ON SOME WEIGHTED INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS RELATED TO FEJÉR'S RESULT

K.-L. TSENG, SHIOW-RU HWANG, AND S.S. DRAGOMIR

Abstract. In this paper, we introduce some functionals associated with weighted integral means for convex functions. Some new Fejér-type inequalities are obtained as well.

1. Introduction

Throughout this paper, let $f:[a,b]\to\mathbb{R}$ be convex, $g:[a,b]\to[0,\infty)$ be integrable and symmetric to $\frac{a+b}{2}$. We define the following mappings on $[0,1]$ that are associated with the well known *Hermite-Hadamard inequality* [1]

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}, \tag{1.1}$$

namely

$$G(t)=\frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right];$$

$$Q(t)=\frac{1}{2}\left[f(ta+(1-t)b)+f(tb+(1-t)a)\right];$$

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx;$$

$$H_g(t)=\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)g(x)\,dx;$$

$$I(t)=\int_a^b\frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx;$$

$$P(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx;$$

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$$P_g(t) = \int_a^b \frac{1}{2}\left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) g\left(\frac{x+a}{2}\right) \right.$$
$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) g\left(\frac{x+b}{2}\right)\right] dx;$$
$$N(t) = \int_a^b \frac{1}{2}\left[f\left(ta + (1-t)\frac{x+a}{2}\right) + f\left(tb + (1-t)\frac{x+b}{2}\right)\right] g(x)\,dx;$$
$$L(t) = \frac{1}{2(b-a)}\int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)]\,dx;$$
$$L_g(t) = \frac{1}{2}\int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)]\,g(x)\,dx$$

and

$$S_g(t) = \frac{1}{4}\int_a^b \left[f\left(ta + (1-t)\frac{x+a}{2}\right) + f\left(ta + (1-t)\frac{x+b}{2}\right)\right.$$
$$\left. + f\left(tb + (1-t)\frac{x+a}{2}\right) + f\left(tb + (1-t)\frac{x+b}{2}\right)\right] g(x)\,dx.$$

Remark 1. *We note that $H = H_g = I, P = P_g = N$ and $L = L_g = S_g$ on $[0,1]$ as $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$.*

For some results which generalize, improve, and extend the famous Hermite-Hadamard integral inequality, see [2] – [19].

In [8], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1) :

Theorem A. *Let f, g be defined as above. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

In [11], Tseng et al. established the following Fejér-type inequalities.

Theorem B. *Let f, g be defined as above. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx \\ &\le \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.3}$$

In [2], Dragomir established the following Hermite-Hadamard-type inequality which refines the first inequality of (1.1).

Theorem C. *Let f, H be defined as above. Then H is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.4}$$

In [15], Yang and Hong obtained the following Hermite-Hadamard-type inequality which is a refinement of the second inequality in (1.1).

Theorem D. *Let f, P be defined as above. Then P is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx = P(0) \le P(t) \le P(1) = \frac{f(a)+f(b)}{2}. \tag{1.5}$$

Yang and Tseng [16] and Tseng et al. [11] established the following Fejér-type inequalities which are weighted generalizations of Theorems C – D.

Theorem E ([16])**.** *Let f, g, H_g, P_g be defined as above. Then H_g, P_g are convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= H_g(0) \le H_g(t) \le H_g(1) \\ &= \int_a^b f(x)\,g(x)\,dx \\ &= P_g(0) \le P_g(t) \le P_g(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.6}$$

Theorem F ([11])**.** *Let f, g, I, N be defined as above. Then I, N are convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= I(0) \le I(t) \le I(1) \\ &= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx \\ &= N(0) \le N(t) \le N(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.7}$$

In [7], Dragomir et al. established the following Hermite-Hadamard-type inequality.

Theorem G. *Let f, H, G, L be defined as above. Then G is convex, increasing on $[0,1]$, L is convex on $[0,1]$, and for all $t \in [0,1]$, we have*

$$H(t) \le G(t) \le L(t) \le \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}. \tag{1.8}$$

In [12] – [13], Tseng et al. obtained the following theorems related to Fejér's result which in their turn are weighted generalizations of the inequality (1.8).

