STOLARSKY AND GINI MEANS

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Abstract. We establish new integral representations of Stolarsky and Gini means. As the first application, we prove the comparison theorem of $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ established by E. Neuman, Zs. Ṕles. As the second application, we prove a monotonicity result for the ratio between two Stolarsky means, and an integral version of Alzer-Martins' inequality can be immediate concluded as consequence.

1. Introduction

Let $r,s \in \mathbb{R}$ and let $a,b>0$. The Stolarsky mean $E_{r,s}(a,b)$ of order (r,s) of a and b $a \neq b$ is defined as

$$E_{r,s}(a,b)=\begin{cases}\left(\dfrac{r}{s}\cdot\dfrac{b^s-a^s}{b^r-a^r}\right)^{1/(s-r)}, & rs(r-s)\neq 0,\\ \exp\left(-\dfrac{1}{r}+\dfrac{a^r\ln a-b^r\ln b}{a^r-b^r}\right), & r=s\neq 0,\\ \left(\dfrac{1}{r}\cdot\dfrac{b^r-a^r}{\ln b-\ln a}\right)^{1/r}, & r\neq 0, s=0,\\ \sqrt{ab}, & r=s=0,\end{cases} \tag{1}$$

with $E_{r,s}(a,a)=a$ (see [38, 39]).

Another class of bivariate means studied in this paper was introduced by Gini [14]. They are defined as

$$G_{r,s}(a,b)=\begin{cases}\left(\dfrac{a^s+b^s}{a^r+b^r}\right)^{1/(s-r)}, & r\neq s,\\ \exp\left(\dfrac{a^r\ln a+b^r\ln b}{a^r+b^r}\right), & r=s\neq 0,\\ \sqrt{ab}, & r=s=0.\end{cases} \tag{2}$$

In [38] it was shown that Stolarsky mean $E_{r,s}(a,b)$ can be extended continuously to the domain

$$\{(r,s,a,b)|r,s\in\mathbb{R},a,b\in\mathbb{R}_+\}, \tag{3}$$

where R_+ denotes the set of positive real numbers. Moreover, the function

$$(r,s,a,b)\mapsto E_{r,s}(a,b)$$

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is infinitely many times differentiable on the domain (3), see [19]. Gini mean $G_{r,s}(a,b)$ can be also extended continuously to the domain (3).

K. B. Stolarsky [38], Leach and Sholander [16] showed that $E_{r,s}(a,b)$ are for $a \neq b$ increasing with both r and s. For the monotonicity of the Gini mean $G_{r,s}(a,b)$ with respect to r or s the reader is referred to [22, 27]. Leach and Sholander [17] and Páles [24] solved the problem of comparison of Stolarsky mean. The problem of comparison of Gini mean was completely solved by Páles [25] (see also the paper by P. Czinder and Zs. Ples [12]). A problem of comparability of Gini and Stolarsky means was addressed by Neuman and Páles [23]. The log-convexity of Gini and Stolarsky means was presented in [22, 29]. The Schur-convexity of Gini and Stolarsky means was considered in [31, 32, 36, 37].

Since $E_{r,s}(a,b)$ are for $a \neq b$ strictly increasing with both r and s, for the particular choices of the parameters r and s, we obtain the following chain of inequalities:

$$E_{0,0}(a,b) < E_{1,0}(a,b) < E_{1,1}(a,b) < E_{2,1}(a,b) \quad \text{for} \quad a \neq b,$$

that is,

$$G(a,b) < L(a,b) < I(a,b) < A(a,b) \quad \text{for} \quad a \neq b,$$

where G, L, I and A are the geometric, logarithmic, identric and arithmetic means, respectively.

