Minkowski-type inequality for Stolarsky and Gini means can be found in [4, 17, 18].

Since $E_{r,s}(a,b)$ are for $a \neq b$ strictly increasing with both r and s, for the particular choices of the parameters r and s, we obtain the following chain of inequalities:

$$E_{0,0}(a,b) < E_{1,0}(a,b) < E_{1,1}(a,b) < E_{2,1}(a,b) \quad \text{for} \quad a \neq b,$$

that is,

$$G(a,b) < L(a,b) < I(a,b) < A(a,b) \quad \text{for} \quad a \neq b,$$

where G, L, I and A are the geometric, logarithmic, identric and arithmetic means, respectively.

But in the literature we can find other means, not belong to the above mentioned two families. One such important mean is the generalized Muirhead (or symmetric) mean. Let $r,s \in \mathbb{R}$ and $a,b>0$, the generalized Muirhead mean $\sum_{r,s}(a,b)$ of a and b is defined by (see, for instance, [19, p. 333] or [20])

$$\sum\nolimits_{r,s}(a,b) = \left(\frac{a^r b^s + a^s b^r}{2}\right)^{1/(r+s)}, r+s \neq 0.$$

The generalized Muirhead mean $\sum_{r,s}(a,b)$ satisfies both (i) and (iii). It is not difficult to see that $\sum_{r,s}(a,b)$ satisfies the mean value property (ii) if and only if $rs > 0$. T. Trif [21] pointed out that, unlike the Stolarsky or Gini means, the generalized Muirhead mean $\sum_{r,s}(a,b)$ can not be extended continuously to the domain (2). T. Trif [21] investigated the monotonicity of $\sum_{r,s}(a,b)$ with respect to r or s. Likewise, a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead mean $\sum_{r,s}(a,b)$ are established.

For $r \in \mathbb{R} \setminus \{0\}$, $\sum_{r,0}(a,b)$ reduces to the power mean of order r of the positive real numbers a and b:

$$\sum\nolimits_{r,0}(a,b) = M_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{1/r}.$$

In the special case when $r + s = 1$, i.e., $r = \alpha, s = 1 - \alpha$, the Muirhead (or symmetric) mean is obtained:

$$\sum\nolimits_{\alpha,1-\alpha}(a,b) = \frac{a^\alpha b^{1-\alpha} + a^{1-\alpha} b^\alpha}{2}.$$

Following A. O. Pittenger [22], we write symmetric mean into the from

$$S_\delta(a,b) = \frac{a^{\frac{1+\sqrt{\delta}}{2}} b^{\frac{1-\sqrt{\delta}}{2}} + a^{\frac{1-\sqrt{\delta}}{2}} b^{\frac{1+\sqrt{\delta}}{2}}}{2}.$$

It is shown in [23] that S_δ is increasing in δ and that for $a \neq b$

$$M_0(a,b) < S_\delta(a,b) < M_1(a,b)$$

provided $0 < \delta < 1$.

The following result is known

$$S_{1/3}(a,b) < L(a,b) < M_{1/3}(a,b). \tag{4}$$

The first inequality in (4) has been established by A. O. Pittenger [22], while the second one was proven in [24, 22]. The first inequality in (4) improves a result of B. C. Carlson [25], who proved $S_{1/4}(a,b) < L(a,b)$.

E. Neuman [26] established the integral representation

$$L(a,b) = \int_0^1 a^t b^{1-t} dt. \tag{5}$$

By applying the Gausss quadrature formula with two knots

$$\int_0^1 f(t)dt = \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right) + \frac{1}{2} f\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) + \frac{1}{4320} f^{(4)}(\xi), \quad 0 < \xi < 1$$

to the function $f(t) = a^t b^{1-t}$, T. Trif [21] presented a very short proof of the first inequality in (4).

Motivated by the technique of T. Trif, we here present a very short proof of the second inequality in (4). To this aim, by applying the Simpson's $\frac{3}{8}$ rule (see [**?**, 27])

$$\begin{aligned} \int_a^b f(t)dt = \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \\ - \frac{(b-a)^5}{6480} f^{(4)}(\xi) \quad \text{for some } \xi \text{ between } a \text{ and } b, \end{aligned} \tag{6}$$

with $a = 0, b = 1$ and $f(t) = a^t b^{1-t} (0 \le t \le 1)$, we get

$$L(a,b) = M_{1/3}(a,b) - \frac{1}{6480} a^\xi b^{1-\xi} (\ln a - \ln b)^4.$$

This yields

$$L(a,b) < M_{1/3}(a,b).$$

We remark that F. Burk [28] obtained the second inequality in (4) by applying (6) to the function $f(t) = e^t$, replacing a and b with $\ln a$ and $\ln b$, respectively.

