A note on a generalization of Lehman's inequality

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Abstract. We extend a theorem from [2] proved for functions of two arguments, to functions of arbitrary arguments.

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1 Introduction

Lehman's inequality of two variables states that (see [2])

$$H(a_1 + \cdots + a_n, b_1 + \cdots + b_n) \geq H(a_1, b_1) + \cdots + H(a_n, b_n), \tag{1}$$

where a_i, b_i $(i = \overline{1,n})$ are positive numbers, and $H(a,b) = \dfrac{2ab}{a+b}$ denotes the harmonic mean of the positive numbers a and b.

In paper [2], the second author proved generalizations of (1), by pointing out connections with subadditivity and convexity. Particularly, the following result appears in [2] (see Theorem 2 there).

Theorem 1 *Let $A = (0,+\infty) \times (0,+\infty)$, and $I = (0,+\infty)$. Define $F(t) = f(1,t)$ for $t \in I$. If f is k-homogenous and F is k-convex (k-concave), then*

$$f(a_1+a_2+\cdots+a_n, b_1+b_2+\cdots+b_n) \underset{(\geq)}{\leq} f(a_1,b_1)+f(a_2+b_2)+\cdots+f(a_n,b_n) \tag{2}$$

for any $(a_i, b_i) \in A$ $(i = \overline{1,n})$.

Recall that f is called *k-homogeneous,* if

$$f(rx, ry) = r^k f(x,y) \tag{3}$$

for any $r > 0$ and $(x,y) \in A$.

Similarly, a function $F : I \to \mathbb{R}$ is called *k-convex (concave),* if

$$F(\lambda a + \mu b) \leq \lambda^k F(a) + \mu^k F(b) \tag{4}$$

for any $a, b \in I$, and any $\lambda, \mu > 0$, $\lambda + \mu = 1$ (see [1], [2]).

Clearly, the definitions (3) and (4) can be extended to functions of many arguments, as it is shown in [2].

Let $f : (0, +\infty)^n \to \mathbb{R}$ be a function with n-arguments, and put $p = (x_1, x_2, \ldots, x_n) \in A = (0, +\infty)^n$. Then f is k-homogeneous, if

$$f(rp) = r^k f(p) \tag{5}$$

for all $r > 0$ and $p \in A$. The function $F : (0, +\infty)^{n-1} \to \mathbb{R}$ is called *k-convex,* if

$$F(\lambda p + \mu p') \leq \lambda^k F(p) + \mu^k F(p'), \tag{6}$$

for any p, $p' \in (0, +\infty)^{n-1}$ and any λ, $\mu > 0$, $\lambda + \mu = 1$.

The aim of this paper is to extend Lehman's inequality (1) as follows

$$\frac{(a_1 + a_2 + \cdots + a_s) \cdot (b_1 + b_2 + \cdots + b_s) \cdot \ldots \cdot (z_1 + z_2 + \cdots + z_s)}{(a_1 + a_2 + \cdots + a_s) + (b_1 + b_2 + \cdots + b_s) + \cdots + (z_1 + z_2 + \cdots + z_s)} \geq$$
$$\geq \frac{a_1 b_1 \cdot \ldots \cdot z_1}{a_1 + b_1 + \cdots + z_1} + \cdots + \frac{a_s b_s \cdot \ldots \cdot z_s}{a_s b_s + \cdots + z_s} \tag{7}$$

In fact, (7) will be a consequence of a more general result of type (2).

2 Main results

The following extension of Theorem 1 of the Introduction will be proved:

Theorem 2 *Let $A = (0, +\infty)^n$, $I = (0, +\infty)^{n-1}$, where $n \geq 2$. Let*

$$F(t_1, t_2, \ldots, t_{n-1}) = f(1, t_1, t_2, \ldots, t_{n-1}),$$

where $f : A \to R$ is a k-homogeneous function. If F is k-convex (concave), then

$$f(a_1 + a_2 + \cdots + a_s, b_1 + b_2 + \cdots + b_s, \cdots, z_1 + z_2 + \cdots + z_s) \underset{(\geq)}{\leq}$$
$$\underset{(\geq)}{\leq} f(a_1, b_1, \ldots, z_1) + \cdots + f(a_s, b_s, \ldots, z_s) \tag{8}$$

for all $s \geq 1$, and all $(a_1, b_1, \ldots, z_1) \in \mathbb{R}^n, \ldots, (a_s, b_s, \ldots, z_s) \in \mathbb{R}^n$.

Proof. First remark that, exactly as in the real-variable case, by induction it follows that (see [2], relation (8), for F being k-convex),

$$F(\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_s p_s) \leq \lambda_1^k F(p_1) + \lambda_2^k F(p_2) + \cdots + \lambda_s^k F(p_s) \tag{9}$$

for any $p_i \in \mathbb{R}^{n-1}$, $(i = \overline{1, s})$, $\lambda_i > 0$, $\lambda_1 + \lambda_2 + \cdots + \lambda_s = 1$.

When F is k-concave, (9) holds with reversed sign of inequality.

