

VECTOR INEQUALITIES FOR POWERS OF SOME OPERATORS IN HILBERT SPACES

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ABSTRACT. Vector inequalities for powers of some operators in Hilbert spaces with applications for operator norm, numerical radius, commutators and self-commutators are given.

1. INTRODUCTION

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers \mathbb{C} given by [13, p. 1]:

$$W(T) = \{ \langle Tx, x \rangle, x \in H, \|x\| = 1 \}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [13, p. 8]:

$$(1.1) \quad w(T) = \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \rightarrow H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [13, p. 9]:

$$(1.2) \quad w(T) \leq \|T\| \leq 2w(T),$$

for any $T \in B(H)$

For more results on numerical radii, see [14], Chapter 11.

For other results and historical comments on the above see [13, p. 39–41]. For recent inequalities involving the numerical radius, see [2]–[10], [15], [19]–[21] and [22].

The Schwarz inequality for positive operators asserts that if T is a positive operator in $B(H)$, then

$$(1.3) \quad |\langle Tx, y \rangle|^2 \leq \langle Tx, x \rangle \langle Ty, y \rangle \text{ for all } x, y \in H.$$

For an arbitrary operator T in $B(H)$ the following "mixed Schwarz" inequality has been established by Kato in [18] (see also [12] and [14, p. 265]):

$$(1.4) \quad |\langle Tx, y \rangle|^2 \leq \langle (T^*T)^\alpha x, x \rangle \langle (TT^*)^{1-\alpha} y, y \rangle \text{ for all } x, y \in H$$

and for $\alpha \in [0, 1]$.

An important consequence of Kato's inequality (1.4) is the famous Heinz inequality (see [1], [16], [17], [18]) which says that if T, A and B are operators in $B(H)$ such that A and B are positive and $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all x, y in H then

$$|\langle Tx, y \rangle| \leq \|A^\alpha x\| \|B^{1-\alpha} y\|$$

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for all $x, y \in H$ and for $\alpha \in [0, 1]$.

In this paper we establish some vector inequalities for powers of various operators in Hilbert spaces. Applications for norm and numerical radius inequalities are provided. Particular cases for commutators and self-commutators are also given.

2. VECTOR INEQUALITIES FOR TWO OPERATORS

The first results concerning powers of two operators is incorporated in:

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$ we have the vector inequality:*

$$(2.1) \quad |\langle Ax, By \rangle|^r \leq \frac{1}{2} [\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r y, y \rangle],$$

where $x, y \in H$, $\|x\| = \|y\| = 1$.

In particular, we have the norm inequality

$$(2.2) \quad \|B^*A\|^r \leq \frac{1}{2} (\|(A^*A)^r\| + \|(B^*B)^r\|)$$

and the numerical radius inequality

$$(2.3) \quad w^r(B^*A) \leq \frac{1}{2} \|(A^*A)^r + (B^*B)^r\|,$$

respectively.

The constant $\frac{1}{2}$ is best possible in all inequalities (2.1), (2.2) and (2.3).

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$(2.4) \quad |\langle B^*Ax, y \rangle| = |\langle Ax, By \rangle| \leq \|Ax\| \cdot \|By\| \\ = \langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*By, y \rangle^{1/2}, \quad x, y \in H.$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$(2.5) \quad \langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*By, y \rangle^{1/2} \leq \frac{\langle A^*Ax, x \rangle + \langle B^*By, y \rangle}{2} \\ \leq \left(\frac{\langle A^*Ax, x \rangle^r + \langle B^*By, y \rangle^r}{2} \right)^{\frac{1}{r}}$$

for any $x, y \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $z \in H$ with $\|z\| = 1$ we have the inequality (see for instance [20])

$$(2.6) \quad \langle Pz, z \rangle^r \leq \langle P^r z, z \rangle.$$

Applying this property to the positive operators A^*A and B^*B , we deduce that

$$(2.7) \quad \left(\frac{\langle A^*Ax, x \rangle^r + \langle B^*By, y \rangle^r}{2} \right)^{\frac{1}{r}} \leq \left(\frac{\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r y, y \rangle}{2} \right)^{\frac{1}{r}}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, on making use of the inequalities (2.4), (2.5) and (2.7), we get the inequality:

$$(2.8) \quad |\langle (B^*A)x, y \rangle|^r \leq \frac{1}{2} [\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r y, y \rangle]$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, which proves (2.1).