For the study of some problems related to means in [29] is used their power series expansion. In fact, for a mean M is considered series of the normalized functions $M(1,1-x), x \in (0,1)$. For example, in [30] is proved that Stolarsky mean $E_{r,s}$ has the following first terms of the power series expansion

$$\begin{aligned} &E_{r,s}(1,1-x)\\ &= 1-\frac{1}{2}x+\frac{r+s-3}{24}x^2+\frac{r+s-3}{48}x^3\\ &\quad -\left[2(r^3+r^2s+rs^2+s^3)-5(r+s)^2-70(r+s)+225\right]\frac{x^4}{5760}\\ &\quad -\left[2(r^3+r^2s+rs^2+s^3)-5(r+s)^2-30(r+s)+105\right]\frac{x^5}{3840}+\dots. \end{aligned} \tag{7}$$

The paper is organized as follows. In Section 2 we establish the asymptotic representations relating to Gini mean and Stolarsky mean, and we prove the monotonicity results for the generalized logarithmic mean and the power mean. In Section 3 we establish the asymptotic representation relating to the generalized Muirhead mean, and we prove Schur-convexity of the generalized Muirhead mean $\sum_{r,s}(a,b)$ with respect to (r,s). In Section 4 we pose three conjectures.

2. Stolarsky and Gini means

Theorem 1. *Let a, b be positive numbers and r, s be real numbers, then the following asymptotic representations hold:*

$$E_{r,s}(x+a,x+b)-x=\frac{a+b}{2}+\frac{(a-b)^2(r+s-3)}{24x}+O(x^{-2}),\quad x\to\infty, \tag{8}$$

$$G_{r,s}(x+a,x+b)-x=\frac{a+b}{2}+\frac{(a-b)^2(r+s-1)}{8x}+O(x^{-2}),\quad x\to\infty. \tag{9}$$

Proof. It is well-known that

$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\dots,\quad |x|<1, \alpha\in\mathbb{R}. \tag{10}$$

By using (10), we have

$$\begin{aligned}
&E_{r,s}(x+a,x+b)-x\\
&=\left[\frac{r}{s}\frac{(x+b)^s-(x+a)^s}{(x+b)^r-(x+a)^r}\right]^{1/(s-r)}-x\\
&=x\left[\frac{(1+b/x)^s-(1+a/x)^s}{s}\right]^{1/(s-r)}\left[\frac{(1+b/x)^r-(1+a/x)^r}{r}\right]^{1/(r-s)}-x\\
&=x\left[1+\frac{(s-1)(b+a)}{2x}+\frac{(s-1)(s-2)(b^2+ab+a^2)}{6x^2}+\dots\right]^{1/(s-r)}\\
&\quad\times\left[1+\frac{(r-1)(b+a)}{2x}+\frac{(r-1)(r-2)(b^2+ab+a^2)}{6x^2}+\dots\right]^{1/(r-s)}-x\\
&=x\left[1+\frac{1}{s-r}A(x)+\frac{\frac{1}{s-r}(\frac{1}{s-r}-1)}{2!}A^2(x)+\dots\right]\\
&\qquad\times\left[1+\frac{1}{r-s}B(x)+\frac{\frac{1}{r-s}(\frac{1}{r-s}-1)}{2!}B^2(x)+\dots\right]-x,
\end{aligned} \tag{11}$$

where

$$A(x)=\frac{(s-1)(b+a)}{2x}+\frac{(s-1)(s-2)(b^2+ab+a^2)}{6x^2}+\dots, \tag{12}$$

$$B(x)=\frac{(r-1)(b+a)}{2x}+\frac{(r-1)(r-2)(b^2+ab+a^2)}{6x^2}+\dots. \tag{13}$$

