A NOTE ON l^p NORMS OF WEIGHTED MEAN MATRICES

PENG GAO

Abstract. We present some results concerning the l^p norms of weighted mean matrices. These results can be regarded as analogues to a result of Bennett concerning weighted Carleman's inequalities.

1. Introduction

Suppose throughout that $p \neq 0, \frac{1}{p}+\frac{1}{q}=1$. For $p \geq 1$, let l^p be the Banach space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$ with norm

$$||\mathbf{a}||_p := (\sum_{n=1}^{\infty}|a_n|^p)^{1/p} < \infty.$$

The celebrated Hardy's inequality ([8, Theorem 326]) asserts that for $p>1$,

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p \leq \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p. \tag{1.1}$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\Big|\Big|C\cdot \mathbf{a}\Big|\Big|_p^p=\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{\infty}c_{n,k}a_k\Big|^p \leq U\sum_{n=1}^{\infty}|a_n|^p, \tag{1.2}$$

in which $C=(c_{n,k})$ and the parameter $p>1$ are assumed fixed, and the estimate is to hold for all complex sequences $\mathbf{a}\in l^p$. The l^p operator norm of C is then defined as

$$||C||_{p,p}=\sup_{||\mathbf{a}||_p=1}\Big|\Big|C\cdot \mathbf{a}\Big|\Big|_p.$$

It follows that inequality (1.2) holds for any $\mathbf{a}\in l^p$ when $U^{1/p}\geq ||C||_{p,p}$ and fails to hold for some $\mathbf{a}\in l^p$ when $U^{1/p}<||C||_{p,p}$. Hardy's inequality thus asserts that the Cesáro matrix operator C, given by $c_{n,k}=1/n, k\leq n$ and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

We say a matrix $A=(a_{n,k})$ is a lower triangular matrix if $a_{n,k}=0$ for $n<k$ and a lower triangular matrix A is a summability matrix if $a_{n,k}\geq 0$ and $\sum^n_{k=1}a_{n,k}=1$. We say a summability matrix A is a weighted mean matrix if its entries satisfy:

$$a_{n,k}=\lambda_k/\Lambda_n,\ 1\leq k\leq n;\ \ \Lambda_n=\sum_{i=1}^{n}\lambda_i, \lambda_i\geq 0, \lambda_1>0. \tag{1.3}$$

Hardy's inequality (1.1) now motivates one to determine the l^p operator norm of an arbitrary summability or weighted mean matrix A. In [7], the author proved the following result:

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Theorem 1.1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.3)*. If for any integer $n \geq 1$, there exists a positive constant $0 < L < p$ such that*

$$\frac{\Lambda_{n+1}}{\lambda_{n+1}} \leq \frac{\Lambda_n}{\lambda_n}\Big(1-\frac{L\lambda_n}{p\Lambda_n}\Big)^{1-p}+\frac{L}{p}\ , \tag{1.4}$$

then $||A||_{p,p} \leq p/(p-L)$.

It is easy to see that the above result implies the following well-known result of Cartlidge [4] (see also [1, p. 416, Theorem C]):

Theorem 1.2. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.3)*. If*

$$L = \sup_n \Big(\frac{\Lambda_{n+1}}{\lambda_{n+1}}-\frac{\Lambda_n}{\lambda_n}\Big) < p\ , \tag{1.5}$$

then $||A||_{p,p} \leq p/(p-L)$.

The above result of Cartlidge are very handy to use when determining l^p norms of certain weighted mean matrices. We refer the readers to the articles [5], [2], [6] and [7] for more recent developments in this area.

We note here that by a change of variables $a_k \rightarrow a_k^{1/p}$ in (1.1) and on letting $p \rightarrow +\infty$, one obtains the following well-known Carleman's inequality [3], which asserts that for convergent infinite series $\sum a_n$ with non-negative terms, one has

$$\sum^{\infty}_{n=1}(\prod^n_{k=1}a_k)^{\frac{1}{n}} \leq e\sum^{\infty}_{n=1}a_n,$$

with the constant e being best possible.

It is then natural to study the following weighted version of Carleman's inequality:

$$\sum^{\infty}_{n=1}\Big(\prod^n_{k=1}a_k^{\lambda_k/\Lambda_n}\Big) \leq E\sum^{\infty}_{n=1}a_n, \tag{1.6}$$

where the notations are as in (1.3). The task here is to determine the best constant E so that inequality (1.6) holds for any convergent infinite series $\sum a_n$ with non-negative terms. Note that Cartlidge's result (Theorem 1.2) implies that when (1.5) is satisfied, then for any $\mathbf{a} \in l^p$, one has

$$\sum^{\infty}_{n=1}\Big|\sum^n_{k=1}\frac{\lambda_k a_k}{\Lambda_n}\Big|^p \leq \Big(\frac{p}{p-L}\Big)^p\sum^{\infty}_{n=1}|a_n|^p. \tag{1.7}$$

Similar to our discussions above, by a change of variables $a_k \rightarrow a_k^{1/p}$ in (1.7) and on letting $p \rightarrow +\infty$, one obtains inequality (1.6) with $E = e^L$ as long as (1.5) is satisfied with p replaced by $+\infty$ there.

