SCHUR CONVEXITY AND SCHUR-GEOMETRICALLY CONCAVITY OF GENERALIZED EXPONENT MEAN

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Abstract. The monotonicity, the Schur-convexity and the Schur-geometrically convexity with variables (x, y) in $\mathbb{R}^2_{++}$ for fixed a of the generalized exponent mean $I_a(x, y)$. Besides, the monotonicity with parameters a in $\mathbb{R}$ for fixed (x, y) of $I_a(x, y)$ is discussed by using the hyperbolic composite function. Furthermore, some new inequalities are obtained.

1. Introduction

Throughout the paper we assume that the set of the real number, the nonnegative real number and the positive real number by $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

Let $(a, b) \in \mathbb{R}^2, (x, y) \in \mathbb{R}^2_{++}$. The extended mean (or Stolarsky mean) of (x, y) is defined in [1, p. 43] as

$$E(a,b;x,y) = \begin{cases} \left(\dfrac{r}{s} \cdot \dfrac{y^a - x^a}{y^b - x^b}\right)^{1/(a-b)}, & ab(a-b)(x-y) \neq 0, \\ \left(\dfrac{1}{a} \cdot \dfrac{y^a - x^a}{\ln y - \ln x}\right)^{1/a}, & a(x-y) \neq 0, b = 0; \\ \dfrac{1}{e^{1/a}} \left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{1/(x^a - y^a)}, & a(x-y) \neq 0, a = b; \\ \sqrt{xy}, & a = b = 0, x \neq y; \\ x, & x = y. \end{cases}$$

In particular, for $a \neq 0$,

$$E(a,a;x,y) = \begin{cases} \dfrac{1}{e^{1/a}} \left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{1/(x^a - y^a)}, & x \neq y; \\ x, & x = y \end{cases}$$

is called the generalized exponent mean, in symbols $I_a(x, y)$.

The Schur-convexity of the extended mean $E(r, s; x, y)$ with (x, y) were discussed in [2] and following conclusion is obtained:

Theorem A. *For fixed* $(a, b) \in \mathbb{R}^2$,

(*i*) *if* $2 < 2a < b$ *or* $2 \leq 2b \leq a$, *then* $E(a, b; x, y)$ *is Schur-convex on* $\mathbb{R}^2_{++}$ *with* (x, y),

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(ii) *if* $(a,b)\in\{a<b\le 2a, 0<a\le 1\}\cup\{b<a\le 2b, 0<b\le 1\}\cup\{0<b<a\le 1\}\cup\{0<a<b\le 1\}\cup\{b\le 2a<0\}\cup\{a\le 2b<0\}$*, then* $E(a,b;x,y)$ *is Schur-concave on* $\mathbb{R}^2_{++}$ *with* (x,y).

But this conclusion is not relate to the case that $a=b$. In other words, the Schur-convexity of the generalized exponent mean $I_a(x,y)$ with (x,y) is not discussed in [2].

In this paper, the monotonicity, the Schur-convexity and the Schur-geometrically convexity with variables (x,y) in $\mathbb{R}^2_{++}$ for fixed a of the generalized exponent mean $I_a(x,y)$. Besides, the monotonicity with parameters a in $\mathbb{R}$ for fixed (x,y) of $I_a(x,y)$ is discussed by using the hyperbolic composite function. Furthermore, some new inequalities are obtained.

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 1 ([3, 4])**.** Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\ge\cdots\ge x_{[n]}$ and $y_{[1]}\ge\cdots\ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) $\boldsymbol{x}\ge\boldsymbol{y}$ means $x_i\ge y_i$ for all $i=1,2,\ldots,n$. let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\ge\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\ge\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

(iii) $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(iv) let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\le\varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

Definition 2 ([5, 6])**.** Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n_{++}$.

