INEQUALITIES FOR THE DOUBLE GAMMA FUNCTION

NECDET BATIR

Abstract. We establish various new upper and lower bounds in terms of the classical gamma and digamma functions for the double gamma function (or Barnes G- function).

1. Introduction

The double gamma function (or Barnes G-function) was introduced and investigated by Barnes [3, 4, 5]. Although the double gamma function has not appeared in the tables of the most well-known special functions and is just cited in an exercise proposed by Whittaker and Watson [16, pg.264], in some recent works it has found interesting applications. It has been appeared in some important integrals, infinite products and a certain classes of infinite series involving the Riemann and Hurwitz zeta functions [1, 6, 17]. The G-function plays also an important role in the study of determinants of Laplacians [11, 13, 15]. In [12] it was obtained a complete asymptotic expansion of $\log(G(z))$ for large z.

Following [12], we define the double gamma function G as the entire function

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{1}{2}z - \frac{1}{2}(\gamma+1)z^2\right) \times \prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \exp\left(-z+\frac{z^2}{2k}\right)\right\}, \tag{1.1}$$

where γ is the Euler-Mascheroni's constant. In this paper we restrict z to positive real numbers x. The G-function satisfies the functional equation $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$. Here Γ is the classical

2000 *Mathematics Subject Classification.* Primary: 33B15, 33C99; Secondary: 26D07,33F99.

Key words and phrases. Double gamma function, Barnes G-function, gamma function, digamma function and inequalities.

gamma function. For sufficiently large real number x and $a \in \mathbb{C}$ we have the Stirling formula for the G-function

$$\log G(x+a+1) = \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O(1/x) \quad (x \to \infty),$$

where A is called Glaisher-Kinkelin constant defined by

$$\log A = \lim_{n\to\infty}\left\{\sum\nolimits_{k=1}^{n} k\log k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4}\right\} = 1.28242713....$$

For positive integers n it satisfies

$$G(n+2) = 1!\,2!...n!.$$

For basic properties of the double gamma function, we refer the readers to [6-11, 14].

There is a reach literature on inequalities for the classical gamma function Γ, but as far as we know it has not been published any inequality for the double gamma function. The aim of this paper is to derive some upper and lower bounds for the double gamma function in terms of the gamma function Γ and its logarithmic derivative known as the digamma or psi function ψ.

2. Main results

The main results of this paper are the following theorems.

Theorem 2.1. *For all positive real numbers x we have*

$$(\Gamma(x))^{\frac{x}{2}} x^x (2\pi)^{\frac{x}{2}} e^{-\frac{x}{2}-\frac{x^2}{2}} < G(x+1) < \left(\frac{\Gamma(x)}{\Gamma(x/2)}\right)^x (8\pi)^{\frac{x}{2}} e^{-\frac{x}{2}-\frac{x^2}{2}}$$

Proof. For a convex function $f : [a,b] \to \mathbb{R}$ the Hadamard's inequality states that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Since $\Gamma(t)$ is logarithmically convex, applying this inequality we find that

$$x\log(\Gamma(x/2+1)) \le \int_0^x \log(\Gamma(t+1))dt \le \frac{x}{2}\log(\Gamma(x+1)). \tag{2.2}$$

On the other hand from [14, pp.32, eq.(42)] we have

$$\int_0^x \log\Gamma(t+1)dt = \frac{1}{2}[\log(2\pi)-1]x - \frac{1}{2}x^2 + x\log\Gamma(x+1) - \log G(x+1) \tag{2.3}$$

Using this relation in (2.2) and rearranging the resulting inequalities we complete the proof of Theorem 2.1. □

Theorem 2.2. *For all positive real number x the following inequality holds:*

$$(2\pi)^{\frac{x}{2}} e^{\frac{x}{2}-\frac{x^2}{2}+\frac{x^2}{2}\psi(x/2)} < G(x+1) < (2\pi)^{\frac{x}{2}} e^{-\frac{x^2}{2}+\frac{x^2}{2}\psi(x)}$$

Proof. From (1.1) we have

$$\log G(x+1) = \frac{x}{2}\log(2\pi) - \frac{1}{2}x - \frac{1}{2}x^2 - \frac{\gamma}{2}x^2 + \sum_{k=1}^{\infty}\left[k\log\left(1+\frac{x}{k}\right) - x + \frac{x^2}{2k}\right]. \tag{2.4}$$

It is very easy to see that

$$k\log\left(1+\frac{x}{k}\right) - x + \frac{x^2}{2k} = \int_0^x \left(\frac{t}{k} - \frac{t}{t+k}\right)dt \tag{2.5}$$

and $g(t) = \frac{t}{k} - \frac{t}{t+k}$ is convex. Hence by virtue of (2.1) we have

$$\frac{x^2}{2}\left(\frac{1}{k} - \frac{1}{x/2+k}\right) < k\log\left(1+\frac{x}{k}\right) - x + \frac{x^2}{2k} < \frac{x^2}{2}\left(\frac{1}{k} - \frac{1}{x+k}\right). \tag{2.6}$$

Employing these inequalities in (2.4) we get

$$\frac{x}{2}\log(2\pi) - \frac{x}{2} - \frac{x^2}{2} + \frac{x^2}{2}\left(-\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x/2+k}\right)\right) < \log G(x+1) < \frac{x}{2}\log(2\pi) - \frac{x}{2} - \frac{x^2}{2} + \frac{x^2}{2}\left(-\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right). \tag{2.7}$$