Substituting (12) and (13) into (11) simplifying leads to

$$\begin{aligned}
&E_{r,s}(x+a,x+b)-x\\
&=\frac{a+b}{2}+\frac{4(b^2+ab+a^2)(r+s-3)+3(b+a)^2(3-r-s)}{24x}+O(x^{-2})\\
&=\frac{a+b}{2}+\frac{(a-b)^2(r+s-3)}{24x}+O(x^{-2}).
\end{aligned}$$

By using (10), we have

$$\begin{aligned}
&G_{r,s}(x+a,x+b)-x\\
&=\left[\frac{(x+b)^s+(x+a)^s}{(x+b)^r+(x+a)^r}\right]^{1/(s-r)}-x\\
&=x\left[\left(1+\frac{b}{x}\right)^s+\left(1+\frac{a}{x}\right)^s\right]^{1/(s-r)}\left[\left(1+\frac{b}{x}\right)^r+\left(1+\frac{a}{x}\right)^r\right]^{1/(r-s)}-x\\
&=x\left[1+\frac{s(b+a)}{2x}+\frac{s(s-1)(b^2+a^2)}{4x^2}+\dots\right]^{1/(s-r)}\\
&\qquad\qquad\times\left[1+\frac{r(b+a)}{2x}+\frac{r(r-1)(b^2+a^2)}{4x^2}+\dots\right]^{1/(r-s)}-x \qquad (14)\\
&=x\left[1+\frac{1}{s-r}C(x)+\frac{\frac{1}{s-r}(\frac{1}{s-r}-1)}{2!}C^2(x)+\dots\right]\\
&\qquad\qquad\times\left[1+\frac{1}{r-s}D(x)+\frac{\frac{1}{r-s}(\frac{1}{r-s}-1)}{2!}D^2(x)+\dots\right]-x,
\end{aligned}$$

where

$$C(x)=\frac{s(b+a)}{2x}+\frac{s(s-1)(b^2+a^2)}{4x^2}+\dots, \tag{15}$$

$$D(x)=\frac{r(b+a)}{2x}+\frac{r(r-1)(b^2+a^2)}{4x^2}+\dots. \tag{16}$$

Substituting (15) and (16) into (14) simplifying leads to

$$\begin{aligned}
&E(r,s;x+a,x+b)-x\\
&=\frac{a+b}{2}+\frac{2(b^2+a^2)(r+s-1)+(b+a)^2(1-r-s)}{8x}+O(x^{-2})\\
&=\frac{a+b}{2}+\frac{(a-b)^2(r+s-1)}{8x}+O(x^{-2}).
\end{aligned}$$

The proof of Theorem 1 is complete. □

It is worth mentioning that

$$G_{0,r}(a,b)=E_{r,2r}(a,b)=M_r(a,b).$$

Thus the classes of Gini and Stolarsky means contain both the power means. Alzer and Ruscheweyh [31] have proven that the joint elements in the classes of Gini and Stolarsky means are exactly the power means.

Taking $s=r, r=1$ in $E_{r,s}(a,b)$, we obtain the generalized logarithmic mean $S_r(a,b)$ of two positive numbers a,b, defined in [2, 3], for $a=b$ by $S_r(a,b)=a$ and for $a\neq b$ by

$$S_r(a,b)=\begin{cases}\left(\dfrac{b^r-a^r}{r(b-a)}\right)^{1/(r-1)}, & r\neq 0,\ 1,\\[2mm] \dfrac{b-a}{\ln b-\ln a}, & r=0,\\[2mm] \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & r=1.\end{cases}$$

It follows from Theorem 1 that

$$M_r(x+a,x+b)-x=\frac{a+b}{2}+\frac{(a-b)^2(r-1)}{8x}+O(x^{-2}),\quad x\to\infty, \tag{17}$$

$$S_r(x+a,x+b)-x=\frac{a+b}{2}+\frac{(a-b)^2(r-2)}{24x}+O(x^{-2}),\quad x\to\infty, \tag{18}$$

Motivated by (17) and (18), we obtain the following theorem.

