SOME COMPANIONS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. In this paper, we establish some companions of Fejér's inequality for convex functions which generalize the inequalities of Hermite-Hadamard type from [2] and [7].

1. Introduction

In what follows we assume that the function $f:[a,b]\to\mathbb{R}$ is convex, $g:[a,b]\to[0,\infty)$ is integrable and symmetric to $\frac{a+b}{2}$ and we define the following associated functions on $[0,1]$ by:

$$G(t)=\frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right];$$

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx;$$

$$I(t)=\int_a^b\frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx;$$

$$L(t)=\frac{1}{2(b-a)}\int_a^b\left[f(ta+(1-t)x)+f(tb+(1-t)x)\right]dx;$$

and

$$S_g(t)=\frac{1}{4}\int_a^b\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(ta+(1-t)\frac{x+b}{2}\right)+f\left(tb+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)\,dx.$$

If f is defined as above, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}\tag{1.1}$$

is known as the Hermite-Hadamard inequality [1].

For some results which generalize, improve, and extend this famous integral inequality see [2] – [16].

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In [2], Dragomir established the following theorem which refines the first inequality of (1.1).

Theorem A. *Let f, H be defined as above. Then H is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

In [7], Dragomir, Milošević and Sándor established inequalities related to (1.1). They are incorporated in the following:

Theorem B. *Let f, H be defined as above. Then:*

(1) *The following inequality holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \le \int_0^1 H(t)\,dt \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned} \tag{1.3}$$

(2) *If f is differentiable on $[a,b]$, then, for all $t \in [0,1]$, we have the inequalities*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)\,dx - H(t) \\ &\le (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned} \tag{1.4}$$

and

$$0 \le \frac{f(a)+f(b)}{2} - H(t) \le \frac{\left(f'(b)-f'(a)\right)(b-a)}{4}. \tag{1.5}$$

Theorem C. *Let f, H, G be defined as above. Then:*

(1) *G is convex and increasing on $[0,1]$.*

(2) *We have*

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2}.$$

(3) *The following inequality holds for all $t \in [0,1]$:*

$$H(t) \le G(t). \tag{1.6}$$

(4) *The following inequality holds:*

$$\begin{aligned} \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\le \int_0^1 G(t)\,dt \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]. \end{aligned} \tag{1.7}$$

(5) *If f is differentiable on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$0\le H(t)-f\left(\frac{a+b}{2}\right)\le G(t)-H(t). \tag{1.8}$$

Theorem D. *Let f,H,G,L be defined as above. Then:*

(1) *L is convex on $[0,1]$.*
(2) *We have the inequality:*

$$G(t)\le L(t)\le \frac{1-t}{b-a}\int_a^b f(x)\,dx+t\cdot\frac{f(a)+f(b)}{2}\le\frac{f(a)+f(b)}{2} \tag{1.9}$$

for all $t\in[0,1]$ and

$$\sup_{t\in[0,1]} L(t)=\frac{f(a)+f(b)}{2}.$$

(3) *For all $t\in[0,1]$, we have the inequalities:*

$$H(1-t)\le L(t)\quad\text{and}\quad \frac{H(t)+H(1-t)}{2}\le L(t).$$

In [8], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

Theorem E. *Let f,g be defined as above. Then we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)g(x)\,dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx, \tag{1.10}$$

which is known as Fejér's inequality.

In [11], Tseng, Hwang and Dragomir established the following theorems related to Fejér-type inequalities.

Theorem F. *Let f,g,I be defined as above. Then I is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have the following Fejér-type inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= I(0)\le I(t)\le I(1)\\ &=\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx. \end{aligned} \tag{1.11}$$

Theorem G. *Let f,g be defined as above. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx\\ &\le \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx\\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.12}$$

In this paper, we establish other Fejér-type inequalities related to the functions G,H,I,L,S_g and therefore generalize Theorems B – D from above.

2. Main Results

In order to prove our main results, we need the following simple lemma:

Lemma 1 (see [10])**.** *Let f be defined as above and let $a \leq A \leq C \leq D \leq B \leq b$ with $A+B=C+D$. Then*

$$f(C)+f(D)\leq f(A)+f(B).$$

Now, we are ready to state and prove our results.