Theorem H ([12]). *Let f, g, G, H_g, L_g be defined as above. Then L_g is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} H_g(t) &\le G(t)\int_a^b g(x)\,dx \\ &\le L_g(t) \\ &\le (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.9}$$

Theorem I ([13]). *Let f, g, G, I, S_g be defined as above. Then S_g is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} I(t) &\le G(t)\int_a^b g(x)\,dx \le S_g(t) \\ &\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\qquad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.10}$$

In this paper, we provide some new Fejér-type inequalities related to the mappings $G, Q, H_g, P_g, I, N, L_g, S_g$ defined above. They generalize known results obtained in relation with the Hermite-Hadamard inequality and therefore are useful in obtaining various results for means when the convex function and the weight take particular forms.

2. Main Results

The following lemmae are needed in the proofs of our main results:

Lemma 2 (see [9]). *Let f be defined as above and let $a \le A \le C \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

The assumptions in Lemma 2 can be weakened as in the following lemma:

Lemma 3. *Let f be defined as above and let $a \le A \le C \le B \le b$ and $a \le A \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

Lemma 4 (see [14]). *Let f, G, Q be defined as above. Then Q is symmetric about $\frac{1}{2}$, Q is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$,*

$$G(2t) \le Q(t) \qquad \left(t \in \left[0, \frac{1}{4}\right]\right),$$

$$G(2t) \ge Q(t) \qquad \left(t \in \left[\frac{1}{4}, \frac{1}{2}\right]\right),$$

$$G\left(2\left(1-t\right)\right) \geq Q\left(t\right) \qquad \left(t \in \left[\frac{1}{2}, \frac{3}{4}\right]\right)$$

and

$$G\left(2\left(1-t\right)\right) \leq Q\left(t\right) \qquad \left(t \in \left[\frac{3}{4}, 1\right]\right).$$

Now, we are ready to state and prove our results.

Theorem 5. *Let $f, g, G, H_g, P_g, L_g, S_g$ be defined as above. Then:*

(1) *The inequality*

$$\begin{aligned} \int_a^b f\left(x\right) g\left(x\right) dx &\leq 2\left[\int_a^{\frac{3a+b}{4}} f\left(x\right) g\left(2x-a\right) dx + \int_{\frac{a+3b}{4}}^b f\left(x\right) g\left(2x-b\right) dx\right] \\ &\leq \int_0^1 P_g\left(t\right) dt \\ &\leq \frac{1}{2}\left[\int_a^b f\left(x\right) g\left(x\right) dx + \frac{f\left(a\right)+f\left(b\right)}{2}\int_a^b g\left(x\right) dx\right] \end{aligned} \tag{2.1}$$

holds.

(2) *The inequalities*

$$\begin{aligned} L_g\left(t\right) &\leq P_g\left(t\right) \\ &\leq \left(1-t\right)\int_a^b f\left(x\right) g\left(x\right) dx + t \cdot \frac{f\left(a\right)+f\left(b\right)}{2}\int_a^b g\left(x\right) dx \\ &\leq \frac{f\left(a\right)+f\left(b\right)}{2}\int_a^b g\left(x\right) dx \end{aligned} \tag{2.2}$$

and

$$0 \leq N\left(t\right) - G\left(t\right)\int_a^b g\left(x\right) dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}\int_a^b g\left(x\right) dx - N\left(t\right) \tag{2.3}$$

hold for all $t \in [0,1]$.

(3) *If f is differentiable on $[a,b]$, then we have the inequalities*

$$\begin{aligned} 0 &\leq t\left[\frac{1}{b-a}\int_a^b f\left(x\right) dx - f\left(\frac{a+b}{2}\right)\right] \cdot \inf_{x\in[a,b]} g\left(x\right) \\ &\leq P_g\left(t\right) - \int_a^b f\left(x\right) g\left(x\right) dx; \end{aligned} \tag{2.4}$$

$$\begin{aligned} 0 &\leq P_g\left(t\right) - f\left(\frac{a+b}{2}\right)\int_a^b g\left(x\right) dx \\ &\leq \frac{\left(f'\left(b\right)-f'\left(a\right)\right)\left(b-a\right)}{4}\int_a^b g\left(x\right) dx; \end{aligned} \tag{2.5}$$

$$0 \leq L_g\left(t\right) - H_g\left(t\right) \leq \frac{\left(f'\left(b\right)-f'\left(a\right)\right)\left(b-a\right)}{4}\int_a^b g\left(x\right) dx; \tag{2.6}$$

$$0 \leq P_g\left(t\right) - L_g\left(t\right) \leq \frac{\left(f'\left(b\right)-f'\left(a\right)\right)\left(b-a\right)}{4}\int_a^b g\left(x\right) dx; \tag{2.7}$$

$$0 \leq P_g(t) - H_g(t) \leq \frac{(f'(b) - f'(a))(b-a)}{4} \int_a^b g(x)\,dx; \tag{2.8}$$

$$0 \leq N(t) - I(t) \leq \frac{(f'(b) - f'(a))(b-a)}{4} \int_a^b g(x)\,dx \tag{2.9}$$

and

$$0 \leq S_g(t) - I(t) \leq \frac{(f'(b) - f'(a))(b-a)}{4} \int_a^b g(x)\,dx \tag{2.10}$$

for all $t \in [0,1]$.