(i) $\Omega\subset\mathbb{R}^n_{++}$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta,\ldots,x_n^\alpha y_n^\beta)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(ii) let $\Omega\subset\mathbb{R}^n_{++}$, φ: $\Omega\to\mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1,\ldots,\ln x_n)\prec(\ln y_1,\ldots,\ln y_n)$ on Ω implies $\varphi(\boldsymbol{x})\le\varphi(\boldsymbol{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Lemma 1 ([3, p. 38])**.** *A function* $\varphi(\boldsymbol{x})$ *is increasing if and only if* $\nabla\varphi(\boldsymbol{x})\ge 0$ *for* $\boldsymbol{x}\in\Omega$*, where* $\Omega\subset\mathbb{R}^n$ *is an open set,* $\varphi:\Omega\to R$ *is differentiable, and*

$$\nabla\varphi(\boldsymbol{x})=\left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1},\ldots,\frac{\partial\varphi(\boldsymbol{x})}{\partial x_n}\right)\in\mathbb{R}^n.$$

Lemma 2 ([3, p. 58])**.** *Let* $\Omega\subset\mathbb{R}^n$ *is symmetric and has a nonempty interior set.* Ω^0 *is the interior of* Ω*.* $\varphi:\Omega\to\mathbb{R}$ *is continuous on* Ω *and differentiable in* Ω^0*. Then* φ *is the* $Schur-convex(Schur-concave)function$*, if and only if* φ *is symmetric on* Ω *and*

$$(x_1-x_2)\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{\partial x_2}\right)\ge 0(\le 0)$$

holds for any $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)\in\Omega^0$.

Lemma 3 ([5, p. 108])**.** *Let $\Omega \subset \mathbb{R}^n_{++}$ is a symmetric and has a nonempty interior geometrically convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}_+$ is continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0)$$

holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is the Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 4. *Let $x \leq y, u(t) = tx+(1-t)y, v(t) = ty+(1-t)x$. If $1/2 \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq 1/2$, then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (x, y). \tag{1}$$

Proof. Case 1. When $1/2 \leq t_2 \leq t_1 \leq 1$, it is easy to see that $u(t_1) \geq v(t_1), u(t_2) \geq v(t_2), u(t_1) \geq u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = x + y$, this is (1) holds.

Case 2. When $0 \leq t_1 \leq t_2 \leq 1$, then $1/2 \leq 1 - t_2 \leq 1 - t_1 \leq 1$, by the Case 1, it follows

$$(u(1-t_2), v(1-t_2)) \prec (u(1-t_1), v(1-t_1)),$$

i.e. $(u(t_2), v(t_2)) \prec (u(t_1), v(t_1))$. □

Lemma 5 ([7])**.** *Let $0 \leq x \leq y, c \geq 0$. Then*

$$\left(\frac{x+c}{x+y+2c}, \frac{y+c}{x+y+2c}\right) \prec \left(\frac{x}{x+y}, \frac{y}{x+y}\right). \tag{2}$$

Lemma 6. *For x in $\mathbb{R}$ with $x \neq 0$, we have*

$$\sinh^2 x > x. \tag{3}$$

Proof. Let $f(x)= \sinh^2 x - x$. Then $f'(x)= \sinh 2x - 2x$. Since $f''(x) = 2(\cosh 2x - 1) > 0$ for $x \in \mathbb{R}$ with $x \neq 0$, $f'(x)$ is strictly increasing. It following that $f'(x) > f(0) = 0$, moreover, $f(x) > f(0) = 0$ for $x > 0$, and $f'(x) < f(0) = 0$, moreover, $f(x) > f(0) = 0$ for $x < 0$. Hence (3) is holds for any $x \in \mathbb{R}$ with $x \neq 0$. □

Lemma 7. *Let (x, y) and $(a, b) \in \mathbb{R}^2_{++}$ with $x < y, a < b, a + b = 1$. Then*

$$ax + by > \frac{x+y}{2}, \tag{4}$$

$$bx + ay < \frac{x+y}{2}, \tag{5}$$

Proof.

$$\begin{aligned} ax + by - \frac{x+y}{2} &= \left(a - \frac{1}{2}\right)x + \left(b - \frac{1}{2}\right)y \\ &= \left(1 - b - \frac{1}{2}\right)x + \left(b - \frac{1}{2}\right)y = -\left(b - \frac{1}{2}\right)x + \left(b - \frac{1}{2}\right)y \\ &= \left(b - \frac{1}{2}\right)(y - x) > 0, \end{aligned}$$

i.e.(4) is holds. (5) can be proved similarly. □

Lemma 8. *Let $(x,y)\in\mathbb{R}^2_{++}$ and $(a,b)\in\mathbb{R}^2$ with $ab(a-b)(x-y)\neq 0$. Then*

$$E(a,b;x,y)=\sqrt{xy}\left(\frac{b\sinh\left(a\ln\sqrt{u}\right)}{a\sinh\left(b\ln\sqrt{u}\right)}\right)^{\frac{1}{a-b}}, \tag{6}$$

where $u=y/x$.