Theorem 2. *Let f,g,I be defined as above. Then:*

(1) *The following inequality holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \\ & \leq 2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,g(4x-2a-b)\,dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\,g(4x-a-2b)\,dx\right] \\ & \leq \int_0^1 I(t)\,dt \\ & \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right. \\ & \qquad \left.+\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\right]. \end{aligned} \tag{2.1}$$

(2) *If f is differentiable on $[a,b]$ and g is bounded on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$\begin{aligned} 0 & \leq \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx-I(t) \\ & \leq (1-t)\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\|g\|_\infty, \end{aligned} \tag{2.2}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}|g(x)|$.

(3) *If f is differentiable on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$\begin{aligned} 0 & \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx-I(t) \\ & \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \end{aligned} \tag{2.3}$$

Proof. (1) Using simple techniques of integration, under the hypothesis of g, we have the following identities:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx=4\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}\right)g(2x-a)\,dt\,dx; \tag{2.4}$$

$$2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,g(4x-2a-b)\,dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\,g(4x-a-2b)\,dx\right]$$
$$=2\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]g(4x-2a-b)\,dx$$
$$=2\int_{a}^{\frac{a+b}{2}}\int_{0}^{\frac{1}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]g(2x-a)\,dtdx; \tag{2.5}$$

$$\begin{aligned}\int_0^1 I(t)\,dt&=\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right.\\&\qquad\left.+f\left((1-t)\frac{x+a}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\&\qquad+\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right.\\&\qquad\left.+f\left((1-t)\frac{x+b}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\&=\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right.\\&\qquad\left.+f\left((1-t)\frac{x+a}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\&\qquad+\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{a+2b-x}{2}+(1-t)\frac{a+b}{2}\right)\right.\\&\qquad\left.+f\left((1-t)\frac{a+2b-x}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\end{aligned}$$
$$\begin{aligned}&=\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(t\frac{a+b}{2}+(1-t)x\right)\right.\\&\qquad\left.+f\left(tx+(1-t)\frac{a+b}{2}\right)\right]g(2x-a)\,dtdx\\&\qquad+\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.\\&\qquad\left.+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)\right]g(2x-a)\,dtdx;\end{aligned} \tag{2.6}$$

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx+\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\right]$$
$$=\frac{1}{2}\int_a^b\int_0^{\frac{1}{2}}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{x+a}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]g(x)\,dtdx$$
$$=\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]g(2x-a)\,dtdx \tag{2.7}$$

$$+\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{a+b}{2}\right)+f(a+b-x)\right]g(2x-a)\,dtdx.$$

By Lemma 1, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$4f\left(\frac{a+b}{2}\right)\le 2\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right] \tag{2.8}$$

holds when $A=\frac{x}{2}+\frac{a+b}{4}$, $C=D=\frac{a+b}{2}$ and $B=\frac{3(a+b)}{4}-\frac{x}{2}$ in Lemma 1.

$$2f\left(\frac{x}{2}+\frac{a+b}{4}\right)\le f\left(t\frac{a+b}{2}+(1-t)\,x\right)+f\left(tx+(1-t)\,\frac{a+b}{2}\right) \tag{2.9}$$

holds when $A=t\frac{a+b}{2}+(1-t)\,x$, $C=D=\frac{x}{2}+\frac{a+b}{4}$ and $B=tx+(1-t)\,\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned}2f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\le\ & f\left(t(a+b-x)+(1-t)\,\frac{a+b}{2}\right)\\ &+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)\end{aligned} \tag{2.10}$$

holds when $A=t(a+b-x)+(1-t)\,\frac{a+b}{2}$, $C=D=\frac{3(a+b)}{4}-\frac{x}{2}$ and $B=t\frac{a+b}{2}+(1-t)(a+b-x)$ in Lemma 1.

