

New upper bounds for Mathieu–type series

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Abstract

The Mathieu’s series $S(r)$ was considered firstly by É.L. Mathieu in 1890; its alternating variant $\tilde{S}(r)$ has been recently introduced by Pogány *et al.* [12] where various bounds have been established for S, \tilde{S} . In this note we obtain new upper bounds over $S(r), \tilde{S}(r)$ with the help of Hardy–Hilbert double integral inequality.

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1. Introduction and preliminaries

The series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \quad (1)$$

is named after Émile Léonard Mathieu (1835–1890), who investigated it in his 1890 book [9] written on the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two–dimensional rectangular domain, see [13, Eq. (54), p. 258]. The alternating version of $S(r)$, that is

$$\tilde{S}(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^2} \quad (2)$$

was introduced following certain Tomovski's ideas and recently discussed by Pogány *et al.* in [12]. Applications of alternating Mathieu series $\tilde{S}(r)$ concerning ODE which solution is the Butzer–Flocke–Hauss Omega function were studied in [3], [11]. The integral representations of $S(r), \tilde{S}(r)$ [6], [12] respectively, reads as follows:

$$S(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{e^x - 1} dx, \quad \tilde{S}(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{e^x + 1} dx. \quad (3)$$

These integral expressions will be the starting points of our investigations.

2. Results required

Let us consider a Hölder pair $(p, q), p^{-1} + q^{-1} = 1, p > 1$, two non-negative functions $f \in L^p(\mathbb{R}_+), g \in L^q(\mathbb{R}_+)$, and let us denote $\|\cdot\|_{L^s(\mathbb{R}_+)} := \|\cdot\|_s$ the usual integral L^s -norm on the set of positive reals. The celebrated Hardy–Hilbert (or Hilbert) integral inequality [10] reads

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y) dx dy}{x+y} \leq \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q. \quad (4)$$

The inequality is strict unless at least one of f, g is zero, and the constant on the right in (4) is the best possible [10].

Consider the scaled parametric integral

$$\mathcal{I}_p = \int_0^\infty \frac{|\sin x|^p}{x^p} \quad (p > 1). \quad (5)$$

We point out that in [2, p. 663] the following estimate has been proved:

$$\mathcal{I}_p \leq \frac{\pi}{2} \sqrt{\frac{2}{p}} \quad (p \geq 2). \quad (6)$$

However, we shall give another estimate over \mathcal{I}_p when $p > 1$.

Lemma 1. *For all $p > 1$ the following estimate holds*

$$\mathcal{I}_p \leq q \quad (7)$$

where q is the conjugate Hölder pair to p .

Proof. Let us write

$$\mathcal{I}_p := \int_0^1 \frac{|\sin x|^p}{x^p} dx + \int_1^\infty \frac{|\sin x|^p}{x^p} dx.$$

Then, by the estimate $\sin x \leq x, x \in [0, 1]$ and by the redundant $|\sin x| \leq 1, x > 1$ respectively, we easily deduce

$$\mathcal{I}_p \leq \int_0^1 dx + \int_1^\infty \frac{dx}{x^p} = 1 + \frac{1}{p-1} = q. \quad (8)$$

This finishes the proof of the Lemma. \square

3. Main results

At first we establish an upper bound for both $S(r), \tilde{S}(r)$ of magnitude $O(r^{-1/2})$.

Theorem 1. *Let (p, q) , $p > 1$ be a Hölder pair. Then we have*

$$S(r) \leq \frac{16\sqrt{\pi} q^{1/(2p)} p^{1/(2q)}}{\sqrt{r} \sin^{1/2}(\pi/p)} \quad \text{and} \quad \tilde{S}(r) \leq \frac{16\sqrt{\pi} q^{1/(2p)} p^{1/(2q)}}{\sqrt{r} \sin^{1/2}(\pi/p)}. \quad (9)$$

Proof. It is sufficient to prove the inequality on the left in (9) since the right one can be proved similarly. First, we give two elementary inequalities:

$$\frac{x}{e^x + 1} \leq \frac{x}{e^x - 1} \leq \frac{2}{e^{x/2}} \quad (x \geq 0) \quad (10)$$

$$\frac{xy(x+y)}{64} \leq \exp \left\{ \frac{x}{4} + \frac{y}{4} + \frac{x+y}{4} \right\} = \exp \left\{ \frac{x+y}{2} \right\} \quad (x, y \geq 0). \quad (11)$$