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.8) and since the operators $(A^*A)^r$ and $(B^*B)^r$ are self-adjoint, we deduce the desired inequality (2.2).

Now, if we take $y = x$ in (2.1), then we get

$$(2.9) \quad |\langle (B^*A)x, x \rangle|^r \leq \frac{1}{2} [\langle [(A^*A)^r + (B^*B)^r]x, x \rangle]$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.9) we get (2.3).

The sharpness of the constant follows by taking $r = 1$ and $B = A$ in all inequalities (2.1), (2.2) and (2.3). The details are omitted. ■

Corollary 1. *For any $A \in B(H)$ and $r \geq 1$ we have the vector inequalities:*

$$(2.10) \quad |\langle Ax, y \rangle|^r \leq \frac{1}{2} [\langle (A^*A)^r x, x \rangle + 1],$$

and

$$(2.11) \quad |\langle A^2x, y \rangle|^r \leq \frac{1}{2} [\langle (A^*A)^r x, x \rangle + \langle (AA^*)^r y, y \rangle],$$

where $x, y \in H$, $\|x\| = \|y\| = 1$.

In particular, we have the norm inequalities

$$(2.12) \quad \|A\|^r \leq \frac{1}{2} (\|(A^*A)^r\| + 1)$$

and

$$(2.13) \quad \|A^2\|^r \leq \frac{1}{2} (\|(A^*A)^r\| + \|(AA^*)^r\|),$$

respectively.

We also have the numerical radius inequalities

$$(2.14) \quad w^r(A) \leq \frac{1}{2} \|(A^*A)^r + I\|$$

and

$$(2.15) \quad w^r(A^2) \leq \frac{1}{2} \|(A^*A)^r + (AA^*)^r\|,$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A, B \in B(H)$, any $\alpha \in (0, 1)$ and $r \geq 1$, we have the vector inequality:*

$$(2.16) \quad |\langle Ax, By \rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1 - \alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have the norm inequality

$$(2.17) \quad \|B^*A\|^{2r} \leq \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + (1 - \alpha) \left\| (B^*B)^{\frac{r}{1-\alpha}} \right\|$$

and the numerical radius inequality

$$(2.18) \quad w^{2r}(B^*A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) (B^*B)^{\frac{r}{1-\alpha}} \right\|,$$

respectively.

Proof. By Schwarz's inequality, we have:

$$(2.19) \quad \begin{aligned} |\langle (B^*A)x, y \rangle|^2 &\leq \langle (A^*A)x, x \rangle \cdot \langle (B^*B)y, y \rangle \\ &= \left\langle \left[(A^*A)^{\frac{1}{\alpha}} \right]^\alpha x, x \right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} y, y \right\rangle, \end{aligned}$$

for any $x, y \in H$.

It is well known that (see for instance [20]) if P is a positive operator and $q \in (0, 1]$ then for any $u \in H$, $\|u\| = 1$, we have

$$(2.20) \quad \langle P^q u, u \rangle \leq \langle Pu, u \rangle^q.$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ ($\alpha \in (0, 1)$), we have

$$(2.21) \quad \begin{aligned} \left\langle \left[(A^*A)^{\frac{1}{\alpha}} \right]^\alpha x, x \right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} y, y \right\rangle \\ \leq \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^\alpha \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y \right\rangle^{1-\alpha}, \end{aligned}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b$, $\alpha \in (0, 1)$, $a, b \geq 0$, we get

$$(2.22) \quad \begin{aligned} \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^\alpha \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y \right\rangle^{1-\alpha} \\ \leq \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y \right\rangle \end{aligned}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

$$\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \quad \alpha \in (0, 1), \quad a, b \geq 0,$$

we deduce that

$$(2.23) \quad \begin{aligned} \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y \right\rangle \\ \leq \left[\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^r + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y \right\rangle^r \right]^{\frac{1}{r}} \\ \leq \left[\alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y \right\rangle \right]^{\frac{1}{r}}, \end{aligned}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, where, for the last inequality we used the inequality (2.6) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.19), (2.21), (2.22) and (2.23), we get

$$(2.24) \quad |\langle (B^*A)x, y \rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y \right\rangle$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, and the inequality (2.16) is proved.

