ON A GENERALISED TRAPEZOIDAL RULE FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. A generalised trapezoidal rule is considered. Error estimates for functions of bounded variation are given. Applications for some particular cases of interest are provided as well.

1. Introduction

In [1], in order to approximate the integral $\int_a^b f(t)\,dt$ for the function $f : [a,b] \to \mathbb{R}$ of bounded variation with the generalised trapezoid rule $f(a)(x-a) + f(b)(b-x), x \in [a,b]$, the authors have considered the *trapezoid error functional*

$$T(f;a,b,x) := \int_a^b f(t)\,dt - [f(a)(x-a) + f(b)(b-x)], \tag{1.1}$$

and obtained the following sharp bound

$$|T(f;a,b,x)| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.2}$$

for each $x \in [a,b]$, where $\bigvee_a^b (f)$ denotes the total variation of f over $[a,b]$.

The best inequality we can derive from (1.2) is the following trapezoid inequality for functions of bounded variation:

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2} \cdot (b-a)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f), \tag{1.3}$$

where the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

For other inequalities for functions of bounded variation, see [2], [3], [4], [5], [6] and the references therein.

The main aim of this paper is to provide an approximation of the integral $\int_a^b f(t)\,dt$ in terms of another integral $\int_a^b g(t)\,dt$, assumed to be simpler to calculate, and in terms of the values of f and g at the end points of the interval $[a,b]$. Applications when g is itself an integral of a weight w are given. Some examples of quadratures for particular g's are provided.

Date: 15th July, 2008.

2000 *Mathematics Subject Classification.* 26D15, 41.

Key words and phrases. Trapezoidal rule, Functions of Bounded variation, Error estimates.

2. Error Estimates

In order to approximate the Riemann integral $\int_a^b f(t)\,dt$ in terms of the integral $\int_a^b g(t)\,dt$ and some values of f and g, we introduce the following error functional

$$E(f,g;a,b) = \int_a^b f(t)\,dt - f(b)\left[\frac{1}{g(b)-g(a)}\cdot\int_a^b g(s)\,ds - g(a)(b-a)\right] - f(a)\left[g(b)(b-a) - \frac{1}{g(b)-g(a)}\cdot\int_a^b g(s)\,ds\right], \tag{2.1}$$

where we assume that $g(b)\neq g(a)$.

For $h:[a,b]\to\mathbb{R}$ a Riemann integrable function with $h(a)\neq h(b)$ we define the *expectation* of h on $[a,b]$ by

$$\mu_h(a,b) := \frac{\int_a^b s\,dh(s)}{h(b)-h(a)}. \tag{2.2}$$

The following result may be stated:

Theorem 1. *For h Riemann integrable on $[a,b]$ and $h(a)\neq h(b)$ define*

$$B(h;a,b) := \begin{cases} b-\mu_h(a,b) & \text{if } \mu_h(a,b)<a, \\ \frac{1}{2}(b-a)+\left|\mu_h(a,b)-\frac{a+b}{2}\right| & \text{if } \mu_h(a,b)\in[a,b], \\ \mu_h(a,b)-a & \text{if } \mu_h(a,b)>b. \end{cases} \tag{2.3}$$

If f,g are Riemann integrable on (a,b), then

$$|E(f,g;a,b)| \leq \begin{cases} \bigvee_a^b(f)\,B(g;a,b) & \text{if } f \text{ is of bounded variation,} \\ \left|\frac{f(b)-f(a)}{g(b)-g(a)}\right|\bigvee_a^b(g)\,B(f;a,b) & \text{if } g \text{ is of bounded variation.} \end{cases} \tag{2.4}$$

Proof. First of all we show that for f and g Riemann integrable on $[a,b]$, we have the identity:

$$[f(b)-f(a)]\int_a^b g(t)\,dt - [g(b)-g(a)]\int_a^b f(t)\,dt + [g(b)f(a)-f(b)g(a)](b-a) = \int_a^b\left(\int_a^b (t-s)\,df(t)\right)dg(s) = \int_a^b\left(\int_a^b (t-s)\,dg(s)\right)df(t). \tag{2.5}$$

Since f is Riemann integrable on $[a,b]$ and $u(t)=t-s$, $s\in[a,b]$ is Lipschitzian, then the Stieltjes integral $\int_a^b (t-s)\,df(t)$ exists for each $s\in[a,b]$ and [1]

$$(b-s)f(b)+(s-a)f(a)-\int_a^b f(t)\,dt = \int_a^b (t-s)\,df(t) \tag{2.6}$$

for any $s\in[a,b]$.