Proof. (1) By using simple integration techniques and the hypothesis of g, we have the following identities

$$\int_a^b f(x)\,g(x)\,dx = 2\int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} [f(x) + f(a+b-x)]\,g(x)\,dtdx; \tag{2.11}$$

$$\begin{aligned} &2\left[\int_a^{\frac{3a+b}{4}} f(x)\,g(2x-a)\,dx + \int_{\frac{a+3b}{4}}^b f(x)\,g(2x-b)\,dx\right] \\ &= 2\int_a^{\frac{3a+b}{4}} [f(x) + f(a+b-x)]\,g(2x-a)\,dx \\ &= 2\int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] g(x)\,dtdx; \end{aligned} \tag{2.12}$$

$$\begin{aligned} \int_0^1 P_g(t)\,dt &= \int_a^{\frac{a+b}{2}} \int_0^1 f(ta + (1-t)x)\,g(x)\,dtdx \\ &\quad + \int_{\frac{a+b}{2}}^b \int_0^1 f(tb + (1-t)x)\,g(x)\,dtdx \\ &= \int_a^{\frac{a+b}{2}} \int_0^1 f(ta + (1-t)x)\,g(x)\,dtdx \\ &\quad + \int_a^{\frac{a+b}{2}} \int_0^1 f(tb + (1-t)(a+b+x))\,g(x)\,dtdx \\ &= \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} [f(tx + (1-t)a) + f(ta + (1-t)x)]\,g(x)\,dtdx \\ &\quad + \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} [f(tb + (1-t)(a+b-x)) \\ &\quad + f(t(a+b-x) + (1-t)b)]\,g(x)\,dtdx \end{aligned} \tag{2.13}$$

and

$$\frac{1}{2}\left[\int_a^b f(x)\,g(x)\,dx+\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\right] \tag{2.14}$$
$$=\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}[f(a)+f(x)]\,g(x)\,dtdx$$
$$+\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}[f(a+b-x)+f(b)]\,g(x)\,dtdx.$$

By Lemma 2, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$f(x)+f(a+b-x)\le f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right) \tag{2.15}$$

holds when $A=\frac{a+x}{2}$, $C=x$, $D=a+b-x$ and $B=\frac{a+2b-x}{2}$ in Lemma 2.

$$f\left(\frac{a+x}{2}\right)\le\frac{1}{2}\left[f(tx+(1-t)\,a)+f(ta+(1-t)\,x)\right] \tag{2.16}$$

holds when $A=tx+(1-t)\,a$, $C=D=\frac{a+x}{2}$ and $B=ta+(1-t)\,x$ in Lemma 2.

$$f\left(\frac{a+2b-x}{2}\right) \tag{2.17}$$
$$\le\frac{1}{2}\left[f(tb+(1-t)(a+b-x))+f(t(a+b-x)+(1-t)\,b)\right]$$

holds when $A=tb+(1-t)(a+b-x)$, $C=D=\frac{a+2b-x}{2}$ and $B=t(a+b-x)+(1-t)\,b$ in Lemma 2.

$$\frac{1}{2}\left[f(tx+(1-t)\,a)+f(ta+(1-t)\,x)\right]\le\frac{f(a)+f(x)}{2} \tag{2.18}$$

holds when $A=a$, $C=tx+(1-t)\,a$, $D=ta+(1-t)\,x$ and $B=x$ in Lemma 2.

$$\frac{1}{2}\left[f(tb+(1-t)(a+b-x))+f(t(a+b-x)+(1-t)\,b)\right] \tag{2.19}$$
$$\le\frac{f(a+b-x)+f(b)}{2}$$

holds as $A=a+b-x$, $C=tb+(1-t)(a+b-x)$, $D=t(a+b-x)+(1-t)\,b$ and $B=b$ in Lemma 2. Multiplying the inequalities (2.15) – (2.19) by $g(x)$ and integrating them over t on $\left[0,\frac{1}{2}\right]$, over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.11) – (2.14), we derive (2.1).