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.24) produces the desired inequality (2.17).

The numerical radius inequality follows from (2.24) written for $y = x$. The details are omitted. ■

The following particular instances are of interest:

Corollary 2. For any $A \in B(H)$ and $\alpha \in (0, 1)$, $r \geq 1$, we have the vector inequalities

$$(2.25) \quad |\langle Ax, y \rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + 1 - \alpha,$$

$$(2.26) \quad |\langle A^2x, y \rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1 - \alpha) \left\langle (AA^*)^{\frac{r}{1-\alpha}} y, y \right\rangle$$

and

$$(2.27) \quad |\langle Ax, Ay \rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1 - \alpha) \left\langle (A^*A)^{\frac{r}{1-\alpha}} y, y \right\rangle,$$

respectively, where $x, y \in H$, $\|x\| = \|y\| = 1$.

We have the norm inequalities

$$(2.28) \quad \|A\|^{2r} \leq \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + 1 - \alpha$$

and

$$(2.29) \quad \|A^2\|^{2r} \leq \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + (1 - \alpha) \left\| (AA^*)^{\frac{r}{1-\alpha}} \right\|,$$

respectively.

We have the numerical radius inequalities

$$(2.30) \quad w^{2r}(A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) I \right\|$$

and

$$(2.31) \quad w^{2r}(A^2) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) (AA^*)^{\frac{r}{1-\alpha}} \right\|,$$

respectively.

Moreover, we have the norm inequality

$$(2.32) \quad \|A\|^{4r} \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) (A^*A)^{\frac{r}{1-\alpha}} \right\|.$$

3. VECTOR INEQUALITIES FOR THE SUM OF TWO PRODUCTS

The following result concerning four operators may be stated:

Theorem 3. For any $A, B, C, D \in B(H)$ and $r, s \geq 1$ we have:

$$(3.1) \quad \left| \left\langle \left[\frac{B^*A + D^*C}{2} \right] x, y \right\rangle \right|^2 \leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$(3.2) \quad \left\| \frac{B^*A + D^*C}{2} \right\|^2 \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(B^*B)^s + (D^*D)^s}{2} \right\|^{\frac{1}{s}}.$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$\begin{aligned}
(3.3) \quad & |\langle (B^*A + D^*C)x, y \rangle|^2 \\
&= |\langle B^*Ax, y \rangle + \langle D^*Cx, y \rangle|^2 \\
&\leq [|\langle B^*Ax, y \rangle| + |\langle D^*Cx, y \rangle|]^2 \\
&\leq \left[\langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*By, y \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y \rangle^{\frac{1}{2}} \right]^2,
\end{aligned}$$

for any $x, y \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2), \quad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$\begin{aligned}
(3.4) \quad & \langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*By, y \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y \rangle^{\frac{1}{2}} \\
&\leq (\langle A^*Ax, x \rangle + \langle C^*Cx, x \rangle) \cdot (\langle B^*By, y \rangle + \langle D^*Dy, y \rangle),
\end{aligned}$$

for any $x, y \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r, s \geq 1$ that

$$\begin{aligned}
(3.5) \quad & (\langle A^*Ax, x \rangle + \langle C^*Cx, x \rangle) \cdot (\langle B^*By, y \rangle + \langle D^*Dy, y \rangle) \\
&\leq 4 \cdot \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}}
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Consequently, by (3.3) – (3.5) we have:

$$\begin{aligned}
(3.6) \quad & \left| \left\langle \left[\frac{B^*A + D^*C}{2} \right] x, y \right\rangle \right|^2 \\
&\leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}}
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which provides the desired result (3.1).

Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ in (3.6) we deduce the desired inequality (3.2). ■

Remark 1. If we make $y = x$ in (3.6) and take the supremum over $\|x\| = 1$, then we get the inequality

$$w^2 \left(\frac{B^*A + D^*C}{2} \right) \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(B^*B)^s + (D^*D)^s}{2} \right\|^{\frac{1}{s}},$$

which is not as good as (3.2) since we always have

$$w^2 \left(\frac{B^*A + D^*C}{2} \right) \leq \left\| \frac{B^*A + D^*C}{2} \right\|^2.$$

Remark 2. If $s = r$, then the inequality (3.1) becomes :

$$(3.7) \quad \left| \left\langle \left[\frac{B^*A + D^*C}{2} \right] x, y \right\rangle \right|^{2r} \\ \leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[\frac{(B^*B)^r + (D^*D)^r}{2} \right] y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ while (3.2) is equivalent with

$$(3.8) \quad \left\| \frac{B^*A + D^*C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(B^*B)^r + (D^*D)^r}{2} \right\|.$$

Corollary 3. For any $A, C \in B(H)$ we have:

$$(3.9) \quad \left| \left\langle \left(\frac{A + C}{2} \right) x, y \right\rangle \right|^{2r} \leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. In particular, we have the norm inequality

$$(3.10) \quad \left\| \frac{A + C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|,$$

where $r \geq 1$.

The result is obvious by choosing $B = D = I$ in Theorem 3.

Corollary 4. For any $A, C \in B(H)$ we have:

$$(3.11) \quad \left| \left\langle \left(\frac{A^2 + C^2}{2} \right) x, y \right\rangle \right|^2 \\ \leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(AA^*)^s + (CC^*)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Also, we have the norm inequality

$$(3.12) \quad \left\| \frac{A^2 + C^2}{2} \right\|^2 \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + (CC^*)^s}{2} \right\|^{\frac{1}{s}}$$

for all $r, s \geq 1$.

If $s = r$, then we have, in particular,

$$(3.13) \quad \left| \left\langle \left(\frac{A^2 + C^2}{2} \right) x, y \right\rangle \right|^{2r} \\ \leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[\frac{(AA^*)^r + (CC^*)^r}{2} \right] y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(3.14) \quad \left\| \frac{A^2 + C^2}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\|$$

for $r \geq 1$.

The result is obvious by choosing $B = A^*$ and $D = C^*$ in Theorem 3.

Another particular result of interest is the following one:

Corollary 5. For any $A, B \in B(H)$ we have:

$$(3.15) \quad \left| \left\langle \left[\frac{B^*A + A^*B}{2} \right] x, y \right\rangle \right|^2 \leq \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(A^*A)^s + (B^*B)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$(3.16) \quad \left\| \frac{B^*A + A^*B}{2} \right\|^2 \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(A^*A)^s + (B^*B)^s}{2} \right\|^{\frac{1}{s}}$$

for any $r, s \geq 1$.

In particular we have

$$(3.17) \quad \left| \left\langle \left[\frac{B^*A + A^*B}{2} \right] x, y \right\rangle \right|^{2r} \leq \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2} \right] x, x \right\rangle \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2} \right] y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$(3.18) \quad \left\| \frac{B^*A + A^*B}{2} \right\|^r \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|$$

where $r \geq 1$.

The proof is obvious by choosing $D = A$ and $C = B$ in Theorem 3.

Another particular case that might be of interest is the following one.

Corollary 6. For any $A, D \in B(H)$ we have:

$$(3.19) \quad \left| \left\langle \left(\frac{A+D}{2} \right) x, y \right\rangle \right|^2 \leq \left\langle \left[\frac{(A^*A)^r + I}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(DD^*)^s + I}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(3.20) \quad \left\| \frac{A+D}{2} \right\|^2 \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(DD^*)^s + I}{2} \right\|^{\frac{1}{s}},$$

where $r, s \geq 1$.

