SOME INEQUALITIES FOR POWER SERIES OF SELFADJOINT OPERATORS IN HILBERT SPACES VIA REVERSES OF THE SCHWARZ INEQUALITY

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Abstract. In this paper we obtain some operator inequalities for functions defined by power series with real coefficients and, more specifically, with nonnegative coefficients. In order to obtain these inequalities some recent reverses of the Schwarz inequality for vectors in inner product spaces are utilized. Natural applications for some elementary functions of interest are also provided.

1. Introduction

Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be an analytic function defined by a power series with nonnegative coefficients a_n, $n \geq 0$ and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. As the most natural examples of such functions we have

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, z \in D(0,1) \text{ and } f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, z \in \mathbb{C}.$$

Other function as power series representations with nonnegative coefficients are, for instance

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \qquad z \in \mathbb{C};$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \qquad z \in \mathbb{C};$$

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1) n!} z^{2n+1}, \qquad z \in D(0,1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1);$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \quad \alpha,\beta,\gamma > 0, z \in D(0,1).$$

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Now, by the help of the power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series.

As some natural examples that are useful for applications, we can point out that, if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z},\qquad z\in D(0,1);$$
$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\qquad z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\qquad z\in\mathbb{C};$$
$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\qquad z\in D(0,1),$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are obviously

$$f_A(z)=\ln\frac{1}{1-z},\qquad g_A(z)=\cosh z,$$
$$h_A(z)=\sinh z\quad\text{and}\quad l_A(z)=\frac{1}{1-z}$$

and they are defined on the same domain as the generating functions.

Before we are able to state the new inequalities for functions of selfadjoint operators we need to recall *the continuous functional calculus* for such operators.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [7, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f)\ \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)}\qquad f(t)\geq g(t)\ \text{ for any } t\in Sp(A)\ \text{ implies that } f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [7] and the references therein. For other results, see [9], [10], [11] and [8].

The main aim of the present paper is to provide some operator inequalities for functions defined by power series with real coefficients and, more specifically, with nonnegative coefficients. In order to obtain these inequalities some recent reverses of the Schwarz inequality for vectors in inner product spaces are utilized. Natural applications for some elementary functions of interest are also considered.

2. Some Operator Inequalities

We use the following vector inequality obtained by the author in [1] (see also [5, pp. 28-29]):

Lemma 1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $z,y\in H$, $\gamma,\Gamma\in\mathbb{K}$ $(\Gamma\neq-\gamma)$ *so that either*

$$\operatorname{Re}\langle\Gamma y-z,z-\gamma y\rangle\geq 0, \tag{2.1}$$

or, equivalently,

$$\left\|z-\frac{\gamma+\Gamma}{2}y\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left\|y\right\| \tag{2.2}$$

holds. Then we have the inequalities

$$\begin{aligned}(0\leq)\left\|z\right\|\left\|y\right\|-\left|\langle z,y\rangle\right| &\leq\left\|z\right\|\left\|y\right\|-\left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle z,y\rangle\right]\right|\\ &\leq\left\|z\right\|\left\|y\right\|-\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle z,y\rangle\right]\leq\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^{2}}{|\Gamma+\gamma|}\left\|y\right\|^{2}.\end{aligned} \tag{2.3}$$

The constant $\frac{1}{4}$ *in the last inequality is best possible.*

Perhaps a more useful particular case of the above lemma is

Corollary 1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $z,y\in H$, $N>n>0$ *so that either*

$$\operatorname{Re}\langle Ny-z,z-ny\rangle\geq 0, \tag{2.4}$$

or, equivalently,

$$\left\|z-\frac{n+N}{2}y\right\|\leq\frac{1}{2}\left(N-n\right)\left\|y\right\| \tag{2.5}$$

holds. Then we have the inequalities

$$\begin{aligned}(0\leq)\left\|z\right\|\left\|y\right\|-\left|\langle z,y\rangle\right| &\leq\left\|z\right\|\left\|y\right\|-\left|\operatorname{Re}\langle z,y\rangle\right|\\ &\leq\left\|z\right\|\left\|y\right\|-\operatorname{Re}\langle z,y\rangle\leq\frac{1}{4}\cdot\frac{(N-n)^{2}}{n+N}\left\|y\right\|^{2}.\end{aligned} \tag{2.6}$$