As well known the digamma or psi function ψ, the logarithmic derivative of the classical gamma function, has the following series representation

$$\psi(u+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+u}\right), \tag{2.8}$$

for $u > 0$, please see[14, pp.14]. Using this identity and the well known functional equation $\psi(x+1) = \frac{1}{x} + \psi(x)$ for the psi function in (2.7) completes the proof of Theorem2.2. □

Theorem 2.3. *For all positive real numbers x we have*

$$(2\pi)^{x/2}\left(\Gamma(x+1)\right)^x \exp\left(-\frac{x}{2} - \frac{x^2}{2} - \frac{x^2}{2}\psi(\alpha(x))\right) < G(x+1) <$$
$$(2\pi)^{x/2}\left(\Gamma(x+1)\right)^x \exp\left(-\frac{x}{2} - \frac{x^2}{2} - \frac{x^2}{2}\psi(\beta(x))\right),$$

where $\alpha(x) = x/3$ *and* $\beta(x) = \frac{x^2}{2}\frac{1}{(x+1)\log(x+1)-x}$.

Proof. Integrating both sides of (2.8) over $0 \leq u \leq t$, we get for $t > 0$

$$\log\Gamma(t+1) = -\gamma t + \sum\nolimits_{n=1}^{\infty}\left(\frac{t}{n} - \log(n+t) + \log n\right). \tag{2.9}$$

It is not difficult to see that this series is uniformly convergent for $t > 0$. Integrating once both sides of (2.9) from $t = 0$ to $t = x$ reveals that

$$\int_0^x \log\Gamma(t+1)dt = -\frac{1}{2}\gamma x^2$$
$$+\sum\nolimits_{n=1}^{\infty}\left(\frac{x^2}{2n} - \left((n+x)\log(n+x) - n\log n\right) + x\log n + x\right). \tag{2.10}$$

Applying extended mean value theorem to the function $f(t) = t\log t$ on the interval $[n, n+x]$ there exist a θ with $0 < \theta(n) < x$ for which

$$(n+x)\log(n+x) - n\log n = x(\log n + 1) + \frac{x^2}{2}\frac{1}{n+\theta(n)}. \tag{2.11}$$

Utilizing (2.11) in (2.10), we obtain

$$\int_0^x \log\Gamma(t+1)dt = -\frac{1}{2}\gamma x^2 + \frac{x^2}{2}\sum\nolimits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\theta(n)}\right). \tag{2.12}$$

Now we shall show that θ is strictly increasing in n. From (2.11) we get

$$\theta(n) = \frac{x^2}{2}\frac{1}{(n+x)\log(n+x) - (n+x)\log n - x} - n.$$

Differentiation with respect to n gives

$$\theta'(n) = -\frac{x^2}{2n^2}\frac{\log 1 + x/n) - x/n}{\left[(1+x/n)\log(1+x/n) - x/n\right]^2} - 1.$$

for a fixed $x > 0$. Setting $t = x/n$, this becomes

$$\theta'(x/t) = \frac{\sigma(t)}{2\left[(1+t)\log(1+t) - t\right]^2}, \tag{2.13}$$

where

$$\sigma(t) = t^3 - 2t^2 + 4t\log(1+t) + 3t^2\log(1+t) - 2(1+t)^2\log^2(1+t).$$

It is easy to see that $\sigma(0) = \sigma'(0) = \sigma''(0) = 0$ and

$$\sigma'''(t) = \frac{\alpha(t)}{(1+t)^3}, \tag{2.14}$$

where $\alpha(t) = 6t^3 + 12t^2 + 8t - 8(1+t)^2\log(1+t)$. A simple computation gives $\alpha(0) = \alpha'(0) = \alpha''(0) = 0$ and $\alpha'''(t) > 0$. Hence α'' is strictly increasing. In view of the fact $\alpha(0) = \alpha'(0) = \alpha''(0) = 0$ this means $\alpha(t) > 0$, implying by (2.14) that $\sigma'''(t) > 0$. Using the facts $\sigma(0) = \sigma'(0) = \sigma''(0) = 0$ again, this yields that $\sigma(t) > 0$. Applying (2.12) we find that θ is strictly increasing. Using monotonic increase of θ in (2.12) we obtain

$$\frac{x^2}{2}\psi(\theta(1)+1) < \int_0^x \log\Gamma(t+1)dt < \frac{x^2}{2}\psi(\theta(\infty)+1). \tag{2.15}$$

It is clear that

$$\theta(1) = \frac{x^2}{2}\frac{1}{(x+1)\log(x+1) - x}.$$

It is not difficult to see $\lim_{n\to\infty}\theta(n) = \frac{x}{3}$. Employing (2.3) and these relations in (2.15) we complete the proof. □

Remark 2.4. For all $x > 0$ it is easy that $x\psi(x/3) < \log(\Gamma(x))$. This implies that the left hand side of the inequality in Theorem 2.3 is better than the left hand side of the inequality in Theorem 2.2.

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Department of Mathematics, Faculty of Arts and Sciences, Yuzuncu Yil University, 65080 Van, Turkey

E-mail address: necdet_batir@hotmail.com