Due to the fact that the left side of (2.6) is a Lipschitzian function on s, then taking the Stieltjes integral with respect to the integrator $g(s)$ produces the identity

$$f(b)\int_a^b (b-s)\,dg(s) + f(a)\int_a^b (s-a)\,dg(s) - [g(b)-g(a)]\int_a^b f(t)\,dt = \int_a^b \left(\int_a^b (t-s)\,df(t)\right) dg(s). \tag{2.7}$$

However, an integration by parts shows that $\int_a^b (b-s)\,dg(s) = \int_a^b g(s)\,ds - (b-a)\,g(a)$ and $\int_a^b (s-a)\,dg(s) = (b-a)\,g(b) - \int_a^b g(s)\,ds$, which, when incorporated in the left hand side of (2.7) produces the first equality in (2.5).

The equality between the first term and the last term in (2.5) follows by the equality

$$\int_a^b (t-s)\,dg(s) = \int_a^b g(s)\,ds - g(b)(b-t) - g(a)(t-a), \qquad t \in [a,b] \tag{2.8}$$

integrated with respect to the integrator $f(t)$. The details are omitted.

Now recall that if $p : [a,b] \to \mathbb{R}$ is continuous and v is of bounded variation, then the Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and $\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v)$. Applying this property for the continuous function $p(t) = \int_a^b (t-s)\,dg(s)$ and for the function of bounded variation $v(t) = f(t)$, $t \in [a,b]$, we get:

$$\begin{aligned} |E(f,g;a,b)| &= \left|\frac{1}{g(b)-g(a)} \cdot \int_a^b \left(\int_a^b (t-s)\,dg(s)\right) df(t)\right| \\ &\le \bigvee_a^b (f) \max_{t\in[a,b]} \left|\frac{\int_a^b (t-s)\,dg(s)}{g(b)-g(a)}\right| = \bigvee_a^b (f) \max_{t\in[a,b]} \left|t - \mu_g(a,b)\right|. \end{aligned} \tag{2.9}$$

Since, obviously

$$\max_{t\in[a,b]} \left|t-\mu_g(a,b)\right| = \begin{cases} b - \mu_g(a,b) & \text{if } \mu_g(a,b) < a, \\ \frac{1}{2}(b-a) + \left|\mu_g(a,b) - \frac{a+b}{2}\right| & \text{if } \mu_g(a,b) \in [a,b], \\ \mu_g(a,b) - a & \text{if } \mu_g(a,b) > b, \end{cases}$$

hence the first inequality in (2.4) is proved.

Also, we have

$$|E(f,g;a,b)| = \left|\frac{f(b)-f(a)}{g(b)-g(a)}\right| \cdot \left|\frac{1}{f(b)-f(a)} \int_a^b \left(\int_a^b (t-s)\,df(t)\right) dg(s)\right|$$

$$\le \left|\frac{f(b)-f(a)}{g(b)-g(a)}\right| \bigvee_a^b (g) \max_{s\in[a,b]} \left|\frac{\int_a^b t\,df(t)}{f(b)-f(a)} - s\right| = \left|\frac{f(b)-f(a)}{g(b)-g(a)}\right| \bigvee_a^b (g)\, B(f;a,b),$$

which proves the second part of (2.4). ∎

Remark 1. *A sufficient condition for* $h : [a,b] \to \mathbb{R}$ *such that* $\mu_h(a,b) \in [a,b]$ *is that* h *is monotonic nondecreasing on the interval* $[a,b]$.

Now, if we assume that f is of bounded variation and g is monotonic nondecreasing, then:

$$|E(f,g;a,b)| \leq \left[\frac{1}{2}(b-a) + \left|\mu_g(a,b) - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \leq (b-a) \bigvee_a^b (f). \tag{2.10}$$

Also, if g is of bounded variation and f is monotonic nondecreasing then:

$$|E(f,g;a,b)| \leq \left[\frac{1}{2}(b-a) + \left|\mu_f(a,b) - \frac{a+b}{2}\right|\right] \frac{f(b)-f(a)}{|g(b)-g(a)|} \bigvee_a^b (g) \tag{2.11}$$
$$\leq (b-a) \cdot \frac{f(b)-f(a)}{|g(b)-g(a)|} \bigvee_a^b (g).$$

Definition 1. *Let $u, v : [a,b] \to \mathbb{R}$ and $L > 0$. The function u will be called $L - v-$Lipschitzian if*

$$|u(t) - u(s)| \leq L |v(t) - v(s)| \quad \text{for each } t, s \in [a,b]. \tag{2.12}$$

Natural examples of such functions are provided by the Cauchy-mean-value theorem:

Proposition 1. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) and $g'(t) \neq 0$ on (a,b). If*

$$L := \sup_{\xi \in [a,b]} \left|\frac{f'(\xi)}{g'(\xi)}\right| < \infty, \tag{2.13}$$

then f is $L - g-$Lipschitzian.