It is known [11, 22, 38] that

$$\ln E_{r,s}(a,b) = \frac{1}{s-r}\int_r^s \ln I_t(a,b)dt, \tag{4}$$

$$\ln G_{r,s}(a,b) = \frac{1}{s-r}\int_r^s \ln J_t(a,b)dt, \tag{5}$$

where

$$I_t(a,b) = E_{t,t}(a,b) \quad \text{and} \quad J_t(a,b) = G_{t,t}(a,b).$$

The paper is organized as follows. In Section 2 we establish new integral representations of Gini and Stolarsky means. In Section 3 we present two applications.

2. Integral representations

Theorem 1. *The following integral formulas hold:*

$$\ln E_{r,s}(a,b) = \frac{1}{s-r}\int_r^s u(t;a,b)dt, \tag{6}$$

$$\ln G_{r,s}(a,b) = \frac{1}{s-r}\int_r^s v(t;a,b)dt, \tag{7}$$

where

$$u(t) \triangleq u(t;a,b) = \ln\sqrt{\frac{a}{b}}\coth\left(t\ln\sqrt{\frac{a}{b}}\right) - \frac{1}{t} + \ln\sqrt{ab}, \tag{8}$$

$$v(t) \triangleq v(t;a,b) = \ln\sqrt{\frac{a}{b}}\tanh\left(t\ln\sqrt{\frac{a}{b}}\right) + \ln\sqrt{ab}. \tag{9}$$

Proof. It is clear that $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ are both homogenous function of order one in its variables, i.e.,

$$E_{r,s}(\lambda a,\lambda b)=\lambda E_{r,s}(a,b) \quad \text{and} \quad G_{r,s}(\lambda a,\lambda b)=\lambda G_{r,s}(a,b), \quad \lambda>0.$$

Therefore,

$$E_{r,s}(a,b)=\sqrt{ab}\,E_{r,s}\left(\sqrt{\frac{a}{b}},\sqrt{\frac{b}{a}}\right)=\sqrt{ab}\,E_{r,s}(e^p,e^{-p}),$$

$$G_{r,s}(a,b)=\sqrt{ab}\,G_{r,s}\left(\sqrt{\frac{a}{b}},\sqrt{\frac{b}{a}}\right)=\sqrt{ab}\,G_{r,s}(e^p,e^{-p}),$$

where $p=\ln\sqrt{\frac{a}{b}}$. Thus,

$$\begin{aligned}\ln E_{r,s}(a,b)&=\ln\sqrt{ab}+\frac{1}{s-r}\left[\ln\frac{e^{ps}-e^{-ps}}{ps}-\ln\frac{e^{pr}-e^{-pr}}{pr}\right]\\&=\ln\sqrt{ab}+\frac{1}{s-r}\int_{pr}^{ps}\left[\coth(x)-\frac{1}{x}\right]dx\\&=\ln\sqrt{ab}+\frac{1}{s-r}\int_{r}^{s}\left[p\coth(pt)-\frac{1}{t}\right]dt\\&=\frac{1}{s-r}\int_{r}^{s}\left[\ln\sqrt{\frac{a}{b}}\coth\left(t\ln\sqrt{\frac{a}{b}}\right)-\frac{1}{t}+\ln\sqrt{ab}\right]dt.\end{aligned}$$

$$\begin{aligned}\ln G_{r,s}(a,b)&=\ln\sqrt{ab}+\frac{1}{s-r}\left[\ln\cosh(ps)-\ln\cosh(pr)\right]\\&=\ln\sqrt{ab}+\frac{1}{s-r}\int_{pr}^{ps}\tanh(x)dx\\&=\ln\sqrt{ab}+\frac{1}{s-r}\int_{r}^{s}p\tanh(pt)dt\\&=\frac{1}{s-r}\int_{r}^{s}\left[\ln\sqrt{\frac{a}{b}}\tanh\left(t\ln\sqrt{\frac{a}{b}}\right)+\ln\sqrt{ab}\right]dt.\end{aligned}$$