The constant $\frac{1}{4}$ *in the last inequality is best possible.*

The following result holds:

Theorem 1. *Let consider the power series* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *with real coefficients* a_n *that is convergent on the open disk* $D(0,R)$ *with* $R>0$*. If the selfadjoint operator* T *on the Hilbert space* H *has the spectrum* $Sp(T)\subset[m,M]$ *and* $0<m<M$ *with* $\frac{M^2}{\sqrt{mM}}<R$*, then for any* $x\in H$ *with* $\|x\|=1$ *we have the inequality*

$$\max\{\|f(T)x\|,\|f_A(T)x\|\}\le\langle f_A(T)x,x\rangle+\frac{1}{8}\cdot\left[f_A\left(\frac{M^2}{\sqrt{mM}}\right)-2f_A\left(\sqrt{mM}\right)+f_A\left(\frac{m^2}{\sqrt{mM}}\right)\right]. \tag{2.7}$$

Proof. Since $Sp(T)\subset[m,M]$ then we have $0<m^kI\le T^k\le M^kI$ in the operator order of $B(H)$ and for any natural number k, which implies that

$$\left\langle M^kx-T^kx,T^kx-m^kx\right\rangle\ge0$$

or, equivalently,

$$\left\|T^kx-\frac{m^k+M^k}{2}x\right\|\le\frac{1}{2}\left(M^k-m^k\right)$$

for any $x\in H$ with $\|x\|=1$.

Now, if we apply the inequality (2.6), then we get

$$\left\|T^kx\right\|\le\left\langle T^kx,x\right\rangle+\frac{1}{4}\cdot\frac{(M^k-m^k)^2}{M^k+m^k} \tag{2.8}$$

for any natural number k and for any $x\in H$ with $\|x\|=1$.

Since $M^k+m^k\ge2\left(\sqrt{mM}\right)^k$, then we have

$$\frac{(M^k-m^k)^2}{M^k+m^k}\le\frac{M^{2k}-2m^kM^k+m^{2k}}{2\left(\sqrt{mM}\right)^k}=\frac{1}{2}\left[\left(\frac{M^2}{\sqrt{mM}}\right)^k-2\left(\sqrt{mM}\right)^k+\left(\frac{m^2}{\sqrt{mM}}\right)^k\right] \tag{2.9}$$

for any natural number k.

Utilising (2.8) and (2.9) we get

$$\left\|T^kx\right\|\le\left\langle T^kx,x\right\rangle+\frac{1}{8}\cdot\left[\left(\frac{M^2}{\sqrt{mM}}\right)^k-2\left(\sqrt{mM}\right)^k+\left(\frac{m^2}{\sqrt{mM}}\right)^k\right] \tag{2.10}$$

for any natural number k and for any $x\in H$ with $\|x\|=1$.

Now, if we multiply (2.10) with $|a_k|$ sum over k from 0 to n and use the triangle inequality, we deduce

$$\begin{aligned}\max\left\{\left\|\sum_{k=0}^{n}a_kT^kx\right\|,\left\|\sum_{k=0}^{n}|a_k|\,T^kx\right\|\right\}&\le\sum_{k=0}^{n}|a_k|\left\|T^kx\right\|\le\left\langle\sum_{k=0}^{n}|a_k|\,T^kx,x\right\rangle\\&+\frac{1}{8}\cdot\left[\sum_{k=0}^{n}|a_k|\left(\frac{M^2}{\sqrt{mM}}\right)^k-2\sum_{k=0}^{n}|a_k|\left(\sqrt{mM}\right)^k+\sum_{k=0}^{n}|a_k|\left(\frac{m^2}{\sqrt{mM}}\right)^k\right]\end{aligned} \tag{2.11}$$

for any natural number n and for any $x\in H$ with $\|x\|=1$.