Note that (1.5) can be regarded as the case $p \rightarrow 1^+$ of (1.4) while the case $p \rightarrow +\infty$ of (1.4) suggests the following result:

Corollary 1.1. *Suppose that*

$$M = \sup_n \frac{\Lambda_n}{\lambda_n}\log\Big(\frac{\Lambda_{n+1}/\lambda_{n+1}}{\Lambda_n/\lambda_n}\Big) < +\infty,$$

then inequality (1.6) *holds with $E = e^M$.*

In fact, the above corollary is a consequence of the following nice result of Bennett (see the proof of [2, Theorem 13]):

Theorem 1.3. *Inequality* (1.6) *holds with*

$$E = \sup_n \frac{\Lambda_{n+1}}{\lambda_{n+1}}\prod^n_{k=1}\Big(\frac{\lambda_k}{\Lambda_k}\Big)^{\lambda_k/\Lambda_n}.$$

It is shown in the proof of Theorem 13 in [2] that Corollary 1.1 follows from the above theorem. It is also easy to see that $M \leq L$ for L defined by (1.5) so that Corollary 1.1 provides a better result than what one can infer from Cartlidge's result as discussed above.

Note that the bound given in Theorem 1.3 is global in the sense that it involves all the λ_n's and it implies the local version Corollary 1.1, in which only the terms Λ_n/λ_n and $\Lambda_{n+1}/\lambda_{n+1}$ are involved. It is then natural to ask whether one can obtain a similar result for the l^p norms for $p > 1$ so that it implies the local version Theorem 1.1. It is our goal in this note to present one such result and as our result is motivated by the result of Bennett, we will first study the limiting case $p \to +\infty$, namely weighted Carleman's inequalities in the next section before we move on to the l^p cases in Section 3.

2. A Discussion on Weighted Carleman's Inequalities

In this section we study weighted Carleman's inequalities. Our goal is to give a different proof of Theorem 1.3 than that given in [2] and discuss some variations of it. It suffices to consider the cases of (1.6) with the infinite summations replaced by any finite summations, say from 1 to $N \geq 1$ here. Our starting point is the following result of Pečarić and Stolarsky [9, (2.4)], which is an outgrowth of Redheffer's approach in [10]:

$$\sum_{n=1}^{N} \Lambda_n (b_n - 1) G_n + \Lambda_N G_N \leq \sum_{n=1}^{N} \lambda_n a_n b_n^{\Lambda_n/\lambda_n}, \tag{2.1}$$

where $\mathbf{b}$ is any positive sequence and

$$G_n = \prod_{k=1}^{n} a_k^{\lambda_k/\Lambda_n}.$$

We now discard the last term on the left-hand side of (2.1) and make a change of variables $\lambda_n a_n b_n^{\Lambda_n/\lambda_n} \mapsto a_n$ to recast inequality (2.1) as

$$\sum_{n=1}^{N} \Lambda_n (b_n - 1) \Big(\prod_{k=1}^{n} \lambda_k^{-\lambda_k/\Lambda_n} \Big) \Big(\prod_{k=1}^{n} b_k^{-\Lambda_k/\Lambda_n} \Big) G_n \leq \sum_{n=1}^{N} a_n.$$

Now, a further change of variables $b_n \mapsto \lambda_{n+1} b_n / \lambda_n$ allows us to recast the above inequality as

$$\sum_{n=1}^{N} \Lambda_n \Big(\frac{b_n}{\lambda_n} - \frac{1}{\lambda_{n+1}} \Big) \prod_{k=1}^{n} b_k^{-\Lambda_k/\Lambda_n} G_n \leq \sum_{n=1}^{N} a_n. \tag{2.2}$$

If one now chooses $b_n = (\Lambda_{n+1}/\lambda_{n+1})/(\Lambda_n/\lambda_n)$ such that $\Lambda_n(b_n/\lambda_n - 1/\lambda_{n+1}) = 1$, then one gets immediately the following:

$$\sum_{n=1}^{N} \frac{\lambda_{n+1}}{\Lambda_{n+1}} \prod_{k=1}^{n} \Big(\frac{\Lambda_k}{\lambda_k} \Big)^{\lambda_k/\Lambda_n} G_n \leq \sum_{n=1}^{N} a_n,$$

from which the assertion of Theorem 1.3 can be readily deduced.

