FEJÉR-TYPE INEQUALITIES (II)

K.-L. TSENG, SHIOW-RU HWANG, AND S.S. DRAGOMIR

Abstract. In this paper, we establish some Fejér-type inequalities for convex functions. They complement the results from the previous recent paper [12].

1. Introduction

Throughout this paper, let $f : [a, b] \to \mathbb{R}$ be convex, $g : [a, b] \to [0, \infty)$ be integrable and symmetric to $\frac{a+b}{2}$ and define the following functions on $[0, 1]$:

$$G(t) = \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)\right];$$

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx;$$

$$H_g(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)g(x)\,dx;$$

$$L(t) = \frac{1}{2(b-a)}\int_a^b \left[f(ta + (1-t)x) + f(tb + (1-t)x)\right]dx;$$

and

$$L_g(t) = \frac{1}{2}\int_a^b \left[f(ta + (1-t)x) + f(tb + (1-t)x)\right]g(x)\,dx.$$

If f is defined as above, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality [1].

For some results which generalize, improve, and extend this famous integral inequality see [2] – [17].

In [2], Dragomir established the following theorem which refines the first inequality of (1.1).

Theorem A. *Let f, H be defined as above. Then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

In [7], Dragomir, Milošević and Sándor established the following inequalities related to (1.1):

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hermite-Hadamard inequality, Fejér inequality, Convex function.

This research was partially supported by grant NSC 97-2115-M-156-002.

Theorem B. *Let f, H be defined as above. Then:*

(1) *The following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \leq \int_0^1 H(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right]. \tag{1.3}$$

(2) *If f is differentiable on $[a,b]$, then, for all $t \in [0,1]$, we have the inequalities*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)\,dx - H(t) \leq (1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right] \tag{1.4}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - H(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4}. \tag{1.5}$$

Theorem C. *Let f, H, G be defined as above. Then:*

(1) *G is convex and increasing on $[0,1]$.*

(2) *We have*

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2}.$$

(3) *The following inequality holds for all $t \in [0,1]$:*

$$H(t) \leq G(t). \tag{1.6}$$

(4) *The following inequality holds:*

$$\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leq \int_0^1 G(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]. \tag{1.7}$$

(5) *If f is differentiable on $[a,b]$, then, for all $t \in [0,1]$, we have the inequality*

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq G(t) - H(t). \tag{1.8}$$

Theorem D. *Let f, H, G, L be defined as above. Then:*

(1) *L is convex on $[0,1]$.*

(2) *We have the inequality:*

$$G(t) \leq L(t) \leq \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2} \tag{1.9}$$

for all $t \in [0,1]$ *and*

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}.$$

(3) *For all* $t \in [0,1]$, *we have the inequalities:*

$$H(1-t) \leq L(t) \quad \text{and} \quad \frac{H(t)+H(1-t)}{2} \leq L(t).$$

In [8], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

Theorem E. *Let* f, g *be defined as above. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \tag{1.10}$$

is known as Fejér inequality.

In [14], Yang and Tseng established the following theorem which refines the first inequality of (1.10) and generalizes Theorem A.

Theorem F. *Let* f, g, H_g *be defined as above. Then* H_g *is convex, increasing on* $[0,1]$, *and for all* $t \in [0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx = H_g(0) \leq H_g(t) \leq H_g(1) = \int_a^b f(x)\,g(x)\,dx. \tag{1.11}$$

In this paper, we establish some Fejér-type inequalities related to the functions G, H, H_g, L, L_g and generalize Theorems B – D. They complement the results from the recent paper [12].

2. Main Results

In order to prove our main results, we need the following lemma:

Lemma 1 (see [9])**.** *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function and let* $a \leq A \leq C \leq D \leq B \leq b$ *with* $A+B = C+D$, *then*

$$f(C) + f(D) \leq f(A) + f(B).$$

Now, we are ready to state and prove our results.

Theorem 2. *Let* f, g, H_g *be defined as above. Then we have the following Fejér-type inequalities:*

(1) *The following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le 2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,g\left(2x-\frac{a+b}{2}\right)dx \le \int_0^1 H_g(t)\,dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx+\int_a^b f(x)\,g(x)\,dx\right]. \tag{2.1}$$

(2) *If f is differentiable on $[a,b]$ and g is bounded on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$0 \le \int_a^b f(x)\,g(x)\,dx - H_g(t) \le (1-t)\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\|g\|_\infty , \tag{2.2}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} |g(x)|$.