In particular we have

$$(3.21) \quad |\langle Ax, y \rangle|^2 \leq \left\langle \left[\frac{(A^*A)^r + I}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(AA^*)^s + I}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(3.22) \quad \|A\|^2 \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + I}{2} \right\|^{\frac{1}{s}}.$$

Moreover, for any $r \geq 1$ we have

$$(3.23) \quad |\langle Ax, y \rangle|^{2r} \leq \left\langle \left[\frac{(A^*A)^r + I}{2} \right] x, x \right\rangle \cdot \left\langle \left[\frac{(AA^*)^r + I}{2} \right] y, y \right\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$(3.24) \quad \|A\|^{2r} \leq \left\| \frac{(A^*A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^r + I}{2} \right\|.$$

The proof of (3.19) is obvious by the Theorem 3 on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D . The details are omitted.

Remark 3. If $T \in B(H)$ and $T = A + iC$, i.e., A and C are its Cartesian decomposition, then we get from (3.9)

$$(3.25) \quad |\langle Tx, y \rangle|^{2r} \leq 2^{2r-1} \langle [(A^*A)^r + (C^*C)^r] x, x \rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. In particular, we have the norm inequality

$$(3.26) \quad \|T\|^{2r} \leq 2^{2r-1} \|(A^*A)^r + (C^*C)^r\|,$$

where $r \geq 1$.

Now, if we use the inequality (3.19) for T, A and B , then we get:

$$(3.27) \quad |\langle Tx, y \rangle|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(A^*A)^r + I] x, x \rangle^{\frac{1}{r}} \cdot \langle [(CC^*)^s + I] y, y \rangle^{\frac{1}{s}}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(3.28) \quad \|T\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(A^*A)^r + I\|^{\frac{1}{r}} \cdot \|(CC^*)^s + I\|^{\frac{1}{s}},$$

where $r, s \geq 1$. In particular, we have

$$(3.29) \quad |\langle Tx, y \rangle|^{2r} \leq 2^{2r-2} \langle [(A^*A)^r + I] x, x \rangle \cdot \langle [(CC^*)^r + I] y, y \rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(3.30) \quad \|T\|^{2r} \leq 2^{2r-2} \|(A^*A)^r + I\| \cdot \|(CC^*)^r + I\|,$$

for any $r \geq 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot, \cdot)$, where, as in [2],

$$w_e(T, U) := \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + |\langle Ux, x \rangle|^2 \right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. For any $A, B, C, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the vector inequality:

$$(3.31) \quad |\langle Ax, By \rangle|^2 + |\langle Cx, Dy \rangle|^2 \leq \langle [(A^*A)^p + (C^*C)^p] x, x \rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] y, y \rangle^{1/q}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have the inequality for the *Euclidean radius*:

$$(3.32) \quad w_e^2(B^*A, D^*C) \leq \|(A^*A)^p + (C^*C)^p\|^{1/p} \cdot \|(B^*B)^q + (D^*D)^q\|^{1/q}.$$

Proof. On utilising the elementary inequality

$$ac + bd \leq (a^p + b^p)^{1/p} \cdot (c^q + d^q)^{1/q}, \quad a, b, c, d \geq 0 \text{ and } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

then for any $x, y \in H$, $\|x\| = \|y\| = 1$ we have the inequalities:

$$\begin{aligned}
& |\langle B^*Ax, y \rangle|^2 + |\langle D^*Cx, y \rangle|^2 \\
& \leq \langle A^*Ax, x \rangle \cdot \langle B^*By, y \rangle + \langle C^*Cx, x \rangle \cdot \langle D^*Dy, y \rangle \\
& \leq (\langle A^*Ax, x \rangle^p + \langle C^*Cx, x \rangle^p)^{1/p} \cdot (\langle B^*By, y \rangle^q + \langle D^*Dy, y \rangle^q)^{1/q} \\
& \leq (\langle (A^*A)^p x, x \rangle + \langle (C^*C)^p x, x \rangle)^{1/p} \cdot (\langle (B^*B)^q y, y \rangle + \langle (D^*D)^q y, y \rangle)^{1/q} \\
& = \langle [(A^*A)^p + (C^*C)^p] x, x \rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] y, y \rangle^{1/q}.
\end{aligned}$$

For the second inequality, let us make the choice $y = x$ to get

$$\begin{aligned}
& |\langle B^*Ax, x \rangle|^2 + |\langle D^*Cx, x \rangle|^2 \\
& \leq \langle [(A^*A)^p + (C^*C)^p] x, x \rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] x, x \rangle^{1/q},
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.32). ■

The following particular case is of interest.

