Generalizations of Some Classical Inequalities via a Special Functional Property

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Abstract

In this paper, by using a special functional property, we generalize some well-known inequalities such as Cauchy-Schwarz's, Cauchy-Buniakowski's, Čebyšev's, Steffensen's and Aczél's inequalities. We also present some interesting corollariles and consequences of the general results derived here.

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1. Introduction

It is well-known that the Cauchy-Schwarz's, Cauchy-Buniakowski's, Čebyšev's, Steffensen's and Aczél's inequalities for integrals and sums are classical and basic in mathematics and have been refined, generalized and applied by a remarkably large number of researchers for different and various motivations. For a detailed information, the interested reader may refer to relevant chapters in [5, 7, 8, 9, 12] and to numerous related references which are cited therein.

The aim of this paper is to generalize each of the above-mentioned inequalities by a specific functional approach. As we will show, this approach may be frequently used and applied to generalize many other inequalities as well. For this purpose, we let $\mathfrak{S}$ denote a linear functional mapping onto the function $f(x)$ and then consider the following special function:

$$F(x; f, \lambda) := f(x) + \lambda \mathfrak{S}\big(f(x)\big) \qquad (\lambda \in \mathbb{R} \setminus \{0\}). \tag{1.1}$$

One of the important properties of the function $F(x; f, \lambda)$ defined by (1.1) is that

$$\begin{aligned} \mathfrak{S}(F) &= \mathfrak{S}\big(f + \lambda \mathfrak{S}(f)\big) = \mathfrak{S}(f) + \lambda \mathfrak{S}(f)\mathfrak{S}(1) \\ &= \big(1 + \lambda \mathfrak{S}(1)\big)\mathfrak{S}(f) = K\mathfrak{S}(f) \qquad \big(K := 1 + \lambda \mathfrak{S}(1)\big). \end{aligned} \tag{1.2}$$

By using the general property (1.2), we show that each of the above-mentioned inequalities can be generalized fairly easily. We note that the property (1.2) provides a general approach which can be applied in cases of many other inequalities as well. Furthermore, the property (1.2) could only be applied for one time, that is, we cannot apply the property (1.2) for more than one time in order to generalize the above-mentioned inequalities again, because we have

$$\begin{gathered} \mathfrak{S}\big(\mathfrak{S}(F)\big) = \mathfrak{S}\big(K\mathfrak{S}(f)\big) = K\mathfrak{S}(f)\mathfrak{S}(1) = K^*\mathfrak{S}(f) \\ \big(K^* := K\mathfrak{S}(1); \;\; K := 1 + \lambda \mathfrak{S}(1)\big). \end{gathered} \tag{1.3}$$

2. A generalization of Cauchy-Schwarz's inequality

It is common knowledge that Cauchy-Schwarz's inequality has a long history spanning over 187 years ever since it appeared in [3]. It may be stated as follows:

If $\{F_k\}_{k=1}^n$ $(n \in \mathbb{N})$ *and* $\{G_k\}_{k=1}^n$ $(n \in \mathbb{N})$ *are two sequences of real numbers, then*

$$\left(\sum_{k=1}^n F_k G_k\right)^2 \leqq \left(\sum_{k=1}^n F_k^2\right)\left(\sum_{k=1}^n G_k^2\right). \tag{2.1}$$

As we observed above, the linear functional applied in (2.1) is

$$\mathfrak{S}(\cdot) = \sum(\cdot).$$

So, by noting (1.1), if we set

$$F_k = a_k - \frac{p}{n}\sum_{k=1}^n a_k, \quad \text{and} \quad G_k = b_k - \frac{q}{n}\sum_{k=1}^n b_k, \tag{2.2}$$

then we obtain

$$\mathfrak{S}\left(F_k\right) = \sum_{k=1}^n \left(a_k - \frac{p}{n}\sum_{k=1}^n a_k\right) = \sum_{k=1}^n a_k - \frac{p}{n}\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n 1\right)$$

$$= (1-p)\sum_{k=1}^{n} a_k = (1-p)\mathfrak{S}(a_k), \tag{2.3}$$

$$\mathfrak{S}\left(F_k^2\right) = \sum_{k=1}^{n} F_k^2 = \sum_{k=1}^{n}\left[a_k^2 + \frac{p^2}{n^2}\left(\sum_{k=1}^{n} a_k\right)^2 - \frac{2p}{n}\, a_k \sum_{k=1}^{n} a_k\right]$$

$$= \sum_{k=1}^{n} a_k^2 - \frac{p(p-2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2 \geqq 0, \tag{2.4}$$