Theorem 2. *Let a,b be positive numbers with $a\neq b$, $c=\min\{a,b\}$ and r,s be real numbers, and define for $x>-c$,*

$$x\mapsto P_r(x)=M_r(x+a,x+b)-x,$$
$$x\mapsto Q_r(x)=S_r(x+a,x+b)-x,$$

where $M_r(a,b)$ is the power mean and $S_r(a,b)$ is the generalized logarithmic mean. Then, the function $x\mapsto P_r(x)$ is $\begin{matrix}\text{decreasing convex}\\ \text{increasing concave}\end{matrix}$ with $x\in(-c,\infty)$ according as $r\gtrless 1$; while the function $x\mapsto Q_r(x)$ is $\begin{matrix}\text{decreasing convex}\\ \text{increasing concave}\end{matrix}$ with $x\in(-c,\infty)$ according as $r\gtrless 2$.

Proof. Differentiating $P_r(x)$ with respect to x gives

$$\begin{aligned}P_r'(x)&=\left(\frac{(x+b)^r+(x+a)^r}{2}\right)^{1/r}\frac{(x+b)^{r-1}+(x+a)^{r-1}}{(x+b)^r+(x+a)^r}-1\\&=\frac{G_{r,0}(x+a,x+b)}{G_{r,r-1}(x+a,x+b)}-1\lesseqgtr 0\quad\text{according as}\quad r\gtrless 1,\end{aligned}\tag{19}$$

since $G_{r,s}(a,b)$ is strictly increasing with both r and s.

$$\begin{aligned}P_r''(x)&=\frac{(r-1)(2x+a+b)(x+a)^{r-2}(x+b)^{r-2}}{[(x+a)^r+(x+b)^r]^2}M_r(x+a,x+b)\\&\gtrless 0\quad\text{according as}\quad r\gtrless 1.\end{aligned}$$

Differentiating $Q_r(x)$ with respect to x gives

$$\begin{aligned}Q_r'(x)&=\frac{r[(x+b)^{r-1}-(x+a)^{r-1}]}{(r-1)[(x+b)^r-(x+a)^r]}\left[\frac{(x+b)^r-(x+a)^r}{r(b-a)}\right]^{1/(r-1)}-1\\&=\frac{E_{r,1}(x+a,x+b)}{E_{r,r-1}x+a,x+b)}-1\lesseqgtr 0\quad\text{according as}\quad r\gtrless 2,\end{aligned}\tag{20}$$

since $E_{r,s}(a,b)$ is strictly increasing with both r and s.

$$\begin{aligned}&\frac{1}{E_{1,r}(x+a,x+b)}Q_r''(x)\\&=\frac{r}{r-1}\left\{\frac{(r-1)[(x+b)^{r-2}-(x+a)^{r-2}]}{(x+b)^r-(x+a)^r}-\frac{r[(x+b)^{r-1}-(x+a)^{r-1}]^2}{[(x+b)^r-(x+a)^r]^2}\right\}\\&\qquad+\left[\frac{r}{r-1}\frac{(x+b)^{r-1}-(x+a)^{r-1}}{(x+b)^r-(x+a)^r}\right]^2\\&=(r-2)\left[\frac{1}{E^2_{r-2,r}(x+a,x+b)}-\frac{1}{E^2_{r-1,r}(x+a,x+b)}\right]\\&\gtrless 0\quad\text{according as}\quad r\gtrless 2.\end{aligned}$$

The proof of Theorem 2 is complete. □

Remark 1. Let a, b be positive numbers, and let r be real parameter. The Lehmer mean $L_r(a,b)$ of a and b is defined as

$$L_r(a,b) = \frac{a^r + b^r}{a^{r-1} + b^{r-1}}.$$

From (19) we obtain

$$M_r(a,b) \lesseqgtr L_r(a,b) \quad \text{according as} \quad r \gtreqless 1,$$

where $M_r(a,b)$ is the power mean.

Remark 2. Let a, b be positive numbers, and let r be real parameter. The one-parameter mean $J_r(a,b)$ of a and b is defined in [32, 33] as

$$\begin{aligned} J_r(a,b) &= \frac{r(b^{r+1} - a^{r+1})}{(r+1)(b^r - a^r)}, a \neq b, r \neq 0, -1; \\ J_0(a,b) &= \frac{b-a}{\ln b - \ln a} = L(a,b); \\ J_{-1}(a,b) &= \frac{[G(a,b)]^2}{L(a,b)}; \\ J_r(a,a) &= a. \end{aligned}$$

From (20) we obtain

$$S_r(a,b) \lesseqgtr J_{r+1}(a,b) \quad \text{according as} \quad r \gtreqless 2,$$

where $S_r(a,b)$ is the generalized logarithmic mean.