$$f\left(t\frac{a+b}{2}+(1-t)\,x\right)+f\left(tx+(1-t)\,\frac{a+b}{2}\right)\le f(x)+f\left(\frac{a+b}{2}\right) \tag{2.11}$$

holds when $A=x$, $C=t\frac{a+b}{2}+(1-t)\,x$, $D=tx+(1-t)\,\frac{a+b}{2}$ and $B=\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned}f\left(t(a+b-x)+(1-t)\,\frac{a+b}{2}\right)+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)&\\ \le f\left(\frac{a+b}{2}\right)+f(a+b-x)&\end{aligned} \tag{2.12}$$

holds for $A=\frac{a+b}{2}$, $C=t(a+b-x)+(1-t)\,\frac{a+b}{2}$, $D=t\frac{a+b}{2}+(1-t)(a+b-x)$ and $B=a+b-x$ in Lemma 1. Multiplying the inequalities (2.8) $-$ (2.12) by $g(2x-a)$ and integrating them over t on $\left[0,\frac{1}{2}\right]$, over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.4) $-$ (2.7), we derive (2.1).

(2) On utilising the integration by parts, we have

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right]dx\\ &=\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\\ &=\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx.\end{aligned} \tag{2.13}$$

Using substitution rules for integration, under the hypothesis of g, we have the following identities

$$\begin{aligned}
&\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right] g(x)\,dx \\
&=\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right] g(x)\,dx \\
&=\int_a^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right] g(2x-a)\,dx \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
I(t)=&\int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right. \\
&\left.+f\left(t\frac{a+2b-x}{2}+(1-t)\frac{a+b}{2}\right)\right] g(x)\,dx \\
=&\int_a^{\frac{a+b}{2}} \left[f\left(tx+(1-t)\frac{a+b}{2}\right)\right. \\
&\left.+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right] g(2x-a)\,dx \qquad (2.15)
\end{aligned}$$

for all $t\in[0,1]$.

Now, by the convexity of f and the hypothesis of g, the inequality

$$\begin{aligned}
&\left[f(x)-f\left(tx+(1-t)\frac{a+b}{2}\right)\right] g(2x-a) \\
&\quad+\left[f(a+b-x)-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right] g(2x-a) \\
&\le (1-t)\left(x-\frac{a+b}{2}\right) f'(x)\,g(2x-a) \\
&\quad+(1-t)\left(\frac{a+b}{2}-x\right) f'(a+b-x)\,g(2x-a) \\
&=(1-t)\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right] g(2x-a) \\
&\le (1-t)\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right] \|g\|_\infty
\end{aligned}$$

holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Integrating the above inequalities over x on $\left[a,\frac{a+b}{2}\right]$ and using (2.13) – (2.15) and (1.11), we derive (2.2).

(3) Using the convexity of f, we have

$$\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2}\le\frac{1}{2}\left(a-\frac{a+b}{2}\right)f'(a)=\frac{a-b}{4}f'(a)$$

and

$$\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2}\le\frac{1}{2}\left(b-\frac{a+b}{2}\right)f'(b)=\frac{b-a}{4}f'(b)$$

and taking their sum we obtain

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\leq\frac{(f'(b)-f'(a))(b-a)}{4}.$$

Thus,

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx-f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \tag{2.16}$$

Finally, (2.3) follows from (1.11), (1.12) and (2.16). This completes the proof. ∎

Remark 3. *Let* $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ *in Theorem 2. Then* $I(t)=H(t)$ $(t\in[0,1])$ *and therefore Theorem 2 reduces to Theorem B.*

In the following theorems, we shall point out some inequalities for the functions G,H,I,S_g considered above:

Theorem 4. *Let* f,g,G,I *be defined as above. Then:*

(1) *The following inequality holds for all* $t\in[0,1]$*:*

$$I(t)\leq G(t)\int_a^b g(x)\,dx. \tag{2.17}$$

(2) *If* f *is differentiable on* $[a,b]$ *and* g *is bounded on* $[a,b]$, *then, for all* $t\in[0,1]$, *we have the inequality*

$$0\leq I(t)-f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\leq(b-a)\left[G(t)-H(t)\right]\|g\|_\infty, \tag{2.18}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}|g(x)|$.