Now, observe that

$$0 < \frac{m^2}{\sqrt{mM}} < \sqrt{mM} < M < \frac{M^2}{\sqrt{mM}} < R,$$

therefore all the series in the equation (2.11) are convergent and taking the limit over $n \to \infty$ in (2.11) we deduce the desired result (2.7). □

The following inequality also holds [2] (see also [5, p. 21]):

Lemma 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $z, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ so that either (2.1) or, equivalently, (2.2) holds true. Then we have the inequalities*

$$\|x\| \|y\| \le \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right) \langle x, y \rangle\right]}{\operatorname{Re}(\Gamma\bar{\gamma})} \le \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{\operatorname{Re}(\Gamma\bar{\gamma})} |\langle x, y \rangle| . \tag{2.12}$$

The constant $\frac{1}{2}$ is best possible in both inequalities.

Corollary 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $z, y \in H$, $N > n > 0$ so that either (2.4) or, equivalently, (2.5) holds true. Then we have*

$$\|x\| \|y\| \le \frac{1}{2} \cdot \frac{N + n}{\sqrt{Nn}} \operatorname{Re} \langle x, y \rangle . \tag{2.13}$$

The constant $\frac{1}{2}$ is best possible.

We are able to provide now another upper bound for $\|f(T)x\|$ with $x \in H$, $\|x\| = 1$ as follows:

Theorem 2. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with real coefficients a_n that is convergent on the open disk $D(0, R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with $M\sqrt{\frac{M}{m}} < R$, then for any $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\max\{\|f(T)x\|, \|f_A(T)x\|\} \le \frac{1}{2}\left\langle \left[f_A\left(\sqrt{\frac{M}{m}} \cdot T\right) + f_A\left(\sqrt{\frac{m}{M}} \cdot T\right)\right] x, x \right\rangle . \tag{2.14}$$

Proof. Since $Sp(T) \subset [m, M]$ then

$$\left\langle M^k x - T^k x, T^k x - m^k x \right\rangle \ge 0$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the inequality (2.13), then we can state that

$$\left\|T^k x\right\| \le \frac{1}{2} \cdot \frac{M^k + m^k}{\sqrt{M^k m^k}} \left\langle T^k x, x \right\rangle = \frac{1}{2}\left[\left(\sqrt{\frac{M}{m}}\right)^k + \left(\sqrt{\frac{m}{M}}\right)^k\right] \left\langle T^k x, x \right\rangle \tag{2.15}$$

for any natural number k and for any $x \in H$ with $\|x\| = 1$.

Now, if we multiply (2.10) with $|a_k|$ sum over k from 0 to n and use the triangle inequality, we deduce

$$\max\left\{\left\|\sum_{k=0}^{n} a_k T^k x\right\|, \left\|\sum_{k=0}^{n} |a_k| T^k x\right\|\right\} \tag{2.16}$$
$$\leq \sum_{k=0}^{n} |a_k| \left\|T^k x\right\| \leq \left\langle \sum_{k=0}^{n} |a_k| T^k x, x\right\rangle$$
$$\leq \frac{1}{2}\left\langle \sum_{k=0}^{n} |a_k| \left(\sqrt{\frac{M}{m}}\right)^k T^k x + \sum_{k=0}^{n} |a_k| \left(\sqrt{\frac{m}{M}}\right)^k T^k x, x\right\rangle$$

for any natural number n and for any $x \in H$ with $\|x\| = 1$.

Since

$$M\sqrt{\frac{m}{M}} < M < M\sqrt{\frac{M}{m}} < R$$

therefore all the series in the equation (2.16) are convergent and taking the limit over $n \to \infty$ in (2.16) we deduce the desired result (2.14). □

In order to obtain another reverse inequality for the operator $f(T)$ we need the following result [3] (see also [5, p. 4]):

Lemma 3. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $z, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ so that either (2.1) or, equivalently, (2.2) holds true. Then we have the inequality*

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot |\Gamma - \gamma|^2 \|y\|^4 . \tag{2.17}$$

The constant $\frac{1}{4}$ is best possible.

The case for positive bounds that will be used in the following is incorporated in

Corollary 3. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $z, y \in H$, $N > n > 0$ so that either (2.4) or, equivalently, (2.5) holds true. Then we have*

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot (N - n)^2 \|y\|^4 . \tag{2.18}$$

The constant $\frac{1}{4}$ is best possible.