$$\mathfrak{S}\left(G_k^2\right) = \sum_{k=1}^{n} G_k^2 = \sum_{k=1}^{n} b_k^2 + \frac{q(q-2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2 \geqq 0, \tag{2.5}$$

and

$$\mathfrak{S}\left(F_k G_k\right) = \sum_{k=1}^{n} F_k G_k = \sum_{k=1}^{n}\left[\left(a_k - \frac{p}{n}\sum_{k=1}^{n} a_k\right)\left(b_k - \frac{q}{n}\sum_{k=1}^{n} b_k\right)\right]$$

$$= \sum_{k=1}^{n} a_k b_k + \frac{pq-(p+q)}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right), \tag{2.6}$$

respectively. By substituting these equations into the inequality (2.1), we are led readily to the following theorem.

Theorem 1. *If $\{a_k\}_{k=1}^{n}$ and $\{b_k\}_{k=1}^{n}$ for $n \in \mathbb{N}$ are two sequences of real numbers and $p, q \in \mathbb{R}$, then*

$$\left[\sum_{k=1}^{n} a_k b_k + \frac{pq-(p+q)}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)\right]^2$$

$$\leqq \left[\sum_{k=1}^{n} a_k^2 + \frac{p(p-2)}{n}\left(\sum_{k=1}^{n} a_k\right)^2\right]\left[\sum_{k=1}^{n} b_k^2 + \frac{q(q-2)}{n}\left(\sum_{k=1}^{n} b_k\right)^2\right]. \tag{2.7}$$

The equality in (2.7) *holds true if*

$$p = q \qquad \text{and} \qquad b_k = r a_k,$$

where r is a constant.

Remark 1. As already mentioned in Section 1, we cannot anymore use the functional property (1.2) in order to *further* generalize the inequality (2.7). Moreover, Cauchy-Schwarz's inequality (2.1) can be deduced from the inequality (2.7) by letting

$$p = q = 2 \qquad \text{or} \qquad p = q = 0.$$

3. A generalization of Cauchy-Buniakowski's inequality

The integral form of the Cauchy-Schwarz's inequality (2.1) is known in the literature as Cauchy-Buniakowski's inequality [2] and can be represented as follows:

$$\left(\int_a^b F(x)G(x)\,\mathrm{d}x\right)^2 \leqq \int_a^b [F(x)]^2\,\mathrm{d}x \int_a^b [G(x)]^2\,\mathrm{d}x, \tag{3.1}$$

where $F,G \in L^2[a,b]$.

Upon setting

$$\mathfrak{S}(\cdot) = \int_a^b (\cdot)\,\mathrm{d}x$$

and

$$F(x) = f(x) - \frac{p}{b-a}\int_a^b f(x)\,\mathrm{d}x \quad \text{and} \quad G(x) = g(x) - \frac{q}{b-a}\int_a^b g(x)\,\mathrm{d}x, \tag{3.2}$$

straightforward calculation gives

$$\mathfrak{S}(F) = \int_a^b \left(f(x) - \frac{p}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\mathrm{d}x = (1-p)\int_a^b f(x)\,\mathrm{d}x = (1-p)\mathfrak{S}(f),$$

$$\begin{aligned}\mathfrak{S}\left(F^2\right) = \int_a^b [F(x)]^2\,\mathrm{d}x &= \int_a^b \left[[f(x)]^2 + \frac{p^2}{(b-a)^2}\left(\int_a^b f(x)\,\mathrm{d}x\right)^2\right.\\ &\qquad \left.- \frac{2p}{b-a}f(x)\int_a^b f(x)\,\mathrm{d}x\right]\mathrm{d}x\\ &= \int_a^b [f(x)]^2\,\mathrm{d}x + \frac{p(p-2)}{b-a}\left(\int_a^b f(x)\,\mathrm{d}x\right)^2,\end{aligned}$$

$$\mathfrak{S}\left(G^2\right) = \int_a^b [G(x)]^2\,\mathrm{d}x = \int_a^b [g(x)]^2\,\mathrm{d}x + \frac{q(q-2)}{b-a}\left(\int_a^b g(x)\,\mathrm{d}x\right)^2,$$

and

$$\begin{aligned}\mathfrak{S}(FG) &= \int_a^b F(x)G(x)\,\mathrm{d}x\\ &= \int_a^b \left[\left(f(x) - \frac{p}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\left(g(x) - \frac{q}{b-a}\int_a^b g(x)\,\mathrm{d}x\right)\right]\mathrm{d}x\\ &= \int_a^b f(x)g(x)\,\mathrm{d}x + \frac{pq-(p+q)}{b-a}\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right).\end{aligned}$$

Substituting these equations into (3.1) produces a generalization of Cauchy-Buniakowski's inequality (3.1) as stated below.