Utilising the above concept we can naturally state the following.

Corollary 1. *If $g : [a,b] \to \mathbb{R}$ is of bounded variation and f is $L - g-$Lipschitzian, then*

$$|E(f,g;a,b)| \leq L \bigvee_a^b (g) B(f;a,b). \tag{2.14}$$

Moreover, if f, g are continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) \neq 0$, $t \in [a,b]$, then

$$|E(f,g;a,b)| \leq \sup_{\xi \in [a,b]} \left|\frac{f'(\xi)}{g'(\xi)}\right| \cdot \int_a^b |g'(s)| \, ds \cdot B(f;a,b). \tag{2.15}$$

3. Applications for $g(x) = \int_a^x w(s)\,ds$

Let $w : [a,b] \to \mathbb{R}$ be Riemann integrable and $W(x) = \int_a^x w(s)\,ds$ with $W(b) \neq 0$. If we choose $g = W$ in (2.1) then we get

$$E(f,W;a,b) = \int_a^b f(t)\,dt - \left\{\left[b - \frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds}\right] f(b) + \left[\frac{\int_a^b t w(t)\,dt}{\int_a^b w(s)\,ds} - a\right] f(a)\right\}.$$

If we denote by

$$\mu_W(a,b) := \frac{\int_a^b s w(s)\,ds}{\int_a^b w(s)\,ds} \tag{3.1}$$

the *expectation* of W, and by

$$F(f,W;a,b):=\int_a^b f(t)\,dt-\{[b-\mu_W(a,b)]\,f(b)+[\mu_W(a,b)-a]\,f(a)\} \tag{3.2}$$

the error functional in approximating the Riemann integral $\int_a^b f(t)\,dt$ by the *generalised trapezoid rule*

$$A(f,W;a,b)=[b-\mu_W(a,b)]\,f(b)+[\mu_W(a,b)-a]\,f(a) \tag{3.3}$$

then, on utilising Theorem 1, we can state the following result:

Proposition 2. *Assume that f and w are Riemann integrable on $[a,b]$. If*

$$B(W,a,b):=\begin{cases} b-\mu_W(a,b) & \text{if } \mu_W(a,b)<a,\\ \frac{1}{2}(b-a)+\left|\mu_W(a,b)-\frac{a+b}{2}\right| & \text{if } \mu_W(a,b)\in[a,b],\\ \mu_W(a,b)-a & \text{if } \mu_W(a,b)>b,\end{cases} \tag{3.4}$$

then:

$$|F(f,W;a,b)| \le \begin{cases} B(W;a,b)\bigvee_a^b(f) & \text{if } f \text{ is of bounded variation;}\\ B(f,a,b)\cdot\frac{|f(b)-f(a)|}{\left|\int_a^b w(s)ds\right|}\int_a^b|w(s)|\,ds, & \text{if } f \text{ is Riemann integrable.}\end{cases} \tag{3.5}$$

Remark 2. *We observe that if $w(s)\ge 0,\ s\in[a,b]$, then $\mu_W[a,b]\in[a,b]$ and*

$$|F(f,W;a,b)| \le \begin{cases} \left[\frac{1}{2}(b-a)+\left|\mu_W(a,b)-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) & \text{if } f \text{ is of bounded variation;}\\ B(f,a,b)\cdot|f(b)-f(a)|. \end{cases} \tag{3.6}$$

The first inequality can also be obtained from the generalised trapezoid inequality (see [1]*):*

$$\left|\int_a^b f(t)\,dt-[(b-x)\,f(b)+(x-a)\,f(a)]\right| \le \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f), \tag{3.7}$$

that holds for functions of bounded variation, by choosing $x=\mu_W(a,b)\in[a,b]$.

We also notice that the second inequality in (3.6) provides a bound which does not depend on the weight function w.

4. Some Particular Cases

In this section we provide some particular cases of interest.