Proof. Without loss of generality, we may assume $0<x<y$, then

$$\begin{aligned}
E(a,b;x,y)&=\left(\frac{b}{a}\cdot\frac{y^a-x^a}{y^b-x^b}\right)^{\frac{1}{a-b}}=\left(\frac{b}{a}\cdot\frac{u^a-1}{u^b-1}x^{a-b}\right)^{1/(a-b)}\\
&=x\left(\frac{b}{a}\cdot\frac{e^{2a\ln\sqrt{u}}-1}{e^{2b\ln\sqrt{u}}-1}\right)^{\frac{1}{a-b}}=x\left(\frac{b}{a}\cdot\frac{\frac{e^{2a\ln\sqrt{u}}-1}{2e^{a\ln\sqrt{u}}}}{\frac{e^{2b\ln\sqrt{u}}-1}{2e^{b\ln\sqrt{u}}}}e^{(a-b)\ln\sqrt{u}}\right)^{\frac{1}{a-b}}\\
&=x\sqrt{u}\left(\frac{b\sinh\left(a\ln\sqrt{u}\right)}{a\sinh\left(b\ln\sqrt{u}\right)}\right)^{\frac{1}{a-b}}=\sqrt{xy}\left(\frac{b\sinh\left(a\ln\sqrt{u}\right)}{a\sinh\left(b\ln\sqrt{u}\right)}\right)^{\frac{1}{a-b}}.
\end{aligned}$$

□

Lemma 9. *Let $(x,y)\in\mathbb{R}^2_{++}$ with $x\neq y$, and let $a\in\mathbb{R}$ with $a\neq 0$. Then*

$$I_a(x,y)=\sqrt{xy}\exp\left\{\frac{t}{\tanh(at)}-\frac{1}{a}\right\} \tag{7}$$

where $t=\ln\sqrt{u}, u=y/x$.

Proof. For $b\in\mathbb{R}$ with $b\neq a$, let

$$v=\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}.$$

Then from Lemma 8 we have

$$\begin{aligned}
I_a(x,y)&=\lim_{b\to a}E(a,b;x,y)=\lim_{b\to a}\sqrt{xy}\left(\frac{b\sinh(a\ln\sqrt{u})}{a\sinh(b\ln\sqrt{u})}\right)^{\frac{1}{a-b}}\\
&=\sqrt{xy}\lim_{b\to a}(1+v)^{\frac{1}{a-b}}\\
&=\sqrt{xy}\lim_{b\to a}\left[(1+v)^{\frac{1}{v}}\right]^{\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a-b}\frac{1}{a\sinh(b\ln\sqrt{u})}}\\
&=\sqrt{xy}\exp\left\{\lim_{b\to a}\frac{b\sinh(a\ln\sqrt{u})-a\sinh(b\ln\sqrt{u})}{a-b}\frac{1}{a\sinh(b\ln\sqrt{u})}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{1}{a\sinh\left(a\ln\sqrt{u}\right)}\lim_{b\to a}\frac{b\sinh\left(a\ln\sqrt{u}\right)-a\sinh\left(b\ln\sqrt{u}\right)}{a-b}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{1}{a\sinh\left(a\ln\sqrt{u}\right)}\lim_{b\to a}\frac{b\sinh\left(a\ln\sqrt{u}\right)-a\left(\ln\sqrt{u}\right)\cosh\left(b\ln\sqrt{u}\right)}{-1}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{a\left(\ln\sqrt{u}\right)\cosh\left(a\ln\sqrt{u}\right)-\sinh\left(a\ln\sqrt{u}\right)}{a\sinh\left(a\ln\sqrt{u}\right)}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{(at)\cosh(at)-\sinh(at)}{a\sinh(at)}\right\}\\
&=\sqrt{xy}\exp\left\{\frac{t}{\tanh(at)}-\frac{1}{a}\right\}.
\end{aligned}$$

□

3. Main results and their proofs

Theorem 1. *For fixed $(x,y)\in\mathbb{R}^2_{++}$, $I_a(x,y)$ is increasing on $\mathbb{R}$ with a.*