(2) Using substitution rules for integration and the hypothesis of g, we have the following identities

$$\begin{aligned} P_g(t) &= \int_a^{\frac{a+b}{2}} f(ta+(1-t)x)\,g(x)\,dx \\ &\quad + \int_{\frac{a+b}{2}}^b f(tb+(1-t)x)\,g(x)\,dx \\ &= \int_a^{\frac{a+b}{2}} [f(ta+(1-t)x) \\ &\quad + f(tb+(1-t)(a+b-x))]\,g(x)\,dx \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} L_g(t) &= \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} f(ta+(1-t)x)\,g(x)\,dx \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^b f(tb+(1-t)x)\,g(x)\,dx\right] \\ &\quad + \frac{1}{2}\left[\int_{\frac{a+b}{2}}^b f(ta+(1-t)x)\,g(x)\,dx \right. \\ &\quad \left. + \int_a^{\frac{a+b}{2}} f(tb+(1-t)x)\,g(x)\,dx\right] \\ &= \frac{1}{2}P_g(t) + \frac{1}{2}\int_a^{\frac{a+b}{2}} [f(ta+(1-t)(a+b-x)) \\ &\quad + f(tb+(1-t)x)]\,g(x)\,dx \end{aligned} \tag{2.21}$$

for all $t \in [0,1]$.

If we choose $A = ta + (1-t)x$, $C = ta + (1-t)(a+b-x)$, $D = tb + (1-t)x$ and $B = tb + (1-t)(a+b-x)$ in Lemma 3, then the inequality

$$\begin{aligned} & f(ta+(1-t)(a+b-x)) + f(tb+(1-t)x) \\ & \qquad \leq f(ta+(1-t)x) + f(tb+(1-t)(a+b-x)) \end{aligned} \tag{2.22}$$

holds for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Multiplying the inequality (2.22) by $g(x)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using identities (2.20) $-$ (2.21), we derive the first inequality of (2.2). The second and third inequalities of (2.2) can be obtained by the convexity of f and (1.2). This proves (2.2).

Again, using substitution rules for integration and the hypothesis of g, we have the following identity

$$\begin{aligned} N(t) &= \int_a^b \frac{1}{2}\left[f\left(ta+(1-t)\frac{x+a}{2}\right)\right. \\ &\quad \left. + f\left(tb+(1-t)\frac{a+2b-x}{2}\right)\right] g(x)\,dx \end{aligned}$$

$$\begin{aligned}
&= \int_a^{\frac{a+b}{2}} [f(ta+(1-t)x) \\
&\qquad + f(tb+(1-t)(a+b-x))]\, g(2x-a)\, dx \qquad (2.23)\\
&= \int_a^{\frac{3a+b}{4}} [f(ta+(1-t)x) \\
&\qquad + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \\
&\qquad + f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right) \\
&\qquad + f(tb+(1-t)(a+b-x))]\, g(2x-a)\, dx \qquad (2.24)
\end{aligned}$$

for all $t \in [0,1]$. By Lemma 2, the following inequalities hold for all $t \in [0,1]$ and $x \in \left[a, \frac{3a+b}{4}\right]$.

$$f(ta+(1-t)x) + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right) \le f(a) + f\left(ta+(1-t)\frac{a+b}{2}\right) \tag{2.25}$$

holds when $A = a$, $C = ta+(1-t)x$, $D = ta+(1-t)\left(\frac{3a+b}{2}-x\right)$ and $B = ta+(1-t)\frac{a+b}{2}$ in Lemma 2.

$$f\left(tb+(1-t)\left(\frac{b-a}{2}+x\right)\right) + f(tb+(1-t)(a+b-x)) \le f\left(tb+(1-t)\frac{a+b}{2}\right) + f(b). \tag{2.26}$$

holds when $A = tb+(1-t)\frac{a+b}{2}$, $C = tb+(1-t)\left(\frac{b-a}{2}+x\right)$, $D = tb+(1-t)(a+b-x)$ and $B = b$ in Lemma 2. Multiplying the inequalities $(2.25) - (2.26)$ by $g(2x-a)$ and integrating them over x on $\left[a, \frac{3a+b}{4}\right]$ and using (2.24), we have

$$N(t) \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + G(t)\right] \int_a^b g(x)\, dx \tag{2.27}$$

for all $t \in [0,1]$. Using (2.27), we derive the second inequality of (2.3).