3. Generalized Muirhead mean

Recall now the definition of Schur-convex functions. Let I be an interval with nonempty interior, and let $f : I^n \to \mathbb{R}$. Then f is called Schur-convex on I^n $(n \geq 2)$ if $f(x) \leq f(y)$ for eath two n-tuples $x = (x_1, x_1, \ldots, x_n)$ and $y = (y_1, y_1, \ldots, y_n)$ of I^n, such that $x \prec y$. The relationship of majorization $x \prec y$ means that

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} &\leq \sum_{i=1}^{k} y_{[i]}, k = 1, 2, \ldots, n-1, \\ \sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}, \end{aligned}$$

and $x_{[i]}$ denotes the ith largest component of x.

A function f is called Schur-concave if $-f$ is Schur-convex. The following characterizations is often used in the theory of Schur-convex functions.

Lemma 1. *Let I be an open interval. Then a continuously differentiable function $f : I^2 \to \mathbb{R}$ is Schue-convex if and only if it is symmetric and satisfies the relation*

$$(y-x)\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right) \geq 0, \quad x, y \in I, x \neq y.$$

See e.g. [34, 35] for more general results, with applications.

Theorem 3. *For fixed a, b with $a, b > 0, a \neq b$. Then the generalized Muirhead mean $\sum_{r,s}(a,b)$ are Schur-convex on $\mathbb{R}_+^2$, and Schur-concave on $\mathbb{R}_-^2$, with respect to (r,s), where $\mathbb{R}_+$ $(\mathbb{R}_-)$ denotes the set of positive (negative) reals.*

Proof. Easy computation yields

$$\frac{1}{\sum_{r,s}(x,y)} \frac{\partial \sum_{r,s}(a,b)}{\partial r} = -\frac{1}{(r+s)^2} \ln\left(\frac{t^r+t^s}{2}\right) + \frac{1}{r+s} \frac{t^r \ln t}{t^r+t^s},$$

$$\frac{1}{\sum_{r,s}(a,b)} \frac{\partial \sum_{r,s}(a,b)}{\partial s} = -\frac{1}{(r+s)^2} \ln\left(\frac{t^r+t^s}{2}\right) + \frac{1}{r+s} \frac{t^s \ln t}{t^r+t^s},$$

where $t = \frac{b}{a}$.

$$\begin{aligned} &(s-r)\left(\frac{\partial \sum_{r,s}(a,b)}{\partial s} - \frac{\partial \sum_{r,s}(a,b)}{\partial r}\right) \\ &= \frac{\sum_{r,s}(a,b)}{r+s} \frac{(s-r)(t^s-t^r)\ln t}{t^r+t^s} \\ &\gtrless 0 \quad \text{for} \quad r,s \gtrless 0. \end{aligned}$$

Thus, Theorem 3 holds by Lemma 1. □

Remark 3.

$$\begin{aligned} &\frac{\partial \sum_{r,s}(a,b)}{\partial s} + \frac{\partial \sum_{r,s}(a,b)}{\partial r} \\ &= \frac{2\sum_{r,s}(a,b)}{(r+s)^2}\left[\frac{r+s}{2}\ln t - \ln\left(\frac{t^r+t^s}{2}\right)\right] < 0. \end{aligned}$$

Theorem 4. *Let a, b be positive numbers with $a \neq b$ and r, s be real numbers, then*

$$\sum\nolimits_{r,s}(x+a, x+b) - x = \frac{a+b}{2} + \frac{(a-b)^2\left[\frac{(r-s)^2}{r+s} - 1\right]}{4x} + O(x^{-2}), \quad x \to \infty. \tag{21}$$

Proof.

$$\begin{aligned} &\sum\nolimits_{r,s}(x+a, x+b) - x \\ &= x\left[\frac{(1+a/x)^r(1+b/x)^s + (1+a/x)^s(1+b/x)^r}{2}\right]^{1/(r+s)} - x. \end{aligned} \tag{22}$$