Thus, we have

$$\begin{aligned} (S(r))^2 &= \frac{1}{r^2} \int_0^\infty \int_0^\infty \frac{xy \sin(rx) \sin(ry)}{(e^x - 1)(e^y - 1)} dx dy \\ &\leq \frac{4}{r^2} \int_0^\infty \int_0^\infty |\sin(rx) \sin(ry)| e^{-(x+y)/2} dx dy \quad (\text{by (10)}) \\ &\leq \frac{256}{r^2} \int_0^\infty \int_0^\infty \frac{|\sin(rx) \sin(ry)|}{xy(x+y)} dx dy. \quad (\text{by (11)}) \end{aligned}$$

Taking $f(x) = x^{-1} |\sin(rx)| = g(x)$ we apply the Hardy–Hilbert inequality to the last expression, such that one transforms into

$$\begin{aligned} (S(r))^2 &\leq \frac{256\pi}{r^2 \sin(\pi/p)} \left(\int_0^\infty \frac{|\sin(rx)|^p}{x^p} dx \right)^{1/p} \left(\int_0^\infty \frac{|\sin(ry)|^q}{y^q} dy \right)^{1/q} \\ &= \frac{256\pi r^{(p-1)/p + (q-1)/q}}{r^2 \sin(\pi/p)} (\mathcal{I}_p)^{\frac{1}{p}} (\mathcal{I}_q)^{\frac{1}{q}} \quad (12) \\ &\leq \frac{256\pi q^{\frac{1}{p}} \cdot p^{\frac{1}{q}}}{r \sin(\pi/p)} \quad (\text{by (7)}) \end{aligned}$$

This is equivalent to the asserted result (9). \square

Now, we will extend this result, scaling the exponent of r in the upper bound (9). The achieved magnitude should be $O(r^{-1/(2p)})$, $p > 1$.

Theorem 2. *Let (p, q) , $p > 1$ be a Hölder pair. Then for all $r > 0, v > 1$ we have*

$$S(r) \leq \frac{C(p, v)}{r^{1/(2p)}} \quad \text{and} \quad \tilde{S}(r) \leq \frac{C(p, v)}{r^{1/(2p)}} \quad (13)$$

where

$$C(p, v) := \frac{2^{(5q+1)/(2q)} \max\{2^{1/(2p)}, 2^{1/(2q)}\} (\pi p)^{1/(2p)} (\Gamma(q)\Gamma(2q))^{1/(2q)}}{q^{3/2} (\sin(\pi/p) (p-1/v)^{1/v} (p-1+1/v)^{1-1/v})^{1/(2p)}}. \quad (14)$$

Proof. For a given Hölder pair (p, q) , $p > 1$ and for some $r > 0$ consider

$$\begin{aligned} (S(r))^2 &= \frac{1}{r^2} \int_0^\infty \int_0^\infty \frac{xy \sin(rx) \sin(ry)}{(e^x - 1)(e^y - 1)} dx dy \\ &= \frac{1}{r^6} \int_0^\infty \int_0^\infty \frac{\sin(x) \sin(y)}{xy(x+y)^{1/p}} \cdot \frac{x^2 y^2 (x+y)^{1/p}}{(e^{x/r} - 1)(e^{y/r} - 1)} dx dy. \end{aligned} \quad (15)$$

By the Hölder inequality we conclude

$$\begin{aligned} (S(r))^2 &\leq \frac{1}{r^6} \left(\int_0^\infty \int_0^\infty \frac{|\sin(x) \sin(y)|^p}{x^p y^p (x+y)} dx dy \right)^{1/p} \\ &\quad \times \left(\int_0^\infty \int_0^\infty \frac{x^{2q} y^{2q} (x+y)^{q-1}}{(e^{x/r} - 1)^q (e^{y/r} - 1)^q} dx dy \right)^{1/q}. \end{aligned} \quad (16)$$

Choosing this time w as the Hölder conjugate pair to given $v > 1$ and specifying

$$f(x) = g(x) = x^{-p} |\sin(x)|^p,$$

we evaluate by the Hardy–Hilbert inequality (4) the first integral from above:

$$\begin{aligned} \mathcal{J} &= \int_0^\infty \int_0^\infty \frac{|\sin(x) \sin(y)|^p}{x^p y^p (x+y)} dx dy \\ &\leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty \frac{|\sin(x)|^{pv}}{x^{pv}} dx \right)^{1/v} \left(\int_0^\infty \frac{|\sin(y)|^{pw}}{y^{pw}} dy \right)^{1/w}. \end{aligned} \quad (17)$$