Another natural choice for the values of b_n's is to set $\prod_{k=1}^{n} b_k^{-\Lambda_k/\Lambda_n} = e^{-M}$. From this we see that $b_n = e^{M\lambda_n/\Lambda_n}$ and substituting these values for b_n's we obtain via (2.2):

$$\sum_{n=1}^{N} \Lambda_n \Big(\frac{e^{M\lambda_n/\Lambda_n}}{\lambda_n} - \frac{1}{\lambda_{n+1}} \Big) G_n \leq e^{M} \sum_{n=1}^{N} a_n. \tag{2.3}$$

One checks easily that the above inequality implies Corollary 1.1.

We now consider a third choice for the b_n's by setting $b_n = e^{(\Lambda_{n+1}/\lambda_{n+1}-\Lambda_n/\lambda_n)/(\Lambda_n/\lambda_n)}$ and it follows from this and (2.2) that

$$\sum_{n=1}^N \Lambda_n\Big(\frac{e^{(\Lambda_{n+1}/\lambda_{n+1}-\Lambda_n/\lambda_n)/(\Lambda_n/\lambda_n)}}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)e^{-\sum_{k=1}^n\frac{\lambda_k}{\Lambda_n}(\frac{\Lambda_{k+1}}{\lambda_{k+1}}-\frac{\Lambda_k}{\lambda_k})}G_n \le \sum_{n=1}^N a_n,$$

from which we deduce the following

Corollary 2.1. *Suppose that*

$$M = \sup_n \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n}(\frac{\Lambda_{k+1}}{\lambda_{k+1}}-\frac{\Lambda_k}{\lambda_k}) < +\infty,$$

then inequality (1.6) *holds with* $E = e^M$.

We point out here that the above corollary also provides a better result than what one can infer from Cartlidge's result and it also follows from Theorem 1.3.

We note here the optimal choice for the b_n's will be to choose them to satisfy

$$\Lambda_n\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\prod_{k=1}^n b_k^{-\Lambda_k/\Lambda_n} = e^{-L}.$$

In general it is difficult to solve for the b_n's from the above equations. But we can solve b_1 to get $b_1 = e^L\lambda_1/((e^L-1)\lambda_2)$ and if we set $b_n = (\Lambda_{n+1}/\lambda_{n+1})/(\Lambda_n/\lambda_n)$ for $n \ge 2$, we can then deduce from (2.2) that

$$e^{-L}G_1 + \sum_{n=2}^N \Big(\frac{\Lambda_2(e^L-1)}{\lambda_1 e^L}\Big)^{\lambda_1/\Lambda_N}\frac{\lambda_{n+1}}{\Lambda_{n+1}}\prod_{k=1}^n\Big(\frac{\Lambda_k}{\lambda_k}\Big)^{\lambda_k/\Lambda_n}G_n \le \sum_{n=1}^N a_n.$$

Note that this gives an improvement upon Theorem 1.3 as long as $\lambda_2/\Lambda_2 > e^{-L}$. Similarly, one obtains

$$G_1 + \sum_{n=2}^N \Big(\frac{\lambda_2(e^L-1)}{\lambda_1}\Big)^{\lambda_1/\Lambda_N}\Lambda_n\Big(\frac{e^{M\lambda_n/\Lambda_n}}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)G_n \le e^M\sum_{n=1}^N a_n.$$

3. The l^p Cases

We now return to the discussions on the general l^p cases. Again it suffices to consider the cases of (1.7) with the infinite summations replaced by any finite summations, say from 1 to $N \ge 1$ here. We may also assume that $a_n \ge 0$ for all n. With the discussions of the previous section in mind, here we seek for an l^p version of (2.2). Fortunately this is available by noting that it follows from inequality (4.3) of [6] that

$$\sum_{n=1}^N\Big(\sum_{k=1}^n w_k\Big)^{-(p-1)}\Big(\frac{w_n^{p-1}}{\lambda_n^p}-\frac{w_{n+1}^{p-1}}{\lambda_{n+1}^p}\Big)\Lambda_n^pA_n^p \le \sum_{n=1}^N a_n^p, \tag{3.1}$$

where w_n's are positive parameters and

$$A_n = \frac{\sum_{k=1}^n\lambda_k a_k}{\Lambda_n}.$$

By a change of variables $w_n \mapsto \lambda_n w_n^{1/(p-1)}$, we can recast inequality (3.1) as

$$\sum_{n=1}^N\Big(\frac{\sum_{k=1}^n\lambda_k w_k^{1/(p-1)}}{\Lambda_n}\Big)^{-(p-1)}\Big(\frac{w_n}{\lambda_n}-\frac{w_{n+1}}{\lambda_{n+1}}\Big)\Lambda_nA_n^p \le \sum_{n=1}^N a_n^p,$$