(3) *If f is differentiable on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$0 \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - H_g(t) \le \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \tag{2.3}$$

Proof. (1) Using simple techniques of integration and the hypothesis of g, we have the following identities:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx = 4\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}\right)g(x)\,dtdx; \tag{2.4}$$

$$2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,g\left(2x-\frac{a+b}{2}\right)dx = 2\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]g(x)\,dtdx; \tag{2.5}$$

$$\begin{aligned}\int_0^1 H_g(t)\,dt &= \int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(t\frac{a+b}{2}+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dtdx \\ &+\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right. \\ &\left.+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)\right]g(x)\,dtdx;\end{aligned} \tag{2.6}$$

and

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx+\int_a^b f(x)\,g(x)\,dx\right] \tag{2.7}$$
$$=\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]g(x)\,dtdx$$
$$+\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{a+b}{2}\right)+f(a+b-x)\right]g(x)\,dtdx.$$

By Lemma 1, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$4f\left(\frac{a+b}{2}\right)\le 2\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right] \tag{2.8}$$

holds when $A=\frac{x}{2}+\frac{a+b}{4}$, $C=D=\frac{a+b}{2}$ and $B=\frac{3(a+b)}{4}-\frac{x}{2}$ in Lemma 1.

$$2f\left(\frac{x}{2}+\frac{a+b}{4}\right)\le f\left(t\frac{a+b}{2}+(1-t)\,x\right)+f\left(tx+(1-t)\,\frac{a+b}{2}\right) \tag{2.9}$$

holds when $A=t\frac{a+b}{2}+(1-t)\,x$, $C=D=\frac{x}{2}+\frac{a+b}{4}$ and $B=tx+(1-t)\,\frac{a+b}{2}$ in Lemma 1.

$$2f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right) \tag{2.10}$$
$$\le f\left(t(a+b-x)+(1-t)\,\frac{a+b}{2}\right)+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)$$

holds when $A=t(a+b-x)+(1-t)\,\frac{a+b}{2}$, $C=D=\frac{3(a+b)}{4}-\frac{x}{2}$ and $B=t\frac{a+b}{2}+(1-t)(a+b-x)$ in Lemma 1.

$$f\left(t\frac{a+b}{2}+(1-t)\,x\right)+f\left(tx+(1-t)\,\frac{a+b}{2}\right)\le f(x)+f\left(\frac{a+b}{2}\right) \tag{2.11}$$

holds when $A=x$, $C=t\frac{a+b}{2}+(1-t)\,x$, $D=tx+(1-t)\,\frac{a+b}{2}$ and $B=\frac{a+b}{2}$ in Lemma 1.

$$f\left(t(a+b-x)+(1-t)\,\frac{a+b}{2}\right)+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right) \tag{2.12}$$
$$\le f\left(\frac{a+b}{2}\right)+f(a+b-x)$$

holds for $A=\frac{a+b}{2}$, $C=t(a+b-x)+(1-t)\,\frac{a+b}{2}$, $D=t\frac{a+b}{2}+(1-t)(a+b-x)$ and $B=a+b-x$ in Lemma 1. Multiplying the inequalities (2.8) – (2.12) by $g(x)$ and integrating them over t on $\left[0,\frac{1}{2}\right]$, over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.4) – (2.7), we derive (2.1).

(2) By integration by parts, we have

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) \left[f'(a+b-x) - f'(x)\right] dx \\ &= \int_a^b \left(x - \frac{a+b}{2}\right) f'(x)\, dx \\ &= \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(x)\, dx. \end{aligned} \tag{2.13}$$

Using substitution rules for integration and the hypothesis of g, we have the following identities

$$\int_a^b f(x) g(x)\, dx = \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x)\right] g(x)\, dx \tag{2.14}$$

and

$$\begin{aligned} H_g(t) = \int_a^{\frac{a+b}{2}} \Bigg[& f\left(tx + (1-t)\frac{a+b}{2}\right) \\ & + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\Bigg] g(x)\, dx. \end{aligned} \tag{2.15}$$