Estimating (17) by (7) we deduce

$$\mathcal{J} \leq \frac{\pi}{\sin(\pi/p)} \frac{pv}{(pv - 1)^{1/v} ((p - 1)v + 1)^{1-1/v}}. \quad (18)$$

The second integral in (16) we evaluate in the following way:

$$\begin{aligned} \mathcal{K} &= \int_0^\infty \int_0^\infty \frac{x^{2q} y^{2q} (x+y)^{q-1}}{(e^{x/r} - 1)^q (e^{y/r} - 1)^q} dx dy \\ &= r^{5q+1} \int_0^\infty \int_0^\infty \frac{x^{2q} y^{2q} (x+y)^{q-1}}{(e^x - 1)^q (e^y - 1)^q} dx dy \\ &\leq r^{5q+1} \max\{2, 2^{q-1}\} \int_0^\infty \frac{x^{3q-1} dx}{(e^x - 1)^q} \int_0^\infty \frac{y^{2q} dy}{(e^y - 1)^q} \end{aligned} \quad (19)$$

$$\leq (2r)^{5q+1} q^{-3q} \max\{2, 2^{q-1}\} \Gamma(q) \Gamma(2q). \quad (20)$$

where in (19) we make use of the estimate (such that follows by (10)):

$$\int_0^\infty \frac{x^\alpha}{(e^x - 1)^q} dx \leq 2^q \int_0^\infty x^{\alpha-q} e^{-qx/2} dx = \frac{2^{\alpha+1}}{q^{\alpha-q+1}} \Gamma(\alpha - q + 1), \quad (21)$$

specified for $\alpha = 3q - 1, 2q$ respectively. So, the upper bound over $S(r)$ in (13) is proved.

Repeating the previous procedure, now for \tilde{S} , we clearly deduce

$$(\tilde{S}(r))^2 \leq \frac{\max\{2^{1/p}, 2^{1/q}\} \mathcal{J}^{1/p}}{r^{1/p}} \left(\int_0^\infty \int_0^\infty \frac{x^{3q-1} y^{2q}}{(e^x + 1)^q (e^y + 1)^q} dx dy \right)^{1/q}. \quad (22)$$

Bearing in mind (10) we easily conclude by (21) that

$$\int_0^\infty \frac{x^\alpha}{(e^x + 1)^q} dx \leq \int_0^\infty \frac{x^\alpha}{(e^x - 1)^q} dx \leq \frac{2^{\alpha+1}}{q^{\alpha-q+1}} \Gamma(\alpha - q + 1).$$

Now, transforming the right-hand expression in (22), we easily arrive at the upper bound concerning \tilde{S} in (13). \square

4. Discussion

A. In this research note we derive upper bounds for $S(r)$, $\tilde{S}(r)$, such that possess the form

$$S(r) \leq \frac{\Phi(\theta)}{r^\alpha} \quad (\alpha > 0).$$

Here $\Phi(\theta)$ is an absolute constant and θ denotes the vector of scaling parameters. We obtain our main results (9) and (13) *via* the Hardy–Hilbert integral inequality.

At first, we recall some ancestor results such that will be compared to our bounds for small r . In [9] Mathieu posed his famous conjecture $S(r) < r^{-2}$, $r > 0$. The conjecture was proved after more than 60 years by Berg [1] and by Makai [8]. Actually they showed more:

$$\frac{1}{r^2 + 1/2} < S(r) < \frac{1}{r^2} \quad (r > 0). \quad (23)$$

Another proof of this upper bound has been given by van der Corput and Heflinger [4]. Diananda [5] improved Mathieu's bound to

$$S(r) \leq \frac{1}{r^2} - \frac{1}{(2r^2 + 2r + 1)(8r^2 + 3r + 3)} \quad (r > 0). \quad (24)$$

Here has to be mentioned Guo's bound of magnitude $O(r^{-2})$, [7, Eq. (10)].