By using the power series expansion (10) we obtain

$$\begin{aligned} &\left(1+\frac{a}{x}\right)^r \left(1+\frac{b}{x}\right)^s \\ &= \left[1 + \frac{ra}{x} + \frac{r(r-1)}{2!}\frac{a^2}{x^2} + \dots\right]\left[1 + \frac{sb}{x} + \frac{s(s-1)}{2!}\frac{b^2}{x^2} + \dots\right] \\ &= 1 + \frac{ra+sb}{x} + \frac{\frac{r(r-1)}{2}a^2 + rsab + \frac{s(s-1)}{2}b^2}{x^2} + \dots \end{aligned} \tag{23}$$

and

$$\left(1+\frac{a}{x}\right)^s \left(1+\frac{b}{x}\right)^r = 1 + \frac{ra+sb}{x} + \frac{\frac{r(r-1)}{2}b^2 + rsab + \frac{s(s-1)}{2}a^2}{x^2} + \dots. \tag{24}$$

Substituting (23) and (24) into (22) simplifying leads to

$$\begin{aligned}&\sum\nolimits_{r,s}(x+a,x+b)-x\\&=x\left[1+\frac{(r+s)(a+b)}{2x}+\frac{(a^2+b^2)(r^2+s^2-r-s)+4abrs}{4x^2}+\dots\right]^{1/(r+s)}-x.\end{aligned} \tag{25}$$

Using (10) again, we get

$$\begin{aligned}&\left[1+\frac{(r+s)(a+b)}{2x}+\frac{(a^2+b^2)(r^2+s^2-r-s)+4abrs}{4x^2}+\dots\right]^{1/(r+s)}\\&=1+\frac{1}{r+s}\left[\frac{(r+s)(a+b)}{2x}+\frac{(a^2+b^2)(r^2+s^2-r-s)+4abrs}{4x^2}+\dots\right]\\&+\frac{\frac{1}{r+s}(\frac{1}{r+s}-1)}{2!}\left[\frac{(r+s)(a+b)}{2x}+\frac{(a^2+b^2)(r^2+s^2-r-s)+4abrs}{4x^2}+\dots\right]^2+\dots\\&=1+\frac{a+b}{2x}+\frac{(a^2+b^2)(r^2+s^2-r-s)+4abrs}{4(r+s)x^2}+\frac{(1-r-s)(a+b)^2}{8x^2}+\dots.\end{aligned} \tag{26}$$

Substituting (26) into (25) simplifying leads to

$$\sum\nolimits_{r,s}(x+a,x+b)-x=\frac{a+b}{2}+\frac{(a-b)^2\left[\frac{(r-s)^2}{r+s}-1\right]}{4x}+O(x^{-2}),\quad x\to\infty.$$

The proof of Theorem 4 is complete. □

In particular, let a,b be positive numbers with $a\neq b$ and α be real number, then

$$\sum\nolimits_{\alpha,1-\alpha}(x+a,x+b)-x=\frac{a+b}{2}+\frac{(a-b)^2\alpha(\alpha-1)}{x}+O(x^{-2}),\quad x\to\infty. \tag{27}$$

4. Conjectures

Define sets A and A^* by

$$A=\{(r,s)|r+s\le 3\quad\text{and}\quad e(r,s)\le e(1,2)\}$$

and

$$A^*=\{(r,s)|r+s\ge 3\quad\text{and}\quad e(r,s)\ge e(1,2)\}$$

(see Figure 1 of [36]), where e is defined by

$$e(\alpha,\beta)=\begin{cases}\dfrac{\alpha-\beta}{\ln(\alpha/\beta)}, & \alpha\beta>0,\ \alpha\neq\beta,\\ 0, & \alpha\beta=0,\end{cases}$$

if $r,s\ge 0$, and by

$$e(\alpha,\beta)=\frac{|\alpha|-|\beta|}{\alpha-\beta},\quad \alpha\neq\beta,$$

if $\min(r,s)<0$. In particular,

$$e(1,2)=\begin{cases}1/\ln 2, & \text{if}\quad \min(r,s)\ge 0,\\ 1, & \text{if}\quad \min(r,s)<0.\end{cases}$$