The following result also holds:

Theorem 3. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with real coefficients a_n that is convergent on the open disk $D(0, R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with $M < R$, then for any $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\max\{\|f(T)x\|, \|f_A(T)x\|\} \leq \langle f_A(T)x, x\rangle + \frac{1}{2}[f_A(M) - f_A(m)] . \tag{2.19}$$

Proof. Since $Sp(T) \subset [m, M]$ then

$$\left\langle M^k x - T^k x, T^k x - m^k x\right\rangle \geq 0$$

for any $x \in H$ with $\|x\| = 1$.

Observe that from (2.18) we get

$$\|x\| \|y\| \leq \sqrt{|\langle x, y\rangle|^2 + \frac{1}{4} \cdot (N - n)^2 \|y\|^4} \leq |\langle x, y\rangle| + \frac{1}{2} \cdot (N - n) \|y\|^2$$

provided that either (2.4) or, equivalently, (2.5) holds true.

Now if we apply this inequality for T^k we can write that

$$\left\|T^k x\right\| \leq \left\langle T^k x, x\right\rangle + \frac{1}{2}\cdot\left(M^k - m^k\right)$$

for any natural number k and for any $x \in H$ with $\|x\| = 1$.

Finally, if we apply a similar argument to the one from the above theorems we derive the desired result (2.19). The details are omitted. □

Finally, in order to obtain the last reverse inequality, we need to use the following Klamkin-McLenaghan inequality for vectors in inner product spaces that has been obtained in [6]:

Lemma 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $z, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ so that either (2.1) or, equivalently, (2.2) holds true. Then we have the inequality*

$$(0 \leq) \left\|x\right\|^2 \left\|y\right\|^2 - \left|\langle x, y\rangle\right|^2 \leq \left[\left|\Gamma + \gamma\right| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right] \left|\langle x, y\rangle\right| \left\|y\right\|^2. \tag{2.20}$$

As a particular case of interest we have

Corollary 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $z, y \in H$, $N > n > 0$ so that either (2.4) or, equivalently, (2.5) holds true. Then we have*

$$(0 \leq) \left\|x\right\|^2 \left\|y\right\|^2 - \left|\langle x, y\rangle\right|^2 \leq \left(\sqrt{N} - \sqrt{n}\right)^2 \left|\langle x, y\rangle\right| \left\|y\right\|^2. \tag{2.21}$$

For a generalisation of (2.12), other similar results and the equlity case analysis, see [4].

Theorem 4. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with real coefficients a_n that is convergent on the open disk $D(0, R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with $M < R$, then for any $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\max\left\{\left\|f(T)x\right\|, \left\|f_A(T)x\right\|\right\} \leq \left\langle f_A(T)x, x\right\rangle + f_A(M) - f_A\left(\sqrt{mM}\right). \tag{2.22}$$

Proof. Notice that from the inequality (2.21) we have

$$\begin{aligned}\|x\|\|y\| &\leq \sqrt{\left|\langle x, y\rangle\right|^2 + \left(\sqrt{N} - \sqrt{n}\right)^2 \left|\langle x, y\rangle\right| \left\|y\right\|^2} \\ &\leq \left|\langle x, y\rangle\right| + \left(\sqrt{N} - \sqrt{n}\right)\sqrt{\left|\langle x, y\rangle\right|}\,\|y\|\end{aligned}$$

provided that either (2.4) or, equivalently, (2.5) holds true.

Now if we apply this inequality for T^k we can write that

$$\begin{aligned}\left\|T^k x\right\| &\leq \left\langle T^k x, x\right\rangle + \left(\sqrt{M^k} - \sqrt{m^k}\right)\sqrt{\left|\left\langle T^k x, x\right\rangle\right|} \\ &\leq \left\langle T^k x, x\right\rangle + \left(\sqrt{M^k} - \sqrt{m^k}\right)\sqrt{M^k}\end{aligned}$$

for any natural number k and for any $x \in H$ with $\|x\| = 1$.