Theorem 2. *If $f, g : [a,b] \to \mathbb{R}$ are functions whose squares are integrable on $[a,b]$ and $p, q \in \mathbb{R}$, then*

$$\begin{aligned}\left[\int_a^b f(x)g(x)\,\mathrm{d}x + \frac{pq-(p+q)}{b-a}\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right)\right]^2 \\ \leqq \left[\int_a^b [f(x)]^2\,\mathrm{d}x + \frac{p(p-2)}{b-a}\left(\int_a^b f(x)\,\mathrm{d}x\right)^2\right] \\ \cdot \left[\int_a^b [g(x)]^2\,\mathrm{d}x + \frac{q(q-2)}{b-a}\left(\int_a^b g(x)\,\mathrm{d}x\right)^2\right]. \end{aligned} \tag{3.3}$$

The equality in (3.3) *holds true if*

$$p = q \qquad \text{and} \qquad g(x) = \alpha f(x),$$

where α is a constant.

It is not difficult to see that, if we set

$$p = q = 2 \qquad \text{or} \qquad p = q = 0$$

in (3.3), then Cauchy-Buniakowski's inequality (3.1) follows fairly easily. Furthermore, a generalization of the inequalities (2.7) and (3.3) was presented in a recent paper by Masjed-Jamei *et al.* [10].

4. A generalization of Čebyšev's integral inequality

The famous Čebyšev's inequality [11, p. 39, Theorem 9] can be expressed as follows:

If $F_k : [a,b] \to \mathbb{R}$ $(1 \leqq k \leqq m)$ are nonnegative and integrable functions which are all either increasing or decreasing on $[a,b]$, then

$$\prod_{k=1}^m \int_a^b F_k(x)\,\mathrm{d}x \leqq (b-a)^{m-1}\int_a^b \prod_{k=1}^m F_k(x)\,\mathrm{d}x. \tag{4.1}$$

Furthermore, if one of the functions $F_k(x)$ $(1 \leqq k \leqq m)$ is increasing and the other ones are decreasing, then the inequality (4.1) *is reversed.*

Assuming that the functions $f_k(x)$ $(1 \leqq k \leqq m)$ are all either increasing or decreasing and substituting

$$F_k(x) = f_k(x) + \frac{p_k}{b-a}\int_a^b f_k(x)\,\mathrm{d}x \qquad (p_k \in \mathbb{R};\ 1 \leqq k \leqq m) \tag{4.2}$$

into (4.1), we get

$$\begin{aligned}\prod_{k=1}^m \left((1+p_k)\int_a^b f_k(x)\,\mathrm{d}x\right) \\ \leqq (b-a)^{m-1}\int_a^b \prod_{k=1}^m \left(f_k(x) + \frac{p_k}{b-a}\int_a^b f_k(x)\,\mathrm{d}x\right). \end{aligned} \tag{4.3}$$

The right-hand side of the inequality (4.3) should still be simplified. In order to do so, it is sufficient to just simplify the related functionals. For instance, if we take $m = 3$ in (4.3), then the integral on the right-hand side of the inequality (4.3) may be rearranged as follows:

$$\begin{aligned}\mathfrak{S}\left(F_1F_2F_3\right) &= \int_a^b \prod_{k=1}^{3}\left(f_k(x) + \frac{p_k}{b-a}\int_a^b f_k(x)\,\mathrm{d}x\right)\mathrm{d}x = \int_a^b \prod_{k=1}^{3} f_k(x)\,\mathrm{d}x \\ &\quad + \frac{p_1p_2 + p_1p_3 + p_2p_3 + p_1p_2p_3}{(b-a)^2}\prod_{k=1}^{3}\int_a^b f_k(x)\,\mathrm{d}x \\ &\quad + \frac{1}{b-a}\Bigg(p_1\int_a^b f_1(x)\,\mathrm{d}x\int_a^b f_2(x)f_3(x)\,\mathrm{d}x \\ &\quad + p_2\int_a^b f_2(x)\,\mathrm{d}x\int_a^b f_1(x)f_3(x)\,\mathrm{d}x + p_3\int_a^b f_3(x)\,\mathrm{d}x\int_a^b f_1(x)f_2(x)\,\mathrm{d}x\Bigg).\end{aligned} \tag{4.4}$$

Substituting from (4.4) into (4.3) would finally lead us to the following corollary.