1. If $g(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $g:[a,b]\to\mathbb{R}$, then for any f a Riemann integrable function we have

$$E(f,g;a,b)=\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a),$$

$$B(g,a,b) := \frac{\int_a^b t dg(t)}{g(b)-g(a)} = \frac{a+b}{2}$$

and $\bigvee_a^b (g) = 2$. On utilising Theorem 1 we deduce the following inequality

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \begin{cases} \frac{1}{2}(b-a)\bigvee_a^b (f) & \text{if } f \text{ is of bounded variation;} \\ B(f;a,b)\cdot |f(b)-f(a)|, & \text{if } f \text{ is Riemann integrable.} \end{cases} \tag{4.1}$$

We remark that the first branch in (4.1) provides the well known trapezoid inequality for functions of bounded variation obtained in [1], while the second part can be restated in a more explicit form as:

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) \right| \leq |f(b)-f(a)| \times \begin{cases} \left(b - \frac{\int_a^b sdf(s)}{f(b)-f(a)}\right) & \text{if } \frac{\int_a^b sdf(s)}{f(b)-f(a)} < a \\ \left[\frac{1}{2}(b-a) + \left|\frac{\int_a^b sdf(s)}{f(b)-f(a)} - \frac{a+b}{2}\right|\right] & \text{if } \frac{\int_a^b sdf(s)}{f(b)-f(a)} \in [a,b] \\ \left(\frac{\int_a^b sdf(s)}{f(b)-f(a)} - a\right) & \text{if } \frac{\int_a^b sdf(s)}{f(b)-f(a)} > b, \end{cases} \tag{4.2}$$

which give new bounds for the error in approximating the integral $\int_a^b f(t)\,dt$ by the trapezoid rule $\frac{f(a)+f(b)}{2}(b-a)$.

2. If $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right) e^{t-\frac{a+b}{2}}, t \in [a,b]$, then

$$\int_a^b g(t)\,dt = \left(e^{\frac{b-a}{2}} - 1\right)^2 e^{-\frac{b-a}{2}},$$

$$E(f,g;a,b) = \int_a^b f(t)\,dt - \frac{f(b)-f(a)}{e^{\frac{b-a}{2}} + e^{-\frac{b-a}{2}}}\left(e^{\frac{b-a}{2}} - 1\right)^2 e^{-\frac{b-a}{2}} - \frac{e^{\frac{b-a}{2}} f(a) + e^{-\frac{b-a}{2}} f(b)}{e^{\frac{b-a}{2}} + e^{-\frac{b-a}{2}}}(b-a),$$

$$\bigvee_a^b (g) = \bigvee_a^{\frac{a+b}{2}} \left(-e^{\cdot - \frac{a+b}{2}}\right) + \bigvee_{\frac{a+b}{2}}^b \left(e^{\cdot - \frac{a+b}{2}}\right) = e^{\frac{b-a}{2}} - e^{-\frac{b-a}{2}},$$

and

$$\mu_g(a,b) = \frac{\int_a^b sdg(s)}{g(b)-g(a)} = \frac{be^{\frac{b-a}{2}} + ae^{-\frac{b-a}{2}} - \left(e^{\frac{b-a}{2}} - 1\right)^2 e^{-\frac{b-a}{2}}}{e^{\frac{b-a}{2}} + e^{-\frac{b-a}{2}}} := E(a,b). \tag{4.3}$$

On utilising Theorem 1 we can state the following inequality:

$$(4.4)\quad \left|\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{e^{(b-a)}+1}\left(e^{\frac{b-a}{2}}-1\right)^2 - \frac{e^{\frac{b-a}{2}}f(a)+e^{-\frac{b-a}{2}}f(b)}{e^{\frac{b-a}{2}}+e^{-\frac{b-a}{2}}}(b-a)\right|$$
$$\leq \begin{cases} B(g;a,b)\bigvee_a^b(f) & \text{if } f \text{ is of bounded variation;} \\ B(f;a,b)\,|f(b)-f(a)|\cdot\frac{e^{\frac{b-a}{2}}-e^{-\frac{b-a}{2}}}{e^{\frac{b-a}{2}}+e^{-\frac{b-a}{2}}}, & \text{if } f \text{ is Riemann integrable,} \end{cases}$$

where $B(g;a,b)$ is defined by (2.3) for the function $g(t)=sgn\left(t-\frac{a+b}{2}\right)\exp\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$, namely

$$B(g;a,b)=\begin{cases} b-E(a,b) & \text{if } E(a,b)<a, \\ \frac{1}{2}(b-a)+\left|E(a,b)-\frac{a+b}{2}\right| & \text{if } E(a,b)\in[a,b], \\ E(a,b)-a & \text{if } E(a,b)>b. \end{cases}$$

where $E(a,b)$ is given by (4.3).