Proof. For $a\neq 0$, set $f(a)=\frac{t}{\tanh(at)}-\frac{1}{a}$, where $t=\ln\sqrt{u}, u=y/x$. Then

$$f^{'}(a)=\frac{-t^2}{\tanh^2(at)\cosh^2(at)}+\frac{1}{a^2}=\frac{-t^2}{\sinh^2(at)}+\frac{1}{a^2}=\frac{\sinh^2(at)-(at)^2}{a^2\sinh^2(at)}.$$

From Lemma 6 it follows that $f^{'}(a)>0$, that is $f(a)$ is increasing on $\mathbb{R}$ with a, then and

$$I_a(x,y)=\sqrt{xy}\exp\left\{\frac{t}{\tanh(at)}-\frac{1}{a}\right\}=\sqrt{xy}e^{f(a)}$$

is increasing on $\mathbb{R}$ with a. The proof of Theorem 1 is completed. □

Theorem 2. *For fixed $a\in\mathbb{R}$, $I_a(x,y)$ is increasing on $\mathbb{R}^2_{++}$ with (x,y).*

Proof. Let $A=x^a, B=y^a$. Then

$$\ln I_a(x,y)=\frac{x^a\ln x-y^a\ln y}{x^a-y^a}-\frac{1}{a}=\frac{1}{a}\left(\frac{A\ln A-B\ln B}{A-B}-1\right).$$

$$\begin{aligned}\frac{\partial\ln I_a}{\partial x}&=\frac{\partial\ln I_a}{\partial A}\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{1}{a}\left(\frac{A\ln A-B\ln B}{A-B}-1\right)ax^{a-1}\\&=\frac{A}{x}\left[\frac{A\ln A-B\ln A+(A-B)-A\ln A+B\ln B}{(A-B)^2}\right]\\&=\frac{A}{x}\left[\frac{(A-B)-B(\ln A-\ln B)}{(A-B)^2}\right]\\&=\frac{A}{x(A-B)}\left(1-\frac{\ln A-\ln B}{A-B}\cdot B\right)\\&=\frac{A}{x(A-B)}\left(1-\frac{B}{\xi}\right)\quad(\text{where }\xi\text{ lies between }A\text{ and }B\text{ })\\&=\frac{A}{x(A-B)}\frac{\xi-B}{\xi}=\frac{A}{x\xi}\cdot\frac{\xi-B}{A-B}\geq 0;\end{aligned}$$

$$\begin{aligned}\frac{\partial\ln I_a}{\partial y}&=\frac{\partial\ln I_a}{\partial B}\frac{\mathrm{d}B}{\mathrm{d}y}=\frac{1}{a}\left(\frac{A\ln A-B\ln B}{A-B}-1\right)ay^{a-1}\\&=\frac{A}{y}\left[\frac{-A\ln B+B\ln B-(A-B)-A\ln A-B\ln B}{(A-B)^2}\right]\\&=\frac{B}{y(A-B)}\left(\frac{\ln A-\ln B}{A-B}\cdot A-1\right)\\&=\frac{A}{y(A-B)}\left(\frac{A}{\xi}-1\right)\quad(\text{where }\xi\text{ lies between }A\text{ and }B\text{ })\\&=\frac{B}{y(A-B)}\frac{A-\xi}{\xi}=\frac{B}{y\xi}\cdot\frac{A-\xi}{A-B}\geq 0.\end{aligned}$$

By Lemma 1, it follows that $\ln I_a(x,y)$ is increasing on $\mathbb{R}^2_{++}$ with (x,y), and then $I_a(x,y)$ is increasing on $\mathbb{R}^2_{++}$ with (x,y) too.

