A NEW PROOF OF INEQUALITIES FOR GAUSS COMPOUND MEAN

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ABSTRACT. In this short note, by using some known identities and inequalities a new and simple proof of inequalities for Gauss compound mean is given..

1. INTRODUCTION

The arithmetic-geometric mean of positive numbers $a, b > 0$ with $a \neq b$ is the limit

$$AGM(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \tag{1.1}$$

where $a_0 = a, b_0 = b$, and for $n = 0, 1, 2, 3, ..., a \neq b$

$$a_{n+1} = A(a_n, b_n) = \frac{a_n + b_n}{2}, b_{n+1} = G(a_n, b_n) = \sqrt{a_n b_n}, \tag{1.2}$$

For a mean value M, the t-modification is defined as

$$M_t(a,b) = M^{1/t}(a^t, b^t).$$

For example, the power mean of $a, b > 0$ with $a \neq b$ is

$$A_t(a,b) = (\frac{a^t + b^t}{2})^{1/t},$$

and the logarithmic mean is

$$L(a,b) = \frac{b-a}{\ln b - \ln a}.$$

The power mean is the t-modification of the arithmetic mean $A(a,b)$.

The connection between mean values and elliptic integrals is provided by Gauss's amazing result

$$AGM(1, r') = \frac{\pi}{2\mathrm{K}(r)}, \tag{1.3}$$

where

$$\mathrm{K}(r) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}; 1; r^2) = \int_0^{\frac{\pi}{2}} (1 - r^2 \sin^2 x)^{-1/2} dx, \tag{1.4}$$

$r' = \sqrt{1 - r^2}$ $(0 \leq r \leq 1)$ [2].

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This formula motivates the question of finding minorant/majorant functions for K(r) in terms of mean values. For a fixed $x > 0$ the function $t \to L_t(1,x), t > 0$, increases with t by [14, Theorem 1.2 (1)]. The two-sided inequalities

$$L(a,b) < AGM(a,b) < L^{\frac{2}{3}}(a^{\frac{3}{2}}, b^{\frac{3}{2}}). \tag{1.5}$$

hold; the first inequality was pointed out in [4], and the second one, due to J. and P. Borwein [3], proves a sharp estimate settling a question raised in connection with [14]. Combined with identity above, this inequality yields a very precise inequality for K(r).

The purpose of this short note is to give a new proof by using certain known identities and inequalities for means.

2. Lemmas

For proof of (1.5), we now introduce the-called power-exponential mean (or symmetric geometric mean) of positive numbers a and b which is a case of limit of Gini mean and denoted by

$$Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} \tag{2.1}$$

Lemma 1. *For positive numbers a, b with $a \neq b$ and $p \geq 2$, the following inequality*

$$Z(a,b) > A_p(a,b), \tag{2.2}$$

is always true, where $A_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$.(see [7], [15])

Lemma 2. *[?] For positive numbers a and b, the following identity*

$$Z^2(\sqrt{a}, \sqrt{b}) = G(a,b)e^{\frac{A(a,b)-G(a,b)}{L(a,b)}} \tag{2.3}$$

is always true, which can be concisely denoted by

$$Z_{\frac{1}{2}} = Ge^{\frac{A-G}{L}}. \tag{2.4}$$

Lemma 3. *[11], [15] For exponential mean or identic mean denoted by*

$$E(a,b) = e^{-1}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, \tag{2.5}$$

the following identity

$$E(a,b) = G(a,b)\exp\left(\frac{A(a,b)}{L(a,b)} - 1\right) \tag{2.6}$$

is always valid, which also can be simply denoted by

$$E = Ge^{\frac{A}{L}-1}. \tag{2.7}$$

Lemma 4. *[13], [1] For $a, b > 0$ with $a \neq b$, both the following inequalities*

$$E(a,b) > A_{\frac{2}{3}}(a,b), \tag{2.8}$$

$$L(a,b) + E(a,b) < A(a,b) + G(a,b) \tag{2.9}$$

hold.