It has been proven [36, Lemma 2.1] that if $(r,s)\in A$, then

$$E_{r,s}(a,b)\le E_{1,2}(a,b)=\frac{a+b}{2},$$

while if $(r,s)\in A^*$, then

$$E_{r,s}(a,b)\ge E_{1,2}(a,b)=\frac{a+b}{2}.$$

This implies

$$E_{r,s}(x+a,x+b)-x\le\frac{a+b}{2}\quad\text{if}\quad(r,s)\in A$$

and

$$E_{r,s}(x+a,x+b)-x\ge\frac{a+b}{2}\quad\text{if}\quad(r,s)\in A^*.$$

Obviously, it follows from (8) that

$$\lim_{x\to\infty}\left[E_{r,s}(x+a,x+b)-x\right]=\frac{a+b}{2}.$$

Hence, it is natural to propose the following conjecture:

Conjecture 1. *Let a,b be positive numbers with $a\ne b$, $c=\min\{a,b\}$ and r,s be real numbers. Then the function*

$$x\mapsto E_{r,s}(x+a,x+b)-x$$

is increasing concave (decreasing convex) with $x\in(-c,\infty)$ if $(r,s)\in A$ ($(r,s)\in A^*$).

Define sets B and B^* by

$$B=\{(r,s)|r+s\le 1\quad\wedge\quad(r\le 0\vee s\le 0)\}$$

and

$$B^*=\{(r,s)|r+s\ge 1,r\ge 0,s\ge 0\}$$

(see Figure 2 of [36]). It has been proven [36, Lemma 4.2] that if $(r,s)\in B$, then

$$G_{r,s}(a,b)\le G_{0,1}(a,b)=\frac{a+b}{2},$$

while if $(r,s)\in B^*$, then

$$G_{r,s}(a,b)\ge G_{0,1}(a,b)=\frac{a+b}{2}.$$

This implies

$$G_{r,s}(x+a,x+b)-x\le\frac{a+b}{2}\quad\text{if}\quad(r,s)\in B$$

and

$$G_{r,s}(x+a,x+b)-x\ge\frac{a+b}{2}\quad\text{if}\quad(r,s)\in B^*.$$

Obviously, it follows from (9) that

$$\lim_{x\to\infty}\left[G_{r,s}(x+a,x+b)-x\right]=\frac{a+b}{2}.$$

Hence, it is natural to propose the following conjecture:

Conjecture 2. *Let a, b be positive numbers with $a \neq b$, $c = \min\{a, b\}$ and r, s be real numbers. Then the function*

$$x \mapsto G_{r,s}(x+a, x+b) - x$$

is increasing concave (decreasing convex) with $x \in (-c, \infty)$ if $(r, s) \in B$ ($(r, s) \in B^*$).

Theorem 2 of [21] states that let $r, s, u, v \in \mathbb{R}$ with $(r+s)(u+v) \neq 0$. The inequality

$$\sum\nolimits_{r,s}(a, b) \leq \sum\nolimits_{u,v}(a, b)$$

holds true for $a, b \in \mathbb{R}_+$ if and only if

$$\frac{|r-s|}{r+s} \leq \frac{|u-v|}{u+v} \quad \text{and} \quad \frac{(r-s)^2}{r+s} \leq \frac{(u-v)^2}{u+v}.$$

In particular, taking $u = 1, v = 0$ (or $u = 0, v = 1$), we obtain the following result: Let $r, s \in \mathbb{R}$ with $r + s \neq 0$, then the inequality

$$\sum\nolimits_{r,s}(x+a, x+b) - x \leq \frac{a+b}{2}$$

holds true for $x + a, x + b \in \mathbb{R}_+$ if and only if

$$\frac{|r-s|}{r+s} \leq 1 \quad \text{and} \quad \frac{(r-s)^2}{r+s} \leq 1. \tag{28}$$

Obviously, it follows from (21) that

$$\lim_{x \to \infty} \left[\sum\nolimits_{r,s}(x+a, x+b) - x\right] = \frac{a+b}{2}.$$

Therefore, it is natural to propose the following conjecture:

Conjecture 3. *Let a, b be positive numbers with $a \neq b$, $c = \min\{a, b\}$ and r, s be real numbers. If r, s satisfy those two inequalities in (28), then the function*

$$\sum\nolimits_{r,s}(x+a, x+b) - x$$

is increasing concave with $x \in (-c, \infty)$. If those two inequalities in (28) are both reversed, then the function

$$\sum\nolimits_{r,s}(x+a, x+b) - x$$

is decreasing convex with $x \in (-c, \infty)$.

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