Corollary 7. *For any $A, C \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$(3.33) \quad |\langle Ax, y \rangle|^2 + |\langle Cx, y \rangle|^2 \leq 2^{1/q} \langle [(A^*A)^p + (C^*C)^p] x, x \rangle^{1/p}$$

for each $x, y \in H$, with $\|x\| = \|y\| = 1$. In particular,

$$w_e^2(A, C) \leq 2^{1/q} \|(A^*A)^p + (C^*C)^p\|^{1/p}.$$

The proof follows from (3.31) and (3.32) for $B = D = I$.

Corollary 8. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$(3.34) \quad |\langle Ax, y \rangle|^2 + |\langle Dx, y \rangle|^2 \leq \langle [(A^*A)^p + I] x, x \rangle^{1/p} \cdot \langle [(DD^*)^q + I] y, y \rangle^{1/q}$$

for each $x, y \in H$, with $\|x\| = \|y\| = 1$. In particular,

$$w_e^2(A, D) \leq \|(A^*A)^p + I\|^{1/p} \cdot \|(DD^*)^q + I\|^{1/q}.$$

4. INEQUALITIES FOR THE COMMUTATOR

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U , by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$(4.1) \quad \|TU - UT\| \leq 2\|T\|\|U\|.$$

In [11], the following result has been obtained as well

$$(4.2) \quad \|TU - UT\| \leq 2 \min\{\|T\|, \|U\|\} \min\{\|T - U\|, \|T + U\|\}.$$

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator:

Proposition 1. For any $T, U \in B(H)$ and $r, s \geq 1$ we have the vector inequality

$$(4.3) \quad \begin{aligned} & |\langle (TU - UT)x, y \rangle|^2 \\ & \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(U^*U)^r + (T^*T)^r]x, x \rangle^{\frac{1}{r}} \cdot \langle [(UU^*)^s + (TT^*)^s]y, y \rangle^{\frac{1}{s}}, \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$(4.4) \quad \|TU - UT\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(U^*U)^r + (T^*T)^r\|^{\frac{1}{r}} \cdot \|(UU^*)^s + (TT^*)^s\|^{\frac{1}{s}}.$$

In particular, we have

$$(4.5) \quad \begin{aligned} & |\langle (TU - UT)x, y \rangle|^{2r} \\ & \leq 2^{2r-2} \langle [(U^*U)^r + (T^*T)^r]x, x \rangle \cdot \langle [(UU^*)^r + (TT^*)^r]y, y \rangle \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(4.6) \quad \|TU - UT\|^{2r} \leq 2^{2r-2} \|(U^*U)^r + (T^*T)^r\| \cdot \|(UU^*)^r + (TT^*)^r\|,$$

for any $r \geq 1$.

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. ■

Now, for $U = T^*$ we can state the following corollary.

Corollary 9. For any $T \in B(H)$ we have the vector inequality for the self commutator:

$$(4.7) \quad \begin{aligned} & |\langle (TT^* - T^*T)x, y \rangle|^2 \\ & \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(TT^*)^r + (T^*T)^r]x, x \rangle^{\frac{1}{r}} \cdot \langle [(TT^*)^s + (T^*T)^s]y, y \rangle^{\frac{1}{s}} \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$(4.8) \quad \|TT^* - T^*T\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(TT^*)^r + (T^*T)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (T^*T)^s\|^{\frac{1}{s}}.$$

In particular we have

$$(4.9) \quad \begin{aligned} & |\langle (TT^* - T^*T)x, y \rangle|^{2r} \\ & \leq 2^{2r-2} \langle [(TT^*)^r + (T^*T)^r]x, x \rangle \cdot \langle [(TT^*)^r + (T^*T)^r]y, y \rangle \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$(4.10) \quad \|TT^* - T^*T\|^r \leq 2^{r-1} \|(TT^*)^r + (T^*T)^r\|,$$

for any $r \geq 1$.

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