The proof of Theorem 1 is complete. □

Comparing (4) with (6), and comparing (5) with (7), we obtain

$$\begin{aligned}u(t)\triangleq u(t;a,b)=\ln I_t(a,b)&=-\frac{1}{t}+\frac{a^t\ln a-b^t\ln b}{a^t-b^t}\\&=\ln\sqrt{\frac{a}{b}}\coth\left(t\ln\sqrt{\frac{a}{b}}\right)-\frac{1}{t}+\ln\sqrt{ab},\end{aligned}$$

$$\begin{aligned}v(t)\triangleq v(t;a,b)=\ln J_t(a,b)&=\frac{a^t\ln a+b^t\ln b}{a^t+b^t}\\&=\ln\sqrt{\frac{a}{b}}\tanh\left(t\ln\sqrt{\frac{a}{b}}\right)+\ln\sqrt{ab}.\end{aligned}$$

Theorem 2. *Let $a,b>0$ with $a\neq b$ and let the functions $u(t)$ and $v(t)$ be defined by (8) and (9), respectively. Then the functions $u(t)$ and $v(t)$ have the following properties:*

(*i*) *For $t\in\mathbb{R}$,*

$$u(t)+u(-t)=2u(0) \quad \textit{and} \quad v(t)+v(-t)=2v(0). \tag{10}$$

(ii) $u(t)$ *and* $v(t)$ *are both strictly increasing on* $\mathbb{R}$.
(iii) $u(t)$ *and* $v(t)$ *are both convex on* $(-\infty, 0)$, *and concave on* $(0, \infty)$.

Proof. Since the functions $t \mapsto \coth(t)$ and $t \mapsto \tanh(t)$ are both odd on $\mathbb{R}$, (i) holds obviously.

We conclude from the inequality (see [15, p. 299])

$$\sinh(x) > x \quad \text{for} \quad x \neq 0.$$

that

$$u'(t) = \frac{1}{t^2} - \frac{4p^2}{(e^{pt} - e^{-pt})^2} = p^2 \left\{ \frac{1}{(pt)^2} - \frac{1}{[\sinh(pt)]^2} \right\} > 0 \quad \text{for} \quad t \neq 0.$$

Obviously,

$$v'(t) = \frac{4p^2}{(e^{pt} + e^{-pt})^2} = \left[\frac{p}{\cosh(pt)} \right]^2 > 0.$$

This proves (ii).

It follows from Lazarevic inequality (see [15, p. 300])

$$\left(\frac{\sinh(x)}{x} \right)^3 > \cosh(x) \quad \text{for} \quad x \neq 0 \tag{11}$$

that

$$\begin{aligned} u''(t) &= \frac{8p^3(e^{pt} + e^{-pt})}{(e^{pt} - e^{-pt})^3} - \frac{2}{t^3} = \frac{2\cosh(pt)}{t^3} \left[\left(\frac{pt}{\sinh(pt)} \right)^3 - \frac{1}{\cosh(pt)} \right] \\ &\gtrless 0 \quad \text{according as} \quad t \lessgtr 0. \end{aligned}$$

It is easy to see that

$$v''(t) = -\frac{2p^4}{[\cosh(pt)]^3} \frac{\sinh(pt)}{pt} \frac{1}{t} \gtrless 0 \quad \text{according as} \quad t \lessgtr 0.$$

This proves (iii). □

Theomer 2 has been proven in [11, 22, 35] by using the representations $u(t) \triangleq u(t; a, b) = \ln I_t(a, b)$ and $v(t) \triangleq u(t; a, b) = \ln J_t(a, b)$, we here present new proof by using the representations (8) and (9).

Lemma 1 ([13, 26])**.** *Let* f *be a continuous function on* I, *and let*

$$F(r, s) = \begin{cases} \dfrac{1}{s - r} \displaystyle\int_r^s f(t)\,\mathrm{d}t, & r \neq s, \\ f(r), & r = s. \end{cases}$$

Then,

(i)If f *is strictly increasing (decreasing) on* I, *then* F *is strictly increasing (decreasing) with both* r *and* s.