B. We obtain easily an upper bound, such that is superior to Mathieu's bound r^{-2} for small r . Indeed, starting with the integral expressions for $S(r)$ and $\tilde{S}(r)$ in (3) we have

$$S(r) \leq \frac{1}{r} \int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi^2}{6r} =: S^*(r) \quad \text{and} \quad \tilde{S}(r) \leq \frac{1}{r} \int_0^\infty \frac{x dx}{e^x + 1} = \frac{\pi^2}{12r}. \quad (25)$$

So, when $r \in (0, 6/\pi]$, it follows $S^*(r) \leq r^{-2}$.

C. Let us denote $S_1(r)$, $S_2(r)$ the upper bounds listed in Theorem 1, 2 respectively. Comparing Mathieu's bound with $S_1(r)$, solving the equation $S_1(r) = r^{-2}$ we find that

$$S_1(r) \leq \frac{1}{r^2} \quad \left(0 < r \leq \frac{\sin^{2/3}(\pi/p)}{4\sqrt[3]{4\pi} p^{1/(3q)} q^{1/(3p)}} := r_1(p) < 1 \right).$$

Therefore, $S_1(r)$ is obviously superior to bounds with $O(r^{-2})$, r small. The similar comparison involving $S_2(r)$ and/or Diananda's (24) and Guo's bounds, we leave to the interested reader. These analyses show that our bounds (9), (13) mainly improve the earlier ones.

D. Let us compare $S_1(r)$ and $S_2(r)$. It is not hard to see that

$$r_0 := r_0(p, v) = \frac{2^{3q-1} \pi p^{2-q} q^{4q-1} ((p-1/v)^{1/v} (p-1+1/v)^{1-1/v})^{q-1}}{\sin(\pi/p) \max\{2, 2^{q-1}\} \Gamma(q) \Gamma(2q)}$$

is the unique positive solution of $S_1(r) = S_2(r)$. Accordingly, it follows that

$$S_2(r) < S_1(r) \quad (r \in (0, r_0)),$$

while for $r > r_0$ the reversed conclusion holds. We point out that r_0 can easily skip 1; for instance $r_0(2, 2) = 512\pi$.

E. Because the alternating Mathieu series has been introduced recently in [12], the here established bounds are unique until now. However, for r large the bounding inequalities presented also in [12] are sharper than the here presented ones.

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References

- [1] L. BERG, Über eine abschätzung von Mathieu, *Math. Nachr.* **7** (1952), 257-259.
- [2] M. BERGER, Convexity, *Amer. Math. Monthly* **97** (1990), No. 8, 650-678.
- [3] P.L. BUTZER, T.K. POGÁNY AND H.M. SRIVASTAVA, A linear ODE for the Omega function associated with the Euler function $E_\alpha(z)$ and the Bernoulli function $B_\alpha(z)$, *Appl. Math. Lett.* **19** (2006), 1073-1077.
- [4] J.G. VAN DER CORPUT AND L.O. HEFLINGER, On the inequality of Mathieu, *Indagationes Math.* **18** (1956), 15-20.
- [5] P.H. DIANANDA, On some inequalities related to Mathieu's, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **716-734** (1981), 22-24.
- [6] O. EMERSLEBEN, Über die Reihe $\sum_{k=1}^{\infty} \frac{k}{(k^2+c^2)^2}$, *Math. Ann.* **125** (1952), 165-171.
- [7] BAI-NI GUO, Note on Mathieu's inequality, *RGMA Res. Rep. Coll.* **3** (2000), No. 3, Art. 5.
- [8] E. MAKAI, On the inequality of Mathieu, *Publ. Math. Debrecen* **5** (1957), 204-205.
- [9] É.L. MATHIEU, *Traité de Physique Mathématique. VI-VII: Théorie de l'Elasticité des Corps Solides* (Part 2), Gauthier-Villars, Paris, 1890.
- [10] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Boston, 1991.
- [11] T.K. POGÁNY AND H.M. SRIVASTAVA, Some two-sided bounding inequalities for the Butzer-Flocke-Hauss Omega function, *Math. Ineq. Appl.* **10** (2007), No. 1, 587-595.
- [12] T.K. POGÁNY, H.M. SRIVASTAVA AND Ž. TOMOVSKI, Some families of Mathieu **a**-series and alternating Mathieu **a**-series, *Appl. Math. Comput.* **173** (2006), 69-108.
- [13] K. SCHRÖDER, Das Problem der eingespannten rechteckigen elastischen Platte I.: Die biharmonische Randwertaufgabe für das Rechteck, *Math. Anal.* **121** (1949), 247-326.