THE MONOTONICITY OF THE RATIO BETWEEN STOLARSKY MEANS ★

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Abstract

A monotonicity result for the ratio between two Stolarsky means is established.

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Given two real parameters r, s, if a, b are positive numbers, then their Stolarsky mean $E_{r,s}(a,b)$ (cf. [24, 25]) is defined by

$$
\begin{aligned}
&E_{r,s}(a,b) = \left(\frac{r}{s}\cdot\frac{b^s-a^s}{b^r-a^r}\right)^{1/(s-r)}, && rs(r-s)(a-b)\neq 0;\\
&E_{r,0}(a,b) = \left(\frac{1}{r}\cdot\frac{b^r-a^r}{\ln b-\ln a}\right)^{1/r}, && r(a-b)\neq 0;\\
&E_{r,r}(a,b) = \frac{1}{e^{1/r}}\left(\frac{a^{a^r}}{b^{b^r}}\right)^{1/(a^r-b^r)}, && r(a-b)\neq 0;\\
&E_{0,0}(a,b) = \sqrt{ab}, && a\neq b;\\
&E_{r,s}(a,a) = a, && a=b.
\end{aligned}
$$

K. B. Stolarsky [24], Leach and Sholander [13] showed that $E_{r,s}(a,b)$ are for $a\neq b$ increasing with both r and s. Leach and Sholander [14] and Páles [18] solved the problem of comparison of Stolarsky mean.

Since $E_{r,s}(a,b)$ are for $a\neq b$ strictly increasing with both r and s, for the particular choices of the parameters r and s, we obtain the following chain of

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inequalities:

$$E_{0,0}(a,b) < E_{1,0}(a,b) < E_{1,1}(a,b) < E_{2,1}(a,b) \quad \text{for} \quad a \neq b,$$

that is,

$$G(a,b) < L(a,b) < I(a,b) < A(a,b) \quad \text{for} \quad a \neq b,$$

where G, L, I and A are the geometric, logarithmic, identric and arithmetic means, respectively.

The main result of this article is the following theorem:

Theorem 1. *Let a,b,c,d be positive numbers with $a \neq b, c \neq d$ and r,s be real numbers, and let*

$$R_{r,s}(a,b,c,d) = \frac{E_{r,s}(a,b)}{E_{r,s}(c,d)}. \tag{1}$$

Then the function $R_{r,s}(a,b,c,d)$ are strictly $\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}$ with both r and s according as

$$\frac{\min\{a,b\}}{\max\{a,b\}} \lessgtr \frac{\min\{c,d\}}{\max\{c,d\}}. \tag{2}$$

In order to prove Theorem 1, we need following Lemma.

Lemma 1 ([19]). *If f is an increasing (decreasing) integrable function on I, then the arithmetic mean of the function f,*

$$F(r,s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r \neq s, \\ f(r), & r = s, \end{cases}$$

is also increasing (decreasing) with both r and s on I.

Proof of Theorem 1. Since Stolarsky mean is symmetric in its variables, without loss of generality, assume that $a < b$ and $c < d$. By integral representation [12, 24]

$$\ln E_{r,s}(a,b) = \frac{1}{s-r}\int_r^s \ln I_t(a,b) dt, \tag{3}$$

where

$$I_t(a,b) = E_{t,t}(a,b), \tag{4}$$

we obtain

$$\ln R_{r,s}(a,b,c,d) = \frac{1}{s-r}\int_r^s p'(t) dt, \tag{5}$$

where

$$p(t) = p(t;a,b,c,d) = \begin{cases} \ln \dfrac{(c-d)(a^t-b^t)}{(a-b)(c^t-d^t)}, & t \neq 0; \\ \dfrac{(c-d)\ln(a/b)}{(a-b)\ln(c/d)}, & t = 0. \end{cases}$$

Easy computation reveals

$$p(-t) = p(t) + +t \ln \frac{cd}{ab}, \tag{6}$$

which implies that $p''(-t) = p''(t)$, and then $p(t)$ has the same convexity (concavity) on both $(-\infty, 0)$ and $(0, \infty)$. Now we are in position to consider convexity (concavity) of $p(t)$ for $t \in (0, \infty)$.

A simple computation yields

$$t^2 p''(t) = -\frac{(a/b)^t[\ln(a/b)^t]^2}{[(a/b)^t-1]^2} + \frac{(c/d)^t[\ln(c/d)^t]^2}{[(c/d)^t-1]^2}.$$

Define for $0 < u < 1$,

$$\omega(u) = \frac{u(\ln u)^2}{(1-u)^2},$$

easy calculation gives

$$\left(\frac{u}{u+1}\ln\frac{1}{u}\right)\cdot\frac{\omega'(u)}{\omega(u)} = \frac{\ln u}{u-1} - \frac{2}{u+1} = \frac{1}{L(u,1)} - \frac{1}{A(u,1)} > 0.$$

Hence, $\omega'(u) > 0$ and $\omega(u)$ is strictly increasing for $0 < u < 1$. Thus, for $t > 0$,

$$p''(t) \gtrless 0 \quad \text{according as} \quad \frac{a}{b} \lessgtr \frac{c}{d}.$$

Further, for $t \in (-\infty, \infty)$,

$$p''(t) \gtrless 0 \quad \text{according as} \quad \frac{a}{b} \lessgtr \frac{c}{d}, \tag{7}$$

since $p(t)$ has the same convexity (concavity) on both $(-\infty, 0)$ and $(0, \infty)$. Now, Lemma 1 combined with (5) and (7), imply that the function $R_{r,s}(a, b, c, d)$ are strictly $\begin{matrix}\text{increasing}\\\text{decreasing}\end{matrix}$ with both r and s according as $\frac{a}{b} \lessgtr \frac{c}{d}$. The proof of Theorem 1 is complete. □

Remark 1. It was shown in [2, 16] that let n be a positive integer, then for $r > 0$,

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{8}$$

We call the left-hand side of (8) Alzer's inequality [2], and the right-hand side of (8) Martins' inequality [16]. Several easy proofs of Alzer's inequality have been published by different authors, see [6, 23, 26]. In [4, 8, 10, 11] Alzer's inequality is extended to all real r. In [3, 9] it was proved that Martins' inequality is reversed for $r < 0$. There have been a lot of literature about these two inequalities and their history, background, extensions and generalizations. For more detailed information, refer to [1, 5] and the references therein.

Let $b > a > 0$ and $\delta > 0$, by Theorem 1, the function $r \mapsto \frac{E_{1,r+1}(a,b)}{E_{1,r+1}(a,b+\delta)}$ is strictly decreasing with $r \in (-\infty, +\infty)$, and then, we present an integral version of Alzer-Martins' inequality (8) as follows:

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a}\int_a^b x^r\,dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,dx}\right)^{1/r} \quad \text{for all real} \quad r, \tag{9}$$

$$\left(\frac{\frac{1}{b-a}\int_a^b x^r\,dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,dx}\right)^{1/r} \lessgtr \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \quad \text{according as} \quad r \gtrless 0. \tag{10}$$

This extends a result given by F. Qi and B.-N. Guo [20, 21], who established the inequalitis (9) and (10) for $r > 0$. In [7, 22], the monotonicity of the function $r \mapsto \frac{E_{1,r+1}(a,b)}{E_{1,r+1}(a,b+\delta)}$ has been shown. For the generalizations of the inequalitis (9) and (10) the reader is referred to [15, 17].

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