Now, using the convexity of f and the hypothesis of g, the inequality

$$\begin{aligned} &\left[f(x) - f\left(tx + (1-t)\frac{a+b}{2}\right)\right] g(x) \\ &\quad + \left[f(a+b-x) - f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] g(x) \\ &\le (1-t)\left(x - \frac{a+b}{2}\right) f'(x) g(x) \\ &\quad + (1-t)\left(\frac{a+b}{2} - x\right) f'(a+b-x) g(x) \\ &= (1-t)\left(\frac{a+b}{2} - x\right)\left[f'(a+b-x) - f'(x)\right] g(x) \\ &\le (1-t)\left(\frac{a+b}{2} - x\right)\left[f'(a+b-x) - f'(x)\right] \|g\|_\infty \end{aligned}$$

holds for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Integrating the above inequalities over x on $\left[a, \frac{a+b}{2}\right]$ and using (2.13) – (2.15) and (1.11), we derive (2.2).

(3) Using the convexity of f, we have

$$\frac{f(a) - f\left(\frac{a+b}{2}\right)}{2} \le \frac{1}{2}\left(a - \frac{a+b}{2}\right) f'(a) = \frac{a-b}{4} f'(a)$$

and

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{2} \le \frac{1}{2}\left(b - \frac{a+b}{2}\right) f'(b) = \frac{b-a}{4} f'(b)$$

and taking their sum we obtain

$$\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{\left(f'(b) - f'(a)\right)(b-a)}{4}.$$

Thus,

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \tag{2.16}$$

Finally, (2.3) follows from (1.10), (1.11) and (2.16). This completes the proof. ∎

Remark 3. *Let* $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ *in Theorem 2. Then* $H_g(t)=H(t)$ $(t\in[0,1])$ *and Theorem 2 reduces to Theorem B.*

In the following theorems, we point out some inequalities for the functions H, H_g, G, L_g, Q considered above:

Theorem 4. *Let* f,g,G,H_g *be defined as above. Then we have the following Fejér-type inequalities:*

(1) *The following inequality holds for all* $t\in[0,1]$*:*

$$H_g(t)\leq G(t)\int_a^b g(x)\,dx. \tag{2.17}$$

(2) *The following inequality holds:*

$$\begin{aligned} 2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,g\left(2x-\frac{a+b}{2}\right)dx &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\int_a^b g(x)\,dx \\ &\leq (b-a)\int_0^1 G(t)\,g((1-t)a+tb)\,dt \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx. \end{aligned} \tag{2.18}$$

(3) *If* f *is differentiable on* $[a,b]$ *and* g *is bounded on* $[a,b]$*, then, for all* $t\in[0,1]$*, we have the inequality*

$$0\leq H_g(t)-f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\leq (b-a)\left[G(t)-H(t)\right]\|g\|_\infty \tag{2.19}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}|g(x)|$*.*

Proof. (1) Using simple techniques of integration and the hypothesis of g, we have that the following identity holds on $[0,1]$:

$$G(t)\int_a^b g(x)\,dx=\int_a^{\frac{a+b}{2}}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx. \tag{2.20}$$

By Lemma 1, the following inequality holds for all $x\in\left[a,\frac{a+b}{2}\right]$:

$$f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\leq f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right). \tag{2.21}$$

It holds when

$$A = ta + (1-t)\frac{a+b}{2}, \qquad C = tx + (1-t)\frac{a+b}{2},$$
$$D = t(a+b-x) + (1-t)\frac{a+b}{2} \quad \text{and} \quad B = tb + (1-t)\frac{a+b}{2}$$

in Lemma 1. Multiplying the inequality (2.21) by $g(x)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using identities (2.15) and (2.20), we derive (2.17).