Again, using Lemma 2, we have

$$f\left(ta+(1-t)\frac{a+b}{2}\right) + f\left(tb+(1-t)\frac{a+b}{2}\right) \le f(ta+(1-t)x) + f(tb+(1-t)(a+b-x)) \tag{2.28}$$

for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Multiplying the inequality (2.28) by $g(2x-a)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using (2.23), we derive the first inequality of (2.3).

This proves (2.3).

(3) Integrating by parts, we have

$$\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[(a-x)f'(x)+(x-a)f'(a+b-x)\right]dx = \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right). \tag{2.29}$$

Using substitution rules for integration, we have the following identity

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]dx. \tag{2.30}$$

Now, using the convexity of f and $g(x)\geq 0$ on $[a,b]$, the inequality

$$\begin{aligned}
&[f(ta+(1-t)x)-f(x)]\,g(x) \\
&\qquad + [f(tb+(1-t)(a+b-x))-f(a+b-x)]\,g(x) \\
&\geq t(a-x)f'(x)g(x)+t(x-a)f'(a+b-x)g(x) \\
&= t(x-a)[f'(a+b-x)-f'(x)]\,g(x) \\
&\geq t(x-a)[f'(a+b-x)-f'(x)]\inf_{x\in[a,b]} g(x)
\end{aligned}$$

holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Integrating the above inequality over x on $\left[a,\frac{a+b}{2}\right]$, dividing both sides by $(b-a)$ and using (1.1), (2.20), (2.29) and (2.30), we derive (2.4).

On the other hand, we have

$$\begin{aligned}
\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2}\int_a^b g(x)\,dx &\leq \frac{1}{2}\left(a-\frac{a+b}{2}\right)f'(a)\int_a^b g(x)\,dx \\
&= \frac{a-b}{4}f'(a)\int_a^b g(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2}\int_a^b g(x)\,dx &\leq \frac{1}{2}\left(b-\frac{a+b}{2}\right)f'(b)\int_a^b g(x)\,dx \\
&= \frac{b-a}{4}f'(b)\int_a^b g(x)\,dx
\end{aligned}$$

and taking their sum we obtain:

$$\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\int_a^b g(x)\,dx \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \tag{2.31}$$

Finally, $(2.5)-(2.10)$ follow from (1.6), (1.7), (1.9), (1.10), (2.2) and (2.31).

This completes the proof. □

Let $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$. Then the following Hermite-Hadamard-type inequalities, which are also given in [14], are natural consequences of Theorem 5.

Corollary 6. *Let f, G, H, L, P be defined as above. Then:*

(1) *The inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{2}{b-a}\int_{\left[a,\frac{3a+b}{4}\right]\cup\left[\frac{a+3b}{4},b\right]} f(x)\,dx$$
$$\le \int_0^1 P(t)\,dt$$
$$\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right]$$

holds.

(2) *The inequalities*

$$L(t) \le P(t) \le \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}$$

and

$$0 \le P(t) - G(t) \le \frac{f(a)+f(b)}{2} - P(t)$$

hold for all $t\in[0,1]$.

(3) *If* f *is differentiable on* $[a,b]$, *then we have the inequalities*

$$0 \le t\left[\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right]$$
$$\le P(t) - \frac{1}{b-a}\int_a^b f(x)\,dx;$$

$$0 \le P(t) - f\left(\frac{a+b}{2}\right) \le \frac{(f'(b)-f'(a))(b-a)}{4};$$

$$0 \le L(t) - H(t) \le \frac{(f'(b)-f'(a))(b-a)}{4};$$

$$0 \le P(t) - L(t) \le \frac{(f'(b)-f'(a))(b-a)}{4}$$

and

$$0 \le P(t) - H(t) \le \frac{(f'(b)-f'(a))(b-a)}{4}$$

for all $t\in[0,1]$.

Remark 7. *In Theorem 5, the inequality* (2.1) *gives a new refinement of the Fejér inequality* (1.2).

Remark 8. *In Theorem 5, the inequality* (2.2) *refines the Fejér-type inequality* (1.9).

In the next theorem, we point out some inequalities for the functions G, Q, H_g, P_g, S_g considered above:

Theorem 9. *Let* $f, g, G, Q, H_g, P_g, S_g$ *be defined as above. Then:*

(1) *The inequalities*

$$H_g(t) \le Q(t) \int_a^b g(x)\,dx \le \frac{f(a)+f(b)}{2} \int_a^b g(x)\,dx \qquad \left(t \in \left[0, \frac{1}{3}\right]\right) \tag{2.32}$$

and

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\,dx \le Q(t) \int_a^b g(x)\,dx \le P_g(t) \qquad \left(t \in \left[\frac{1}{3}, 1\right]\right) \tag{2.33}$$

hold for all $t \in [0,1]$.