5. Some Numerical Experiments

Consider the following quantities

$$B_1(f;a,b)=\frac{1}{2}(b-a)\bigvee_a^b(f)$$

and

$$B_2(f;a,b)=|f(b)-f(a)|\times\begin{cases} \left(b-\frac{\int_a^b s\,df(s)}{f(b)-f(a)}\right) & \text{if } \frac{\int_a^b s\,df(s)}{f(b)-f(a)}<a \\ \left[\frac{1}{2}(b-a)+\left|\frac{\int_a^b s\,df(s)}{f(b)-f(a)}-\frac{a+b}{2}\right|\right] & \text{if } \frac{\int_a^b s\,df(s)}{f(b)-f(a)}\in[a,b] \\ \left(\frac{\int_a^b s\,df(s)}{f(b)-f(a)}-a\right) & \text{if } \frac{\int_a^b s\,df(s)}{f(b)-f(a)}>b. \end{cases}$$

provided by the inequality (4.1) in order to give *a priori* bounds for the error in approximating the integral $\int_a^b f(t)\,dt$ by the trapezoid rule $\frac{f(a)+f(b)}{2}(b-a)$.

We will show that there are examples of functions for which one is better than the other one.

Let $f_n:[0,1]\to\mathbb{R}$ be defined by $f_n(s):=s/(s^n+1)$ and put

$$\Delta_n:=B_1(f_n;0,1)-B_2(f_n;0,1).$$

Utilising the computer package Maple we obtain the plot shown in Figure 1.

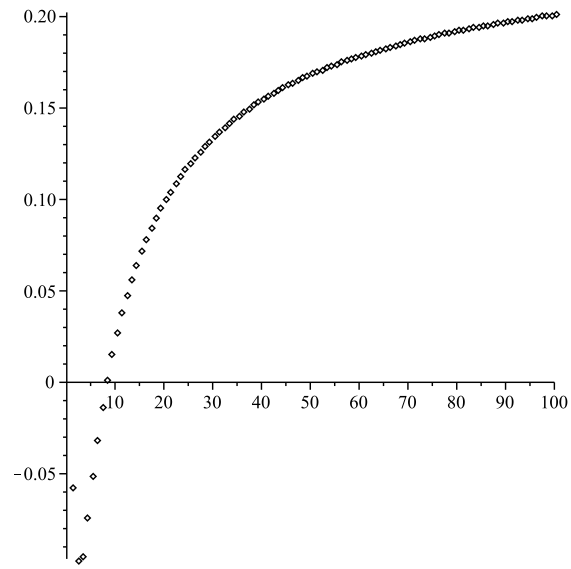

Figure 1: Plot for the difference Δ_n for n ranging from 1 to 100

The above example shows that the bound $B_1(f_n; 0, 1)$ is better than the bound $B_2(f_n; 0, 1)$ for $1 \leq n \leq 7$ while the conclusion is the other way around for $8 \leq n \leq 100$.

Now if we consider another family of functions, namely $g_t : [0, 1] \to \mathbb{R}$, $g_t(s) := \sin(ts)$ and put

$$\Delta_t := B_1(g_t; 0, 1) - B_2(g_t; 0, 1)$$

then the plot of Δ_t for t in steps of 0.01 from 1 to 10.0 incorporated in Figure 2 shows that $B_2(g_t; 0, 1)$ is better than $B_1(g_t; 0, 1)$ for $t \geq 2.6$ while the conclusion is the other way around for t less than that value.

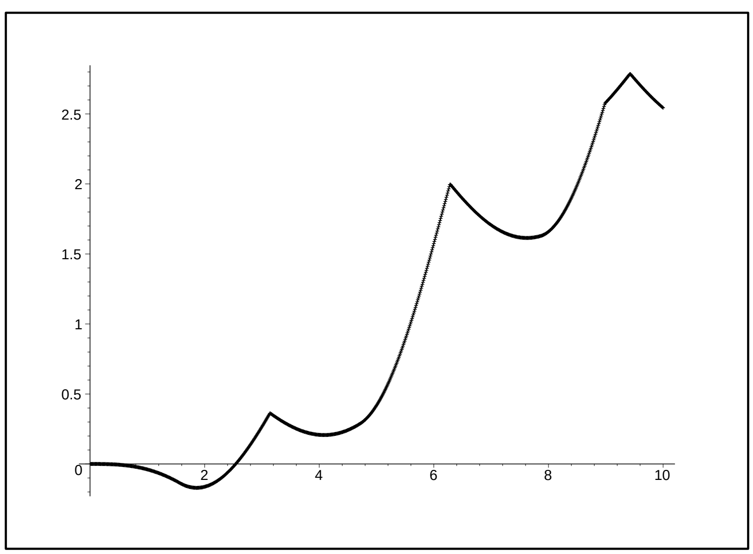

Figure 2: Plot for the difference Δ_t for t in steps of 0.01 from 1 to 10.0.

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