The proof of Theorem 2 is completed. □

Theorem 3. *If $0<a\leq 1$, then $I_a(x,y)$ is Schur-concave on $\mathbb{R}^2_{++}$ with (x,y).*

Proof. For $(x,y)\in\mathbb{R}^2_{++}, 0<a\le 1$, let $A=x^a, B=y^a$. When $x\ne y$, we have

$$\frac{\partial \ln I_a}{\partial x}=\frac{A}{x}\cdot\frac{(A-B)-B(\ln A-\ln B)}{(A-B)^2}$$
$$\frac{\partial \ln I_a}{\partial y}=\frac{B}{y}\cdot\frac{A(\ln A-\ln B)-(A-B)}{(A-B)^2}$$

and then

$$\begin{aligned}
\Delta :&= (x-y)\left(\frac{\partial \ln I_a}{\partial x}-\frac{\partial \ln I_a}{\partial y}\right)\\
&=(x-y)\left[\frac{A}{x}\cdot\frac{(A-B)-B(\ln A-\ln B)}{(A-B)^2}-\frac{B}{y}\cdot\frac{A(\ln A-\ln B)-(A-B)}{(A-B)^2}\right]\\
&=\frac{x-y}{(A-B)^2}\left[\frac{A}{x}(A-B)-\frac{AB}{x}(\ln A-\ln B)-\frac{AB}{y}(\ln A-\ln B)+\frac{B}{y}(A-B)\right]\\
&=\frac{x-y}{(A-B)^2}\left[\left(\frac{A}{x}+\frac{B}{y}\right)(A-B)-AB\left(\frac{1}{x}+\frac{1}{y}\right)(\ln A-\ln B)\right]\\
&=\frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left[\left(\frac{A}{x}+\frac{B}{y}\right)\frac{A-B}{\ln A-\ln B}-AB\left(\frac{1}{x}+\frac{1}{y}\right)\right]\\
&=\frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left(\frac{A}{x}+\frac{B}{y}\right)\left[\frac{A-B}{\ln A-\ln B}-\frac{\left(\frac{1}{x}+\frac{1}{y}\right)AB}{\frac{A}{x}+\frac{B}{y}}\right]\\
&=\frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left(\frac{A}{x}+\frac{B}{y}\right)\left(\frac{x^a-y^a}{\ln x^a-\ln y^a}-\frac{y^{a-1}x^a+x^{a-1}y^a}{x^{a-1}+y^{a-1}}\right)\\
&=\frac{x-y}{A-B}\cdot\frac{\ln A-\ln B}{A-B}\left(\frac{A}{x}+\frac{B}{y}\right)\left[L\left(x^a,y^a\right)-\left(\frac{y^{a-1}}{x^{a-1}+y^{a-1}}x^a+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}y^a\right)\right]
\end{aligned}$$

where L denote the logarithm mean.

Without loss of generality, we may assume $0<x<y$. When $0<a<1$ we have

$$\frac{y^{a-1}}{x^{a-1}+y^{a-1}}<\frac{x^{a-1}}{x^{a-1}+y^{a-1}}$$

and

$$\frac{y^{a-1}}{x^{a-1}+y^{a-1}}+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}=1,$$

and then by Lemma 7, it follows that

$$\frac{y^{a-1}}{x^{a-1}+y^{a-1}}x^a+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}y^a>\frac{x^a+y^a}{2}=A\left(x^a,y^a\right).$$

Furthermore notice that $L\left(x^a,y^a\right)<A\left(x^a,y^a\right)$, we have

$$L\left(x^a,y^a\right)-\left(\frac{y^{a-1}}{x^{a-1}+y^{a-1}}x^a+\frac{x^{a-1}}{x^{a-1}+y^{a-1}}y^a\right)<L\left(x^a,y^a\right)-A\left(x^a,y^a\right)<0.$$

Hence $\Delta<0$ for $0<a<1$. It is easy to see that $\Delta<0$ for $a=1$. By Lemma 2, it follows that for $0<a\le 1$, $\ln I_a(x,y)$ is Schur-concave on $\mathbb{R}^2_{++}$ with (x,y), and then $I_a(x,y)$ is Schur-concave on $\mathbb{R}^2_{++}$ with (x,y) too.

The proof of Theorem 3 is completed. □

Theorem 4. *If $a>0$, then $I_a(x,y)$ is Schur-geometrically convex on $\mathbb{R}^2_{++}$ with (x,y); If $a<0$, then $I_a(x,y)$ is Schur-geometrically concave on $\mathbb{R}^2_{++}$ with (x,y).*