Finally, if we apply a similar argument to the one from the above theorems we derive the desired result (2.19). The details are omitted. □

Problem 1. *It is an open problem whether or not the above inequalities (2.7), (2.14), (2.19) and (2.22) are sharp, and if they are not, which bound is better for different examples of functions.*

3. Applications

1. Consider the function $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$. If T is a bounded linear operator on the Hilbert space $(X, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(T) \subset [m, M]$ and $0 < M^3 < m < M < 1$ then by the inequality (2.7) we have

$$(0 \leq) \left\| (I-T)^{-1} x \right\| - \left\langle (I-T)^{-1} x, x \right\rangle \leq \frac{1}{8} \cdot \left[\frac{\sqrt{mM}}{\sqrt{mM} - M^2} - \frac{2}{1-\sqrt{mM}} + \frac{\sqrt{mM}}{\sqrt{mM} - m^2} \right] \tag{3.1}$$

while by (2.13) we have

$$\left\| (I-T)^{-1} x \right\| \leq \frac{1}{2} \cdot \left\langle \left[\left(I - \sqrt{\frac{M}{m}} \cdot T \right)^{-1} x + \left(I - \sqrt{\frac{m}{M}} \cdot T \right)^{-1} x \right], x \right\rangle \tag{3.2}$$

for any $x \in H$ with $\|x\| = 1$.

If $0 < m < M < 1$, then by (2.19) and (2.22) we have the simpler inequalities

$$(0 \leq) \left\| (I-T)^{-1} x \right\| - \left\langle (I-T)^{-1} x, x \right\rangle \leq \frac{1}{2} \cdot \frac{M-m}{(1-m)(1-M)} \tag{3.3}$$

and

$$(0 \leq) \left\| (I-T)^{-1} x \right\| - \left\langle (I-T)^{-1} x, x \right\rangle \leq \frac{M - \sqrt{mM}}{(1-M)\left(1-\sqrt{mM}\right)} \tag{3.4}$$

respectively, for any $x \in H$ with $\|x\| = 1$.

2. Consider the function $f(z) = \ln\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n}$, $z \in D(0,1)$. If T is a bounded linear operator on the Hilbert space $(X, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(T) \subset [m, M]$ and $0 < M^3 < m < M < 1$ then by the inequality (2.7) we have

$$(0 \leq) \left\| \ln (I-T)^{-1} x \right\| - \left\langle \ln (I-T)^{-1} x, x \right\rangle \leq \frac{1}{8} \cdot \ln \left[\frac{mM\left(1-\sqrt{mM}\right)^2}{\left(\sqrt{mM} - M^2\right)\left(\sqrt{mM} - m^2\right)} \right] \tag{3.5}$$

while by (2.13) we have

$$\left\| \ln (I-T)^{-1} x \right\| \leq \frac{1}{2} \cdot \left\langle \ln \left[\left(I - \sqrt{\frac{M}{m}} \cdot T \right)^{-1} \left(I - \sqrt{\frac{m}{M}} \cdot T \right)^{-1} \right] x, x \right\rangle \tag{3.6}$$

for any $x \in H$ with $\|x\| = 1$.

If $0 < m < M < 1$, then by (2.19) and (2.22) we have the simpler inequalities

$$(0 \leq) \left\| \ln (I-T)^{-1} x \right\| - \left\langle \ln (I-T)^{-1} x, x \right\rangle \leq \frac{1}{2} \cdot \ln \left(\frac{1-m}{1-M} \right) \tag{3.7}$$

and

$$(0 \leq) \left\| \ln (I-T)^{-1} x \right\| - \left\langle \ln (I-T)^{-1} x, x \right\rangle \leq \ln \left(\frac{1-\sqrt{mM}}{1-M} \right) \tag{3.8}$$

respectively, for any $x \in H$ with $\|x\| = 1$.