(2) As for (1), we have the following identities:

$$2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\, g\left(2x - \frac{a+b}{2}\right) dx = \int_a^{\frac{a+b}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right)\right] g(x)\, dx; \tag{2.22}$$

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\int_a^b g(x)\, dx = \int_a^{\frac{a+b}{2}} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] g(x)\, dx; \tag{2.23}$$

$$\begin{aligned}
&(b-a)\int_0^1 G(t)\, g((1-t)a + tb)\, dt \\
&= \frac{b-a}{2}\left[\int_{\frac{1}{2}}^1 f\left(ta + (1-t)\frac{a+b}{2}\right) g(ta + (1-t)b)\, dt\right. \\
&\quad + \int_0^{\frac{1}{2}} f\left(ta + (1-t)\frac{a+b}{2}\right) g((1-t)a + tb)\, dt \\
&\quad + \int_0^{\frac{1}{2}} f\left(tb + (1-t)\frac{a+b}{2}\right) g((1-t)a + tb)\, dt \\
&\quad \left.+ \int_{\frac{1}{2}}^1 f\left(tb + (1-t)\frac{a+b}{2}\right) g(ta + (1-t)b)\, dt\right] \\
&= \int_a^{\frac{a+b}{2}} \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{2a+b-x}{2}\right)\right. \\
&\quad \left.+ f\left(\frac{b+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] g(x)\, dx;
\end{aligned} \tag{2.24}$$

and

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\, dx = \int_a^{\frac{a+b}{2}} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] g(x)\, dx. \tag{2.25}$$

By Lemma 1, the following inequalities hold for all $x \in \left[a, \frac{a+b}{2}\right]$.

$$f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \le f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \tag{2.26}$$

holds when $A=\frac{3a+b}{4}$, $C=\frac{x}{2}+\frac{a+b}{4}$, $D=\frac{3(a+b)}{4}-\frac{x}{2}$ and $B=\frac{a+3b}{4}$ in Lemma 1.

$$f\left(\frac{3a+b}{4}\right)\leq\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{2a+b-x}{2}\right)\right] \tag{2.27}$$

holds when $A=\frac{x+a}{2}$, $C=D=\frac{3a+b}{4}$ and $B=\frac{2a+b-x}{2}$ in Lemma 1.

$$f\left(\frac{a+3b}{4}\right)\leq\frac{1}{2}\left[f\left(\frac{b+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right] \tag{2.28}$$

holds when $A=\frac{b+x}{2}$, $C=D=\frac{a+3b}{4}$ and $B=\frac{a+2b-x}{2}$ in Lemma 1.

$$f\left(\frac{x+a}{2}\right)+f\left(\frac{2a+b-x}{2}\right)\leq f\left(a\right)+f\left(\frac{a+b}{2}\right) \tag{2.29}$$

holds when $A=a$, $C=\frac{x+a}{2}$, $D=\frac{2a+b-x}{2}$ and $B=\frac{a+b}{2}$ in Lemma 1.

$$f\left(\frac{b+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\leq f\left(\frac{a+b}{2}\right)+f\left(b\right) \tag{2.30}$$

holds when $A=\frac{a+b}{2}$, $C=\frac{b+x}{2}$, $D=\frac{a+2b-x}{2}$ and $B=b$ in Lemma 1. Multiplying the inequalities $(2.26)-(2.30)$ by $g\left(x\right)$, integrating both sides over x on $\left[a,\frac{a+b}{2}\right]$ and using identities $(2.22)-(2.25)$, we derive (2.18).

(3) By integration by parts, we have

$$\begin{aligned}
&t\int_{a}^{\frac{a+b}{2}}\left[\left(x-\frac{a+b}{2}\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)\right.\\
&\qquad\left.+\left(\frac{a+b}{2}-x\right)f'\left(t\left(a+b-x\right)+(1-t)\frac{a+b}{2}\right)\right]dx\\
&=t\int_{a}^{b}\left(x-\frac{a+b}{2}\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)dx\\
&=(b-a)\left[G\left(t\right)-H\left(t\right)\right]. \qquad (2.31)
\end{aligned}$$

Now, using the convexity of f and the hypothesis of g, the inequality

$$\begin{aligned}
&\left[f\left(tx+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right]g\left(x\right)\\
&\qquad+\left[f\left(t\left(a+b-x\right)+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right]g\left(x\right)\\
&\leq t\left(x-\frac{a+b}{2}\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)g\left(x\right)\\
&\qquad+t\left(\frac{a+b}{2}-x\right)f'\left(t\left(a+b-x\right)+(1-t)\frac{a+b}{2}\right)g\left(x\right)
\end{aligned}$$

$$= t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.-f'\left(tx+(1-t)\frac{a+b}{2}\right)\right]g(x)$$
$$\leq t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.-f'\left(tx+(1-t)\frac{a+b}{2}\right)\right]\|g\|_\infty$$

holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Integrating the above inequality over x on $\left[a,\frac{a+b}{2}\right]$ and using (2.31) and (1.11), we derive (2.17). This completes the proof. ∎