Put now in (9) the following values:

$$\lambda_1 = \frac{a_1}{a_1 + a_2 + \cdots + a_s}, \quad p_1 = \left(\frac{b_1}{a_1}, \frac{c_1}{a_1}, \ldots, \frac{z_1}{a_1}\right) \in \mathbb{R}^{n-1},$$
$$\lambda_2 = \frac{a_2}{a_1 + a_2 + .. + a_s}, \quad p_2 = \left(\frac{b_2}{a_2}, \frac{c_2}{a_2}, \ldots, \frac{z_2}{a_2}\right) \in \mathbb{R}^{n-1},$$
$$\vdots$$
$$\lambda_s = \frac{a_s}{a_1 + a_2 + \cdots + a_s}, \quad p_s = \left(\frac{b_s}{a_s}, \frac{c_s}{a_s}, \ldots, \frac{z_s}{a_s}\right) \in \mathbb{R}^{n-1}.$$

Now, we can write

$$\begin{aligned}
&\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_s p_s = \\
=&\left(\frac{b_1}{a_1 + a_2 + \cdots + a_s}, \frac{c_1}{a_1 + a_2 + \cdots + a_s}, \ldots, \frac{z_1}{a_1 + a_2 + \cdots + a_s}\right) + \cdots + \\
+&\left(\frac{b_s}{a_1 + a_2 + \cdots + a_s}, \frac{c_s}{a_1 + a_2 + \cdots + a_s}, \ldots, \frac{z_s}{a_1 + a_2 + \cdots + a_s}\right) = \\
=&\left(\frac{b_1 + b_2 + \cdots + b_s}{a_1 + a_2 + \cdots + a_s}, \frac{c_1 + c_2 + \cdots + c_s}{a_1 + a_2 + \cdots + a_s}, \ldots, \frac{z_1 + z_2 + \cdots + z_s}{a_1 + a_2 + \cdots + a_s}\right).
\end{aligned}$$

Thus, by (9) one has

$$\begin{aligned}
&F\left(\frac{b_1 + b_2 + \cdots + b_s}{a_1 + a_2 + \cdots + a_s}, \frac{c_1 + c_2 + \cdots + c_s}{a_1 + a_2 + \cdots + a_s}, \ldots, \frac{z_1 + z_2 + \cdots + z_s}{a_1 + a_2 + \cdots + a_s}\right) \leq \\
\leq&\frac{a_1^k}{(a_1 + a_2 + \cdots + a_s)^k} \cdot F\left(\frac{b_1}{a_1}, \frac{c_1}{a_1}, \ldots, \frac{z_1}{a_1}\right) + \cdots + \\
+&\frac{a_s^k}{(a_1 + a_2 + \cdots + a_s)^k} \cdot F\left(\frac{b_s}{a_s}, \frac{c_s}{a_s}, \ldots, \frac{z_s}{a_s}\right).
\end{aligned}$$

Since f is k-homogeneous, we get

$$\begin{aligned}
&f(a_1 + a_2 + \cdots + a_s, b_1 + b_2 + \cdots + b_s, z_1 + z_2 + \cdots + z_s) = \\
=&(a_1 + a_2 + \cdots + a_s)^k f\left(1, \frac{b_1 + b_2 + \cdots + b_s}{a_1 + a_2 + \cdots + a_s}, \ldots, \frac{z_1 + z_2 + \cdots + z_s}{a_1 + a_2 + \cdots + a_s}\right) = \\
=&(a_1 + a_2 + \cdots + a_s)^k F\left(\frac{b_1 + b_2 + \cdots + b_s}{a_1 + a_2 + \cdots + a_s}, \ldots, \frac{z_1 + z_2 + \cdots + z_s}{a_1 + a_2 + \cdots + a_s}\right) \leq \\
\leq&a_1^k F\left(\frac{b_1}{a_1}, \frac{c_1}{a_1}, \ldots, \frac{z_1}{a_1}\right) + \cdots + a_s^k F\left(\frac{b_s}{a_s}, \frac{c_s}{a_s}, \ldots, \frac{z_s}{a_s}\right) = \\
=&f(a_1, b_1, \ldots, z_1) + \cdots + f(a_s, b_s, \ldots, z_s),
\end{aligned}$$

and the proof of Theorem 2 is completed. ■

Corollary 3 *Inequality* (7) *holds true.*

Proof. Put $f(a_1, b_1, \dots, z_1) = \dfrac{a_1 b_1 \cdot \ldots \cdot z_1}{a_1 + b_1 + \cdots + z_1}$. It is immediate that f is $n-1$ homogeneous. Since $F(t_1, \dots, t_{n-1}) = \dfrac{t_1 \cdot \ldots \cdot t_{n-1}}{1 + t_1 + \cdots + t_{n-1}}$, we have to show that F is $n-1$ concave. But, as F is a continuous function, this is equivalent to the $n-1$-Jensen concavity of this function (see [1], [2]), i.e.

$$F\left(\frac{a+b}{2}\right) \geq \frac{F(a) + F(b)}{2^{n-1}}. \tag{10}$$

Let $a = (a_1, \dots, a_{n-1})$, $b = (b_1, \dots, b_{n-1})$. Then (10) is equivalent to

$$\frac{2(a_1 + b_1) \cdot \ldots \cdot (a_{n-1} + b_{n-1})}{2 + a_1 + b_1 + a_{n-1} + b_{n-1}} \geq \frac{a_1 \cdot \ldots \cdot a_{n-1}}{1 + a_1 + \cdots + a_{n-1}} + \frac{b_1 \cdot \ldots \cdot b_{n-1}}{1 + b_1 + \cdots + b_{n-1}} \tag{11}$$

Put $n - 1 = k$. For $k = 1$, the relation (11) follows by

$$H(a_1, 1) + H(b_1, 1) \leq H(a_1 + b_1, 2)$$

of Lehman's inequality (1).

Now, accepting (11) for $k \geq 1$, we can prove it also for $k + 1$ which is a simple exercise. So (11) follows by mathematical induction. Thus, inequality (7) follows as an application of Theorem 2. ■

References

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