Proof. (1) Using simple techniques of integration, under the hypothesis of g, we have that the following identity holds on $[0,1]$:

$$G(t)\int_a^b g(x)\,dx=\int_a^{\frac{a+b}{2}}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]g(2x-a)\,dx. \tag{2.19}$$

By Lemma 1, the following inequality holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right) \leq f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right) \tag{2.20}$$

holds when $A=ta+(1-t)\frac{a+b}{2}$, $C=tx+(1-t)\frac{a+b}{2}$, $D=t(a+b-x)+(1-t)\frac{a+b}{2}$ and $B=tb+(1-t)\frac{a+b}{2}$ in Lemma 1. Multiplying the inequality (2.20) by $g(2x-a)$, integrating both sides over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.15) and (2.19), we derive (2.17).

(2) Using an integration by parts, we have that the following identity holds on $[0,1]$:

$$
\begin{aligned}
& t\int_a^{\frac{a+b}{2}} \left[\left(x-\frac{a+b}{2}\right) f'\left(tx+(1-t)\frac{a+b}{2}\right)\right. \\
& \qquad \left. +\left(\frac{a+b}{2}-x\right) f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right] dx \\
& = t\int_a^b \left(x-\frac{a+b}{2}\right) f'\left(tx+(1-t)\frac{a+b}{2}\right) dx \\
& = (b-a)\left[G(t)-H(t)\right]. \qquad (2.21)
\end{aligned}
$$

Now, by the convexity of f, under the hypothesis of g, the inequality

$$
\begin{aligned}
& \left[f\left(tx+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right] g(2x-a) \\
& \qquad +\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right] g(2x-a) \\
& \leq t\left(x-\frac{a+b}{2}\right) f'\left(tx+(1-t)\frac{a+b}{2}\right) g(2x-a) \\
& \qquad +t\left(\frac{a+b}{2}-x\right) f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right) g(2x-a) \\
& = t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right. \\
& \qquad \left. -f'\left(tx+(1-t)\frac{a+b}{2}\right)\right] g(2x-a) \\
& \leq t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right. \\
& \qquad \left. -f'\left(tx+(1-t)\frac{a+b}{2}\right)\right] \|g\|_\infty
\end{aligned}
$$

holds for all $t \in [0,1]$ and $x \in \left[a, \frac{a+b}{2}\right]$. Integrating the above inequality over x on $\left[a, \frac{a+b}{2}\right]$ and using (2.21) and (1.11), we derive (2.18). This completes the proof. ∎

Remark 5. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 4. Then* $I(t) = H(t)$ $(t \in [0,1])$ *and the inequalities* (2.17) *and* (2.18) *reduce to the inequalities* (1.6) *and* (1.8), *respectively.*

Theorem 6. *Let* f, g, G, I, S_g *be defined as above. Then we have the following results:*

(1) S_g *is convex on* $[0,1]$.

(2) *The following inequalities hold for all* $t\in[0,1]$*:*

$$\begin{aligned} G(t)\int_a^b g(x)\,dx &\le S_g(t) \\ &\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\quad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx; \end{aligned} \tag{2.22}$$

$$I(1-t)\le S_g(t); \tag{2.23}$$

$$\frac{I(t)+I(1-t)}{2}\le S_g(t). \tag{2.24}$$

(3) *The following inequality holds:*

$$\sup_{t\in[0,1]} S_g(t)=\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2.25}$$

Proof. (1) It is easily observed from the convexity of f that S_g is convex on $[0,1]$.

(2) As for (1) of Theorem 4, we have that the following identity holds on $[0,1]$:

$$\begin{aligned} S_g(t)=\frac{1}{2}\int_a^{\frac{a+b}{2}} &[f(ta+(1-t)x)+f(ta+(1-t)(a+b-x)) \\ &+ f(tb+(1-t)x)+f(tb+(1-t)(a+b-x))]\,g(2x-a)\,dx. \end{aligned} \tag{2.26}$$

By Lemma 1, the following inequalities hold for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$2f\left(ta+(1-t)\frac{a+b}{2}\right)\le f(ta+(1-t)x)+f(ta+(1-t)(a+b-x)) \tag{2.27}$$

holds when $A=ta+(1-t)x$, $C=D=ta+(1-t)\frac{a+b}{2}$ and $B=ta+(1-t)(a+b-x)$ in Lemma 1.