Lemma 5. *$L(a,b)$ is increasing in either a or b on $(0,\infty)$.*

3. Proof

To prove first inequality of (1.5), it is enough to prove that

$$L(a,b) < L(A,G). \tag{3.1}$$

In fact, by (2.2) and (2.4) we have

$$Z_{\frac{1}{2}} = Ge^{\frac{A-G}{L}} > A, \tag{3.2}$$

which is equivalent to

$$\ln G + \tfrac{A-G}{L} > \ln A,$$

i.e. (3.1) is valid.

Next let us prove that

$$L(a^{\frac{3}{2}}, b^{\frac{3}{2}}) > L(A^{\frac{3}{2}}, G^{\frac{3}{2}}). \tag{3.3}$$

In fact, (2.7) can be written as

$$\ln E - \ln G = \frac{A}{L} - 1 = \frac{A-L}{L}; \tag{3.4}$$

On the other hand, inequality (2.9) can be written as

$$E - G < A - L. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$L(E,G) = \frac{E-G}{\ln E - \ln G} = \frac{E-G}{A-L} L < L(a,b). \tag{3.6}$$

By (2.8) and Lemma (5) we have

$$L(a,b) > L(E,G) > L(A_{\frac{2}{3}}, G). \tag{3.7}$$

Replacing a, b by $a^{\frac{3}{2}}, b^{\frac{3}{2}}$ and $a^{\frac{2}{3}}, b^{\frac{2}{3}}$ by a, b in (3.7), we get

$$L(a^{\frac{3}{2}}, b^{\frac{3}{2}}) > L(A^{\frac{3}{2}}, G^{\frac{3}{2}}), \tag{3.8}$$

which shows that

$$L(a^{\frac{3}{2}}, b^{\frac{3}{2}}) > (AGM)^{\frac{3}{2}}, \tag{3.9}$$

i.e. second inequality of (1.5) holds true.

Thus we complete the proof.

Remark 1. *By proof above, it is easy to obtain a more precise two-side inequalities estimating $AGM(a,b)$:*

$$L(A,G) < AGM(a,b) < L^{\frac{2}{3}}(A^{\frac{3}{2}}, G^{\frac{3}{2}}). \tag{3.10}$$

References

[1] Horst Alzer, Ungleichungen für Mittelwerte, *Arch. Math.* **47**(1986), 422-426.
[2] J. M. Borwein AND P. B. Borwein and , *Pi and the AGM*, John Wiley and Sons, 1987. (New York)
[3] J. M. Borwein AND P. B. Borwein, Inequalities for compound mean iterations with logarithmic asymptotes, *J. Math. Anal. Appl.* , **177**(1993), no. 2, 572-582.
[4] B. C. Carlson AND M. Vuorinen, An inequality of the AGM and the logarithmic mean, *SIAM Rev.* , **33**(1991), 655, Problem 91-117.
[5] J. Karamata, quélques problémes posés par Ramanujan *J. Indian Math. Soc.* (N. S.) **24**(1960), 343-365.
[6] J.-Ch. Kuang, *Applied Inequalities, 2nd ed.*, Hunan Education Press, Changsha City, Hunan Province, China, 1993. (Chinese)

[7] E. Neuman AND J. Sándor, Inequalities for sums of powers, *J. Math. Anal. Appl.* **131**(1) (1988), 265–270.
[8] J. Sándor, On the identric and logarithmic means. *Aequations Math.* **40**(1990), 261-270.
[9] J. Sándor, On certain identities for means, *Studia Univ. Babes-Bolyai*, **38**(1993), 7-141.
[10] J. Sándor AND W.- W., On certain identities for means II, *Journal of Chengdu University (Natural science),* **20**(2)(2001), 6-8.
[11] H.-J. Seiffert, Comment to problem 1365, *Math. Mag.* , **65**(1992), 356.
[12] K. B. Stolarsky, Generalizations of the logarithmic mean, *Math. Mag.* **48**(1975), 87-92.
[13] K. B. Stolarsky, The power and generalized logarithmic means, *Amer. Math. monthly*, **87**(1980), 545-548.
[14] M. K. Vamanamurthy AND M. Vuorinen, Inequalities for means, *J. Math. Anal. Appl.* **183**(1)(1994), 155-166.
[15] Zh.-H. Yang, Some identities for means and applications, *RGMIA Res. Rep. Coll.*, **8**(2005), no. 3 Art. 17; Available online at `http://rgmia.vu.edu.au/v8n3.html`.

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