Corollary. *If $f, g, h : [a, b] \to \mathbb{R}$ are nonnegative and integrable functions which are all either increasing or decreasing on $[a, b]$ and $p, q, r \in \mathbb{R}$, then*

$$\begin{aligned}&(1 + p + q + r)\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x\int_a^b h(x)\,\mathrm{d}x \\ &\leqq (b-a)^2\int_a^b f(x)g(x)h(x)\,\mathrm{d}x + (b-a)\Bigg(p\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)h(x)\,\mathrm{d}x \\ &\quad + q\int_a^b g(x)\,\mathrm{d}x\int_a^b f(x)h(x)\,\mathrm{d}x + r\int_a^b h(x)\,\mathrm{d}x\int_a^b f(x)g(x)\,\mathrm{d}x\Bigg).\end{aligned} \tag{4.5}$$

If one of the functions f, g and h is increasing and the others are decreasing, then the inequality (4.5) *is reversed.*

Remark 2. If we let $p = q = r = 0$ in (4.5), then the inequality (4.1) for $m = 3$ follows. Furthermore, in its special case when

$$q = 0 \qquad \text{and} \qquad r = -p = b - a,$$

the inequality (4.5) is reduced to the following form:

$$\begin{aligned}&\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x\int_a^b h(x)\,\mathrm{d}x \leqq (b-a)^2\Bigg(\int_a^b f(x)g(x)h(x)\,\mathrm{d}x \\ &\quad - \int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)h(x)\,\mathrm{d}x + \int_a^b h(x)\,\mathrm{d}x\int_a^b f(x)g(x)\,\mathrm{d}x\Bigg),\end{aligned} \tag{4.6}$$

which can *directly* be deduced by setting

$$F_1(x) = f(x) - \int_a^b f(x)\,\mathrm{d}x, \qquad F_2(x) = g(x),$$

and

$$F_3(x) = h(x) + \int_a^b h(x)\,\mathrm{d}x$$

in (4.1) when $m = 3$. We should also note that the case $m = 2$ of (4.3) generates Čebyšev's inequality (4.1) for $m = 2$.

We find it to be worthwhile to point out that many discrete and important inequalities have been constructed in [16] by using Čebyšev's inequality (4.1) for $m = 2$ and its appropriately weighted forms.

5. A generalization of Steffensen's integral inequality

The noted Steffensen's integral inequality [21] states that, if $F(x)$ and $G(x)$ are integrable functions such that $F(x)$ is a non-increasing function and if

$$0 \leqq G(x) \leqq 1 \qquad \big(x \in (a,b)\big),$$

then

$$\int_{b-\sigma}^b F(x)\,\mathrm{d}x \leqq \int_a^b F(x)G(x)\,\mathrm{d}x \leqq \int_a^{a+\sigma} F(x)\,\mathrm{d}x, \tag{5.1}$$

where

$$\sigma = \int_a^b G(x)\,\mathrm{d}x.$$

By letting

$$F(x) = f(x) + p\int_a^b f(x)\,\mathrm{d}x \quad \text{and} \quad G(x) = g(x) + \frac{q-1}{b-a}\int_a^b g(x)\,\mathrm{d}x \tag{5.2}$$

for $p, q \in \mathbb{R}$ and computing directly, we obtain

$$\int_{b-\sigma}^b F(x)\,\mathrm{d}x = \int_{b-\sigma}^b f(x)\,\mathrm{d}x + p\sigma\int_a^b f(x)\,\mathrm{d}x, \tag{5.3}$$

$$\int_a^{a+\sigma} F(x)\,\mathrm{d}x = \int_a^{a+\sigma} f(x)\,\mathrm{d}x + p\sigma\int_a^b f(x)\,\mathrm{d}x, \tag{5.4}$$

$$\sigma = \int_a^b G(x)\,\mathrm{d}x = \int_a^b \left(g(x) + \frac{q-1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right)\mathrm{d}x = q\int_a^b g(x)\,\mathrm{d}x, \tag{5.5}$$

and

$$\begin{aligned}\int_a^b F(x)G(x)\,\mathrm{d}x &= \int_a^b \left[\left(f(x) + p\int_a^b f(x)\,\mathrm{d}x\right)\right.\\ &\qquad \left.\cdot\left(g(x) + \frac{q-1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right)\right]\mathrm{d}x\\ &= \int_a^b f(x)g(x)\,\mathrm{d}x + \left(\frac{q-1}{b-a} + pq\right)\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x\end{aligned}$$

$$= \int_a^b f(x)g(x)\,\mathrm{d}x + \left[\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) + p\sigma\right]\int_a^b f(x)\,\mathrm{d}x. \tag{5.6}$$

Upon substituting these results into (5.1), we are led eventually to the following generalization.