$$2f\left(tb+(1-t)\frac{a+b}{2}\right)\le f(tb+(1-t)x)+f(tb+(1-t)(a+b-x)) \tag{2.28}$$

holds when $A=tb+(1-t)x$, $C=D=tb+(1-t)\frac{a+b}{2}$ and $B=tb+(1-t)(a+b-x)$ in Lemma 1. Multiplying the inequalities (2.27) – (2.28) by $g(2x-a)$, integrating them over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.19) and (2.26), we derive the first inequality of (2.22). Using the convexity of f and the inequality (1.12), the last part of (2.22) holds. Next, by the convexity of f and the identity (2.26), we get

$$\begin{aligned}
I_g(1-t) &= \int_a^{\frac{a+b}{2}} \left[f\left((1-t)x + t\frac{a+b}{2} \right) \right. \\
&\qquad \left. + f\left((1-t)(a+b-x) + t\frac{a+b}{2} \right) \right] g(2x-a)\,dx \\
&= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{1}{2}(ta+(1-t)x) + \frac{1}{2}(tb+(1-t)x) \right) \right. \\
&\qquad + f\left(\frac{1}{2}(ta+(1-t)(a+b-x)) \right. \\
&\qquad \left. \left. + \frac{1}{2}(tb+(1-t)(a+b-x)) \right) \right] g(2x-a)\,dx \\
&\le S_g(t) \qquad (2.29)
\end{aligned}$$

and the inequality (2.23) is proved. From (2.17), (2.22) and (2.23), we obtain (2.24).

(3) Using (2.22), the inequality (2.25) holds. This completes the proof. ∎

Remark 7. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 6. Then* $I(t) = H(t)$ $(t \in [0,1])$, $S_g(t) = L(t)$ $(t \in [0,1])$ *and Theorem 6 reduces to Theorem D.*

References

[1] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.,* **58** (1893), 171–215.

[2] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* **167** (1992), 49–56.

[3] S. S. Dragomir, A refinement of Hadamard's inequality for isotonic linear functionals, *Tamkang. J. Math.*

[4] S. S. Dragomir, On the Hadamard's inequality for convex on the co-ordinates in a rectangle from the plane, *Taiwanese J. Math.*, **5**(4) (2001), 775–788.

[5] S. S. Dragomir, Further properties of some mappings associated with Hermite-Hadamard inequalities, *Tamkang. J. Math.,* **34**(1) (2003), 45–57.

[6] S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. Appl.,* **245** (2000), 489–501.

[7] S. S. Dragomir, D.S. Milošević and J. Sándor, On some refinements of Hadamard's inequalities and applications, *Univ. Belgrad. Publ. Elek. Fak. Sci. Math.,* **4** (1993), 3–10.

[8] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.,* **24** (1906), 369–390. (Hungarian).

[9] D. Y. Hwang, K. L. Tseng and G. S. Yang, Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane, *Taiwanese J. Math.*, **11**(1) (2007), 63–73.

[10] K. L. Tseng, S. R. Hwang and S. S. Dragomir, On some new inequalities of Hermite-Hadamard-Fejér type involving convex functions, *Demonstratio Math.,* **XL**(1) (2007), 51–64.

[11] K. L. Tseng, S. R. Hwang and S. S. Dragomir, Fejér-type inequalities (I). Preprint, *RGMIA Res. Rep. Coll.*, **12**(2009), No. 4, Article 5. [Online http://www.staff.vu.edu.au/RGMIA/v12n4.asp].

[12] G. S. Yang and M. C. Hong, A note on Hadamard's inequality, *Tamkang. J. Math.,* **28**(1) (1997), 33–37.

[13] G. S. Yang and K. L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities, *J. Math. Anal. Appl.,* **239** (1999), 180–187.

[14] G. S. Yang and K. L. Tseng, Inequalities of Hadamard's type for Lipschitzian mappings, *J. Math. Anal. Appl.,* **260** (2001), 230–238.

[15] G. S. Yang and K. L. Tseng, On certain multiple integral inequalities related to Hermite-Hadamard inequalities, *Utilitas Math.,* **62** (2002), 131–142.

[16] G. S. Yang and K. L. Tseng, Inequalities of Hermite-Hadamard-Fejér type for convex functions and Lipschitzian functions, *Taiwanese J. Math.*, **7**(3) (2003), 433–440.

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