Proof. For $(x, y) \in \mathbb{R}^2_{++}, a \in \mathbb{R}$, let $A = x^a, B = y^a$. When $x \neq y$, we have

$$\frac{\partial \ln I_a}{\partial x} = \frac{A}{x} \cdot \frac{(A-B) - B(\ln A - \ln B)}{(A-B)^2}$$

$$\frac{\partial \ln I_a}{\partial y} = \frac{B}{y} \cdot \frac{A(\ln A - \ln B) - (A-B)}{(A-B)^2}$$

and then

$$\begin{aligned}
\Lambda :&= (x-y)\left(x\frac{\partial \ln I_a}{\partial x} - y\frac{\partial \ln I_a}{\partial y}\right) \\
&= \frac{x-y}{(A-B)^2}\left[A(A-B) - AB(\ln A - \ln B) - AB(\ln A - \ln B) + B(A-B)\right] \\
&= \frac{x-y}{(A-B)^2}\left[(A+B)(A-B) - 2AB(\ln A - \ln B)\right] \\
&= \frac{(x-y)(A+B)(\ln A - \ln B)}{(A-B)^2}\left(\frac{A-B}{\ln A - \ln B} - \frac{2AB}{A+B}\right) \\
&= \frac{(x-y)(A+B)(\ln A - \ln B)}{(A-B)^2}\left(L(A,B) - H(A,B)\right).
\end{aligned}$$

where H denote the harmonic mean.

For $(x, y) \in \mathbb{R}^2_{++}$ with $x \neq y$ and $a \in \mathbb{R}$, we have $L(A, B) > H(A, B)$. If $a > 0(< 0)$, then $(x - y)(\ln A - \ln B) = a(x - y)(\ln x - \ln y) > 0(< 0)$, and then $\Lambda > 0(< 0)$. By Lemma 3, it follows that $\ln I_a(x, y)$ is Schur-geometrically convex (concave) on $\mathbb{R}^2_{++}$ with (x, y), and then $I_a(x, y)$ is Schur-geometrically convex (concave) on $\mathbb{R}^2_{++}$ with (x, y) too.

The proof of Theorem 4 is completed. □

4. Applications

Theorem 5. *Let $0 < a \leq 1$, and let $x \leq y, u(t) = tx + (1-t)y, v(t) = ty + (1-t)x$. If $1/2 \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq 1/2$, then we have*

$$\begin{aligned}
G(x,y) &\leq I_a\left(x^{u(t_1)}y^{v(t_1)}, x^{v(t_1)}y^{u(t_1)}\right) \leq I_a\left(x^{u(t_2)}y^{v(t_2)}, x^{v(t_2)}y^{u(t_2)}\right) \\
&\leq I_a(x,y) \leq I_a\left(u(t_2), v(t_2)\right) \leq I_a\left(u(t_1), v(t_1)\right) \leq A(x,y). \qquad (8)
\end{aligned}$$

Proof. Combining Lemma 4 with Theorem 3, we have

$$\begin{aligned}
I_a(x,y) &\leq I_a\left(u(t_2), v(t_2)\right) \leq I_a\left(u(t_1), v(t_1)\right) \\
&\leq I_a\left((x+y)/2, (x+y)/2\right) = A(x,y).
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
(\ln\sqrt{xy}, \ln\sqrt{xy}) &\prec \left(\ln x^{u(t_1)}y^{v(t_1)}, \ln x^{v(t_1)}y^{u(t_1)}\right) \\
&\prec \left(\ln x^{u(t_2)}y^{v(t_2)}, \ln x^{v(t_2)}y^{u(t_2)}\right) \prec (\ln x, \ln y),
\end{aligned}$$

from Theorem 4, it follows

$$\begin{aligned}
G(x,y) = I_a\left(\sqrt{xy}, \sqrt{xy}\right) &\leq I_a\left(x^{u(t_1)}y^{v(t_1)}, x^{v(t_1)}y^{u(t_1)}\right) \\
&\leq I_a\left(x^{u(t_2)}y^{v(t_2)}, x^{v(t_2)}y^{u(t_2)}\right) \leq I_a(x,y).
\end{aligned}$$

The proof is complete. □

Theorem 6. *Let* $0 \le x \le y, c \ge 0,\ 0 < a \le 1$. *Then*

$$I_a\left(\frac{x+c}{x+y+2c}, \frac{y+c}{x+y+2c}\right) \ge I_a\left(\frac{x}{x+y}, \frac{y}{x+y}\right). \tag{9}$$

Proof. By Lemma 5 and Theorem 3, it follows that (9) is holds.

The proof is complete. □

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