(2) *The inequality*

$$0 \le S_g(t) - G(t) \int_a^b g(x)\,dx \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + Q(t)\right] \int_a^b g(x)\,dx - S_g(t) \tag{2.34}$$

holds for all $t \in [0,1]$.

Proof. (1) We discuss the following two cases.

Case 1. $t \in \left[0, \frac{1}{3}\right]$.

Using substitution rules for integration and the hypothesis of g, we have the following identity

$$H(t) = \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] g(x)\,dx. \tag{2.35}$$

If we choose $A = (1-t)a + tb$, $C = tx + (1-t)\frac{a+b}{2}$, $D = t(a+b-x) + (1-t)\frac{a+b}{2}$ and $B = ta + (1-t)b$ in Lemma 2, then the inequality

$$f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \le f((1-t)a + tb) + f(ta + (1-t)b) \tag{2.36}$$

holds for all $t \in \left[0, \frac{1}{3}\right]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Multiplying the inequality (2.36) by $g(x)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using identity (2.35), we derive the first inequality of (2.32). From Lemma 4, we have

$$\sup_{t \in \left[0, \frac{1}{3}\right]} Q(t) = \frac{f(a)+f(b)}{2}.$$

Then the second inequality of (2.32) can be obtained. This proves (2.32).

Case 2. $t \in \left[\frac{1}{3}, 1\right]$.

If we choose $A = ta + (1-t)x$, $C = ta + (1-t)b$, $D = (1-t)a + tb$ and $B = tb + (1-t)(a+b-x)$ in Lemma 3, then the inequality

$$f(ta + (1-t)b) + f(tb + (1-t)a) \leq f(ta + (1-t)x) + f(tb + (1-t)(a+b-x)) \tag{2.37}$$

holds for all $t \in \left[\frac{1}{3}, 1\right]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Multiplying the inequality (2.37) by $g(x)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using identity (2.20), we obtain the second inequality of (2.33). From Lemma 4, we have

$$\inf_{t \in \left[\frac{1}{3}, 1\right]} Q(t) = f\left(\frac{a+b}{2}\right).$$

Then the first inequality of (2.33) can be obtained. This proves (2.33).

(2) Using substitution rules for integration and the hypothesis of g, we have the following identity

$$\begin{aligned} 2S_g(t) &= \int_a^{\frac{a+b}{2}} \left[f(ta + (1-t)x) + f(tb + (1-t)x)\right] g(2x-a)\,dx \\ &\quad + \int_{\frac{a+b}{2}}^{b} \left[f(ta + (1-t)x) + f(tb + (1-t)x)\right] g(2x-b)\,dx \\ &= \int_a^{\frac{a+b}{2}} \left[f(ta + (1-t)x) + f(tb + (1-t)x) \right. \\ &\quad \left. + f(ta + (1-t)(a+b-x)) + f(tb + (1-t)(a+b-x))\right] \\ &\quad \times g(2x-a)\,dx \\ &= \int_a^{\frac{3a+b}{4}} \left[f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \right. \\ &\quad + f\left(ta + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f(ta + (1-t)(a+b-x)) \\ &\quad + f(tb + (1-t)x) + f\left(tb + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \\ &\quad \left. + f\left(tb + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f(tb + (1-t)(a+b-x))\right] \\ &\quad \times g(2x-a)\,dx \end{aligned} \tag{2.38}$$

for all $t \in [0,1]$.

By Lemma 2, the following inequalities hold for all $t \in [0,1]$ and $x \in \left[a, \frac{3a+b}{4}\right]$.

$$f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \leq f(a) + f\left(ta + (1-t)\frac{a+b}{2}\right) \tag{2.39}$$

holds when $A = a$, $C = ta + (1-t)x$, $D = ta + (1-t)\left(\frac{3a+b}{2} - x\right)$ and $B = ta + (1-t)\frac{a+b}{2}$ in Lemma 2.

$$f\left(ta + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f\left(ta + (1-t)(a+b-x)\right) \leq f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(ta + (1-t)b\right) \tag{2.40}$$

holds when $A = ta+(1-t)\frac{a+b}{2}$, $C = ta+(1-t)\left(\frac{b-a}{2}+x\right)$, $D = ta+(1-t)(a+b-x)$ and $B = ta + (1-t)b$ in Lemma 2.