(ii)If f *is convex (concave) on* I, *then* F *is convex (concave) with both* r *and* s.

(iii) F *is Schur-convex on* I^2 *if and only if* f *is convex on* I.

Recall the definition of Schur-convex function. Let I be an interval with nonempty interior, and let $f : I^n \to \mathbb{R}$. Then f is called Schur-convex on I^n $(n \geq 2)$ if

$f(x) \le f(y)$ for eath two n-tuples $x = (x_1, x_1, \dots, x_n)$ and $y = (y_1, y_1, \dots, y_n)$ of I^n, such that $x \prec y$. The relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, k = 1, 2, \dots, n-1,$$
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Theorem 2, combined with Lemma 1 and the integral representations (4) and (5) (or (6) and (7)), imply the following theorem.

Theorem 3. *For fixed a, b with $a \neq b$, then*

(i) $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ are strictly increasing with both r and s.

(ii) If $r, s > 0$, then $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ are log-concave in both r and s; If $r, s < 0$, then $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ are log-convex in both r and s.

(iii) $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ are Schur-concave on R_+^2, and Schur-convex on R_-^2, with respect to (r,s), where R_+ (R_-) denotes the set of positive (negative) reals.

3. Applications

As the first application of the integral representations (6) and (7), we prove the comparison theorem of $E_{r,s}(a,b)$ and $G_{r,s}(a,b)$ established by E. Neuman, Zs. Píes [23]. As the second application of (6) and (7), we prove a monotonicity result for the ratio between two Stolarsky means, and an integral version of Alzer-Martins' inequality can be immediate concluded as consequence.

Theorem 4. *Let $a, b > 0$ with $a \neq b$ and let $r, s \in \mathbb{R}$. Then*

$$G_{r,s}(a,b) \gtreqqless E_{r,s}(a,b) \quad \textit{according as} \quad r + s \gtreqqless 0. \tag{12}$$

Proof. Set $p = \ln \sqrt{\frac{a}{b}}$. By integral representation (6) and (7), we obtain

$$\ln \frac{G_{r,s}(a,b)}{E_{r,s}(a,b)} = \frac{1}{s-r} \int_r^s w(t)dt, \tag{13}$$

where

$$w(t) = p \tanh(pt) - p \coth(pt) + \frac{1}{t}.$$

Obviously,

$$w(-t) = -w(t). \tag{14}$$

Easy computation reveals

$$w(t) = \frac{1}{t}\left[1 - \frac{2pt}{\sinh(2pt)}\right] \gtrless 0 \quad \text{according as} \quad t \gtrless 0, \tag{15}$$

since the even function $x \mapsto \frac{x}{\sinh(x)}$ is strictly increasing on $(-\infty, 0)$, and strictly decreasing on $(0, \infty)$, the function $x \mapsto \frac{x}{\sinh(x)}$ takes its maximum 1 at $x = 0$.

If $r + s = 0$, then it follows from (1) and (2) that

$$G_{r,-r}(a,b) = E_{r,-r}(a,b) = \sqrt{ab}.$$

When $r = s \neq 0$, we get

$$\begin{aligned} & G_{r,r}(a,b) - E_{r,r}(a,b) \\ &= \frac{1}{r}\left[\frac{x\ln x + y\ln y}{x+y} - \frac{x\ln x - y\ln y}{x-y} + 1\right] \quad (\text{Set} \quad x = a^r, y = b^r) \\ &= \frac{1}{r}\left[1 - \frac{2xy(\ln x - \ln y)}{(x+y)(x-y)}\right] = \frac{1}{r}\left[1 - \frac{H(x,y)}{L(x,y)}\right] \\ &\gtreqless 0 \quad \text{according as} \quad r \gtreqless 0. \end{aligned}$$

Now, (13), combined with (14) and (15), imply that

$$G_{r,s}(a,b) \gtreqless E_{r,s}(a,b) \quad \text{according as} \quad r+s \gtreqless 0.$$