Remark 5. *Let* $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ *in Theorem 4. Then* $H_g(t)=H(t)$ $(t\in[0,1])$ *and Theorem 4 reduces to Theorem C.*

Theorem 6. *Let* f,g,G,H_g,L_g *be defined as above. Then we have the following results:*

(1) L_g *is convex on* $[0,1]$.
(2) *The following inequalities hold for all* $t\in[0,1]$*:*

$$G(t)\int_a^b g(x)\,dx\leq L_g(t) \tag{2.32}$$
$$\leq(1-t)\int_a^b f(x)g(x)\,dx+t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx$$
$$\leq\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx;$$

and

$$H_g(1-t)\leq L_g(t); \tag{2.33}$$

$$\frac{H_g(t)+H_g(1-t)}{2}\leq L_g(t). \tag{2.34}$$

(3) *The following bound is true:*

$$\sup_{t\in[0,1]}L_g(t)=\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2.35}$$

Proof. (1) It is easily observed from the convexity of f that L_g is convex on $[0,1]$.

(2) As for (1) in Theorem 4, we have that the following identity holds on $[0,1]$:

$$L_g(t)=\frac{1}{2}\int_a^{\frac{a+b}{2}}\left[f(ta+(1-t)x)+f(ta+(1-t)(a+b-x))\right. \tag{2.36}$$
$$\left.+f(tb+(1-t)x)+f(tb+(1-t)(a+b-x))\right]g(x)\,dx.$$

By Lemma 1, the following inequalities hold for all $x\in\left[a,\frac{a+b}{2}\right]$.

$$2f\left(ta+(1-t)\frac{a+b}{2}\right)\leq f(ta+(1-t)x)+f(ta+(1-t)(a+b-x)) \tag{2.37}$$

holds when $A = ta+(1-t)\,x, C = D = ta+(1-t)\,\frac{a+b}{2}$ and $B = ta+(1-t)\,(a+b-x)$ in Lemma 1.

$$2f\left(tb+(1-t)\frac{a+b}{2}\right) \le f\left(tb+(1-t)\,x\right)+f\left(tb+(1-t)\,(a+b-x)\right) \tag{2.38}$$

holds when $A = tb+(1-t)\,x, C = D = tb+(1-t)\,\frac{a+b}{2}$ and $B = tb+(1-t)\,(a+b-x)$ in Lemma 1. Multiplying the inequalities $(2.37)-(2.38)$ by $g(x)$, integrating them over x on $\left[a, \frac{a+b}{2}\right]$ and using identities (2.20) and (2.36), we derive the first inequality of (2.32). Using the convexity of f and the inequality (1.10), the last part of (2.32) holds. Again from the convexity of f, we get

$$\begin{aligned} H_g(1-t) &= \int_a^b f\left((1-t)\,x+t\frac{a+b}{2}\right)g(x)\,dx \\ &= \int_a^b f\left(\frac{ta+(1-t)\,x}{2}+\frac{tb+(1-t)\,x}{2}\right)g(x)\,dx \\ &\le L_g(t) \end{aligned} \tag{2.39}$$

and (2.33) is proved. From (2.17), (2.32) and (2.33), we get (2.34).

(3) Using (2.32), the inequality (2.35) holds. This completes the proof. ∎

Remark 7. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 6. Then* $H_g(t) = H(t)$ $(t \in [0,1])$, $L_g(t) = L(t)$ $(t \in [0,1])$ *and Theorem 6 reduces to Theorem D.*

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Department of Mathematics, Aletheia University, Tamsui, Taiwan 25103.
E-mail address: kltseng@email.au.edu.tw

China University of Science and Technology, Nankang, Taipei, Taiwan 11522
E-mail address: shru@ccs.cust.edu.tw

School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: sever.dragomir@.vu.edu.au
URL: http://www.staff.vu.edu.au/RGMIA/dragomir/