$$f\left(tb + (1-t)x\right) + f\left(tb + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \leq f\left(tb + (1-t)a\right) + f\left(tb + (1-t)\frac{a+b}{2}\right) \tag{2.41}$$

holds when $A = tb + (1-t)a$, $C = tb + (1-t)x$, $D = tb + (1-t)\left(\frac{3a+b}{2} - x\right)$ and $B = tb + (1-t)\frac{a+b}{2}$ in Lemma 2.

$$f\left(tb + (1-t)\left(\frac{b-a}{2} + x\right)\right) + f\left(tb + (1-t)(a+b-x)\right) \leq f\left(tb + (1-t)\frac{a+b}{2}\right) + f(b) \tag{2.42}$$

holds when $A = tb+(1-t)\frac{a+b}{2}$, $C = tb+(1-t)\left(\frac{b-a}{2}+x\right)$, $D = tb+(1-t)(a+b-x)$ and $B = b$ in Lemma 2. Multiplying the inequalities (2.39) – (2.42) by $g(2x-a)$, integrating them over x on $\left[a, \frac{3a+b}{4}\right]$ and using identity (2.38), we have

$$2S_g(t) \leq G(t)\int_a^b g(x)\,dx + \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + Q(t)\right]\int_a^b g(x)\,dx \tag{2.43}$$

for all $t \in [0,1]$. Using (1.10) and (2.43), we derive (2.34). This completes the proof. □

Let $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$. Then the following Hermite-Hadamard-type inequalities, which are given in [14], are natural consequences of Theorem 9.

Corollary 10. *Let f, G, H, L, P be defined as above. Then:*

(1) *The inequalities*

$$H(t) \leq Q(t) \leq \frac{f(a)+f(b)}{2} \qquad \left(t \in \left[0, \frac{1}{3}\right]\right)$$

and

$$f\left(\frac{a+b}{2}\right) \leq Q(t) \leq P(t) \qquad \left(t \in \left[\frac{1}{3}, 1\right]\right)$$

hold for all $t \in [0,1]$.

(2) *The inequality*

$$0 \leq L(t) - G(t) \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + Q(t)\right] - L(t)$$

holds for all $t \in [0,1]$.

The following Fejér-type inequalities are natural consequences of Theorems A – B, E – I, 5, 9 and Lemma 4 and we shall omit their proofs.

Theorem 11. *Let $f, g, G, H_g, P_g, I, L_g, S_g$ be defined as above.*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le G(t)\int_a^b g(x)\,dx \le S_g(t) \\ &\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\quad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le I(t) \le G(t)\int_a^b g(x)\,dx \\ &\le L_g(t) \le P_g(t) \\ &\le (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned}$$

Theorem 12. *Let f, g, G, Q, H_g, I be defined as above. Then, for all $t \in \left[0, \frac{1}{4}\right]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le H_g(2t) \le G(2t)\int_a^b g(x)\,dx \\ &\le Q(t)\int_a^b g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le I(t) \le I(2t) \le G(2t)\int_a^b g(x)\,dx \\ &\le Q(t)\int_a^b g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned}$$

Theorem 13. *Let $f, g, G, Q, H_g, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{1}{4}, \frac{1}{3}\right]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le Q(t)\int_a^b g(x)\,dx \le G(2t)\int_a^b g(x)\,dx \\ &\le L_g(2t) \le P_g(2t) \\ &\le (1-2t)\int_a^b f(x)\,g(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le H_g(t) \le Q(t)\int_a^b g(x)\,dx \\
&\le G(2t)\int_a^b g(x)\,dx \le S_g(2t) \\
&\le (1-2t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\quad + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}$$

Theorem 14. *Let $f, g, G, Q, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{1}{3}, \frac{1}{2}\right]$, we have*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,d &\le Q(t)\int_a^b g(x)\,dx \\
&\le G(2t)\int_a^b g(x)\,dx \le L_g(2t) \le P_g(2t) \\
&\le (1-2t)\int_a^b f(x)\,g(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx;
\end{aligned}$$

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,d &\le Q(t)\int_a^b g(x)\,dx \\
&\le G(2t)\int_a^b g(x)\,dx \le S_g(2t) \\
&\le (1-2t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\quad + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \le P_g(t) \le P_g(2t) \\
&\le (1-2t)\int_a^b f(x)\,g(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}$$