The proof of Theorem 4 is complete. □

Theorem 5. *Let a, b, c, d be positive numbers with $a \neq b, c \neq d$ and r, s be real numbers, and let*

$$R_{r,s}(a,b,c,d) = \frac{E_{r,s}(a,b)}{E_{r,s}(c,d)}. \tag{16}$$

Then the function $R_{r,s}(a,b,c,d)$ are strictly $\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}$ with both r and s according as

$$\frac{\max\{a,b\}}{\min\{a,b\}} \gtrless \frac{\max\{c,d\}}{\min\{c,d\}}. \tag{17}$$

Proof. Since Stolarsky mean is symmetric in its variables, without loss of generality, assume that $a < b$ and $c < d$. Set $p = \ln\sqrt{\frac{b}{a}}$ and $q = \ln\sqrt{\frac{d}{c}}$. By integral representation (6), we obtain

$$\ln R_{r,s}(a,b,c,d) = \frac{1}{s-r}\int_r^s \delta(t)dt, \tag{18}$$

where

$$\delta(t) = p\coth(pt) - q\coth(qt) + \ln\sqrt{\frac{ab}{cd}}.$$

Easy computation reveals

$$\begin{aligned} \delta'(t) &= \frac{4q^2}{(e^{qt} - e^{-qt})^2} - \frac{4p^2}{(e^{pt} - e^{-pt})^2} \\ &= \frac{1}{t^2}\left[\left(\frac{qt}{\sinh(qt)}\right)^2 - \left(\frac{pt}{\sinh(pt)}\right)^2\right] \\ &\gtreqless 0 \quad \text{according as} \quad p \gtreqless q, \end{aligned}$$

since the even function $x \mapsto \frac{x}{\sinh(x)}$ is strictly increasing on $(-\infty, 0)$, and strictly decreasing on $(0, \infty)$. Now, Lemma 1 combined with (18), imply that the function $R_{r,s}(a,b,c,d)$ are strictly $\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}$ with both r and s, according as $\frac{b}{a} \gtrless \frac{d}{c}$. The proof of Theorem 5 is complete. □

Remark 1. It was shown in [2, 20] that let n be a positive integer, then for $r > 0$,

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{19}$$

We call the left-hand side of (19) Alzer's inequality [2], and the right-hand side of (19) Martins' inequality [20]. Several easy proofs of Alzer's inequality have been published by different authors, see [6, 34, 40]. In [4, 8, 10] Alzer's inequality is extended to all real r. In [3, 9] it was proved that Martins' inequality is reversed for $r<0$. There have been a lot of literature about these two inequalities and their history, background, extensions and generalizations. For more detailed information, refer to [1, 5] and the references therein.

Let $b>a>0$ and $\delta>0$, by Theorem 5, the function $r\mapsto \frac{E_{1,r+1}(a,b)}{E_{1,r+1}(a,b+\delta)}$ is strictly decreasing with $r\in(-\infty,+\infty)$, and then, we present an integral version of Alzer-Martins' inequality (19) as follows:

$$\frac{b}{b+\delta}<\left(\frac{\frac{1}{b-a}\int_a^b x^r dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r dx}\right)^{1/r} \quad \text{for all real} \quad r, \tag{20}$$

$$\left(\frac{\frac{1}{b-a}\int_a^b x^r dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r dx}\right)^{1/r} \lesseqgtr \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \quad \text{according as} \quad r \gtrless 0. \tag{21}$$

This extends a result given by F. Qi and B.-N. Guo [28, 30], who established the inequalitis (20) and (21) for $r>0$. In [7, 33], the monotonicity of the function $r\mapsto \frac{E_{1,r+1}(a,b)}{E_{1,r+1}(a,b+\delta)}$ has been shown. For the generalizations of the inequalitis (20) and (21) the reader is referred to [18, 21].

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