3. Consider the exponential function $f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$. If T is a bounded linear operator on the Hilbert space $(X, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M < \infty$ then by the inequality (2.7) we have

$$\begin{aligned} (0 \leq) & \|\exp(T) x\| - \langle \exp(T) x, x \rangle \\ & \leq \frac{1}{8} \cdot \left[\exp\left(\frac{M^2}{\sqrt{mM}} \right) - 2\exp\left(\sqrt{mM} \right) + \exp\left(\frac{m^2}{\sqrt{mM}} \right) \right] \end{aligned} \tag{3.9}$$

while by (2.13) we have

$$\|\exp(T) x\| \leq \frac{1}{2} \left\langle \left[\exp\left(\sqrt{\frac{M}{m}} \cdot T \right) + \exp\left(\sqrt{\frac{m}{M}} \cdot T \right) \right] x, x \right\rangle \tag{3.10}$$

for any $x \in H$ with $\|x\| = 1$.

Also, by (2.19) and (2.22) we have the simpler inequalities

$$(0 \leq) \|\exp(T) x\| - \langle \exp(T) x, x \rangle \leq \frac{1}{2} \cdot [\exp(M) - \exp(m)] \tag{3.11}$$

and

$$(0 \leq) \|\exp(T) x\| - \langle \exp(T) x, x \rangle \leq \exp(M) - \exp\left(\sqrt{mM} \right) \tag{3.12}$$

for any $x \in H$ with $\|x\| = 1$.

4. Consider the function $f(z) = \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$, $z \in \mathbb{C}$. Obviously, we have $f_A(z) = \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$, $z \in \mathbb{C}$. If T is a bounded linear operator on the Hilbert space $(X, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M < \infty$ then by the inequality (2.7) we have

$$\begin{aligned} & \|\cos(T) x\| \leq \langle \cosh(T) x, x \rangle \\ & + \frac{1}{8} \cdot \left[\cosh\left(\frac{M^2}{\sqrt{mM}} \right) - 2\cosh\left(\sqrt{mM} \right) + \cosh\left(\frac{m^2}{\sqrt{mM}} \right) \right] \end{aligned} \tag{3.13}$$

while by (2.13) we have

$$\|\cos(T) x\| \leq \frac{1}{2} \left\langle \left[\cosh\left(\sqrt{\frac{M}{m}} \cdot T \right) + \cosh\left(\sqrt{\frac{m}{M}} \cdot T \right) \right] x, x \right\rangle \tag{3.14}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, by (2.19) and (2.22) we have the simpler inequalities

$$\|\cos(T) x\| \leq \langle \cosh(T) x, x \rangle + \frac{1}{2} \cdot [\cosh(M) - \cosh(m)] \tag{3.15}$$

and

$$\|\cos(T) x\| \leq \langle \cosh(T) x, x \rangle + \cosh(M) - \cosh\left(\sqrt{mM} \right) \tag{3.16}$$

respectively, for any $x \in H$ with $\|x\| = 1$.

If we apply the same inequalities only for the function with positive coefficients cosh then we also have

$$0 \leq \|\cosh(T)x\| - \langle \cosh(T)x, x\rangle \leq \frac{1}{8} \cdot \left[\cosh\left(\frac{M^2}{\sqrt{mM}}\right) - 2\cosh\left(\sqrt{mM}\right) + \cosh\left(\frac{m^2}{\sqrt{mM}}\right)\right] \tag{3.17}$$

and

$$\|\cosh(T)x\| \leq \frac{1}{2}\left\langle \left[\cosh\left(\sqrt{\frac{M}{m}} \cdot T\right) + \cosh\left(\sqrt{\frac{m}{M}} \cdot T\right)\right] x, x\right\rangle \tag{3.18}$$

and

$$0 \leq \|\cosh(T)x\| - \langle \cosh(T)x, x\rangle \leq \frac{1}{2} \cdot [\cosh(M) - \cosh(m)] \tag{3.19}$$

and

$$0 \leq \|\cosh(T)x\| - \langle \cosh(T)x, x\rangle \leq \cosh(M) - \cosh\left(\sqrt{mM}\right) \tag{3.20}$$

respectively, for any $x \in H$ with $\|x\| = 1$.

Remark 1. *The reader may obtain other operator inequalities by choosing appropriate examples of functions defined by series with positive coefficients. However, the details are omitted.*

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