With another change of variables, $w_n/w_{n+1} \mapsto b_n$, we can further recast the above inequality as

$$\sum_{n=1}^{N}\Big(\frac{\sum_{k=1}^{n}\lambda_k\prod_{i=k}^{n}b_i^{1/(p-1)}}{\Lambda_n}\Big)^{-(p-1)}\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Lambda_n A_n^p \leq \sum_{n=1}^{N}a_n^p. \tag{3.2}$$

Note that if one makes a change of variables $a_n^p \mapsto a_n$, then inequality (2.2) follows from the above inequality upon letting $p \to +\infty$.

One can then deduce Theorem 1.1 by choosing $b_n = (1 - L\lambda_n/(p\Lambda_n))^{-(p-1)}$ in (3.2) (see [7]). It is easy to see that this gives back inequality (2.3) upon letting $p \to +\infty$ by a change of variables $a_n^p \mapsto a_n$ and setting $L = M$. We note here that the b_n's are so chosen so that the following relations are satisfied:

$$\frac{\sum_{k=1}^{n}\lambda_k\prod_{i=k}^{n}b_i^{1/(p-1)}}{\Lambda_n} = \frac{p}{p-L}. \tag{3.3}$$

Now, to get the l^p analogues of Theorem 1.3, we just need to note that in the $p \to +\infty$ case, one obtains Corollary 1.1 by setting $\prod_{k=1}^{n}b_k^{-\Lambda_k/\Lambda_n} = e^{-M}$ and the conclusion of Corollary 1.1 follows by requiring that $\Lambda_n(b_n/\lambda_n - 1/\lambda_{n+1}) \geq 1$ for the so chosen b_n's. If one instead chooses the b_n's so that the conditions $\Lambda_n(b_n/\lambda_n - 1/\lambda_{n+1}) = 1$ are satisfied, then Theorem 1.3 will follow. Now, in the l^p cases, the choice of the b_n's so that the conditions (3.3) are satisfied implies Theorem 1.1 as (1.4) implies that for the so chosen b_n's,

$$\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Lambda_n \geq 1-\frac{L}{p}.$$

Thus, in order to obtain an result analogue to Theorem 1.3 for the l^p cases, we are then motivated to take the b_n's so that the following conditions are satisfied:

$$\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Lambda_n = 1-\frac{L}{p}. \tag{3.4}$$

We then easily deduce the following l^p analogue of Theorem 1.3:

Theorem 3.1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.3)*. If for any integer $n \geq 1$, there exists a positive constant $0 < L < p$ such that*

$$\sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_n}\prod_{i=k}^{n}\Big(\frac{\Lambda_{i+1}/\lambda_{i+1}-L/p}{\Lambda_i/\lambda_i}\Big)^{1/(p-1)} \leq \frac{p}{p-L},$$

then $||A||_{p,p} \leq p/(p-L)$.

It is easy to see by induction that Theorem 3.1 implies Theorem 1.1. Of course one should really choose the b_n's so that the following relations are satisfied:

$$\Big(\frac{\sum_{k=1}^{n}\lambda_k\prod_{i=k}^{n}b_i^{1/(p-1)}}{\Lambda_n}\Big)^{-(p-1)}\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Lambda_n = \Big(\frac{p}{p-L}\Big)^{-p}.$$

In general it is difficult to determine the b_n's this way but one can certainly solve for b_1 and by choosing other b_n's so that (3.4) are satisfied, one can obtain a slightly better result than Theorem 3.1, we shall leave the details to the reader.

We note here that the choice of the b_n's satisfying (3.4) corresponds to the following choice for the a_n's in Section 4 of [7] (these a_n's are not to be confused with the a_n's used in the rest of the paper):

$$a_n = \Big(\frac{\Lambda_{n+1}/\lambda_{n+1}-L/p}{\Lambda_n/\lambda_n}\Big)^{1/(p-1)}a_{n+1}, \quad a_1 = 1.$$

We also note here that in the case of $\lambda_n = L = 1$, on choosing b_n's to satisfy (3.4), we obtain via (3.2) that

$$\sum_{n=1}^{N}\Big(\frac{\sum_{k=1}^n\prod_{i=k}^n(1+(1-1/p)/i)^{1/(p-1)}}{n}\Big)^{-(p-1)}A_n^p \leq \frac{p}{p-1}\sum_{n=1}^{N}a_n^p.$$

This gives an improvement of Hardy's inequality (1.1).

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Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371 Singapore

E-mail address: penggao@ntu.edu.sg