Theorem 3. *If $f(x)$ and $g(x)$ are integrable functions such that $f(x)$ is non-increasing and*

$$-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \leqq g(x) \leqq 1 - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \tag{5.7}$$

on (a,b), where $q \neq 0$ and

$$\sigma = q\int_a^b g(x)\,\mathrm{d}x, \tag{5.8}$$

then

$$\begin{aligned}\int_{b-\sigma}^b f(x)\,\mathrm{d}x - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)\,\mathrm{d}x &\leqq \int_a^b f(x)g(x)\,\mathrm{d}x \\ &\leqq \int_a^{a+\sigma} f(x)\,\mathrm{d}x - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)\,\mathrm{d}x. \end{aligned}\tag{5.9}$$

It is clear that Steffensen's integral inequality (5.1) is a special case of the inequality (5.9) when $q = 1$. Moreover, one of the advantages of the latter is that the quantity σ is no longer dependent upon the value

$$\int_a^b g(x)\,\mathrm{d}x$$

directly, since there exists a free parameter $q \neq 0$ in (5.8).

We note in passing that the constant p appearing in (5.2) has no influence on the inequality (5.9), since it is automatically omitted in the process of computations.

Another kind of generalizations of Steffensen's integral inequality (5.1) was presented recently in [23].

Remark 3. The Steffensen pairs are a concept related to Steffensen's discrete inequality and have been investigated in [6, 13, 14, 15, 17], and in several related references cited therein, by making use of properties of the function

$$\frac{b^x - a^x}{x}$$

established in [19, 20]. In the sequel, the logarithmically convex properties of the function

$$\frac{b^x - a^x}{x}$$

and some applications are elegantly discovered in [18, 24] and several pending manuscripts by (among others) Qi and his collaborators.

6. A generalization of Aczél's inequality

Aczél [1] proved the following inequality in 1956:

$$\left(F_1^2-\sum_{k=2}^{n}F_k^2\right)\left(G_1^2-\sum_{k=2}^{n}G_k^2\right)\leqq\left(F_1G_1-\sum_{k=2}^{n}F_kG_k\right)^2, \tag{6.1}$$

where $\{F_k\}_{k=1}^n$ and $\{G_k\}_{k=1}^n$ are positive sequences such that either

$$\left(F_1^2-\sum_{k=2}^{n}F_k^2\right)>0 \qquad \text{or} \qquad \left(G_1^2-\sum_{k=2}^{n}G_k^2\right)>0. \tag{6.2}$$

Some improvements of Aczél type inequalities have been carried out in some recent papers such as [4, 22].

By letting

$$F_1=a_1,\ \ F_k=a_k-\frac{p}{n}\sum_{k=2}^{n}a_k,\ \ G_1=b_1,\quad \text{and}\quad G_k=b_k-\frac{q}{n}\sum_{k=2}^{n}b_k, \tag{6.3}$$

and employing the equations (2.3) to (2.6), we get a generalization of Aczél's inequality (6.1) as follows.

Theorem 4. *If $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are positive sequences such that*

$$a_1^2-\sum_{k=2}^{n}a_k^2-\frac{p(p-2)}{n}\left(\sum_{k=2}^{n}a_k\right)^2>0 \tag{6.4}$$

or

$$b_1^2-\sum_{k=2}^{n}b_k^2-\frac{q(q-2)}{n}\left(\sum_{k=2}^{n}b_k\right)^2>0 \tag{6.5}$$

for $p,q\in\mathbb{R}$, then

$$\begin{aligned}\left[a_1^2-\sum_{k=2}^{n}a_k^2-\frac{p(p-2)}{n}\left(\sum_{k=2}^{n}a_k\right)^2\right]\left[b_1^2-\sum_{k=2}^{n}b_k^2-\frac{q(q-2)}{n}\left(\sum_{k=2}^{n}b_k\right)^2\right]\\ \leqq\left[a_1b_1-\sum_{k=2}^{n}a_kb_k-\frac{pq-(p+q)}{n}\left(\sum_{k=2}^{n}a_k\right)\left(\sum_{k=2}^{n}b_k\right)\right]^2.\end{aligned} \tag{6.6}$$

It is clear that Aczél's inequality (6.1) is a special case of the inequality (6.6) when

$$p=q=2 \qquad \text{or} \qquad p=q=0.$$

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