Theorem 15. *Let $f, g, G, Q, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \le G(2(1-t))\int_a^b g(x)\,dx \\ &\le L_g(2(1-t)) \le P_g(2(1-t)) \\ &\le (2t-1)\int_a^b f(x)\,g(x)\,dx + 2(1-t)\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \\ &\le G(2(1-t))\int_a^b g(x)\,dx \le S_g(2(1-t)) \\ &\le (2t-1)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\qquad + 2(1-t)\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned}$$

Theorem 16. *Let $f, g, G, Q, H_g, P_g, L_g, S_g$ be defined as above. Then, for all $t \in \left[\frac{2}{3}, \frac{3}{4}\right]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \\ &\le G(2(1-t))\int_a^b g(x)\,dx \\ &\le G(t)\int_a^b g(x)\,dx \le L_g(t) \le P_g(t) \\ &\le (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le Q(t)\int_a^b g(x)\,dx \le G(2(1-t))\int_a^b g(x)\,dx \\ &\le G(t)\int_a^b g(x)\,dx \le S_g(t) \end{aligned}$$

$$
\begin{aligned}
&\leq (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&\qquad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}
$$

Theorem 17. *Let $f, g, G, Q, H_g, P_g, I, S_g$ be defined as above. Then, for all $t \in \left[\frac{3}{4}, 1\right]$, we have*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq H_g(2(1-t)) \leq G(2(1-t))\int_a^b g(x)\,dx \\
&\leq Q(t)\int_a^b g(x)\,dx \leq P_g(t) \\
&\leq \frac{1-t}{b-a}\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq I(2(1-t)) \leq G(2(1-t))\int_a^b g(x)\,dx \\
&\leq Q(t)\int_a^b g(x)\,dx \leq P_g(t) \\
&\leq \frac{1-t}{b-a}\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.
\end{aligned}
$$

Let $g(x) = \frac{1}{b-a}\ (x \in [a,b])$. Then the following Hermite-Hadamard-type inequalities are natural consequences of Theorems $11-17$, which are given in [14].

Corollary 18. *Let f, Q, G, H, P, L be defined as above. Then we have:*

(1) *For all $t \in \left[0, \frac{1}{4}\right]$ one has the inequality*

$$
f\left(\frac{a+b}{2}\right) \leq H(t) \leq H(2t) \leq G(2t) \leq Q(t) \leq \frac{f(a)+f(b)}{2}.
$$

(2) *For all $t \in \left[\frac{1}{4}, \frac{1}{3}\right]$ one has the inequality*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq H(t) \leq Q(t) \leq G(2t) \leq L(2t) \leq P(2t) \\
&\leq \frac{1-2t}{b-a}\int_a^b f(x)\,dx + 2t\cdot\frac{f(a)+f(b)}{2} \\
&\leq \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

(3) *For all* $t \in \left[\frac{1}{3}, \frac{1}{2}\right]$ *one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq G(2t) \leq L(2t) \leq P(2t) \\ &\leq \frac{1-2t}{b-a}\int_a^b f(x)\,dx + 2t \cdot \frac{f(a)+f(b)}{2} \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq P(t) \leq P(2t) \\ &\leq \frac{1-2t}{b-a}\int_a^b f(x)\,dx + 2t \cdot \frac{f(a)+f(b)}{2} \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

(4) *For all* $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$ *one has the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq G(2(1-t)) \leq L(2(1-t)) \leq P(2(1-t)) \\ &\leq \frac{2t-1}{b-a}\int_a^b f(x)\,dx + 2(1-t) \cdot \frac{f(a)+f(b)}{2} \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

(5) *For all* $t \in \left[\frac{2}{3}, \frac{3}{4}\right]$ *one has the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq Q(t) \leq G(2(1-t)) \leq G(t) \leq L(t) \leq P(t) \\ &\leq \frac{1-t}{b-a}\int_a^b f(x)\,dx + t \cdot \frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

(6) *For all* $t \in \left[\frac{3}{4}, 1\right]$ *one has the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq H(2(1-t)) \leq G(2(1-t)) \leq Q(t) \leq P(t) \\ &\leq \frac{1-t}{b-a}\int_a^b f(x)\,dx + t \cdot \frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

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Department of Mathematics, Aletheia University, Tamsui, Taiwan 25103.
E-mail address: kltseng@email.au.edu.tw

China University of Science and Technology, Nankang, Taipei, Taiwan 11522
E-mail address: hsru@cc.cust.edu.tw

School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: sever.dragomir@.vu.edu.au
URL: http://www.staff.vu.edu.au/RGMIA/dragomir/