Generalizations and analogues of the Nesbitt's inequality

Fuhua Wei and Shanhe Wu *

Department of Mathematics and Computer Science, Longyan University, Longyan, Fujian 364012, P. R. China

E-mail: wushanhe@yahoo.com.cn

*Corresponding Author

Abstract: The Nesbitt's inequality is generalized by introducing exponent and weight parameters. Several Nesbitt-type inequalities for n variables are provided. Finally, two analogous forms of Nesbitt's inequality are given.

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1 Introduction

The Nesbitt's inequality states that if x, y, z are positive real numbers, then

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}, \tag{1}$$

the equality occurs if and only if the three variables are equal ([1], see also [2]).

It is well known that this cyclic sum inequality has many applications in the proof of fractional inequalities. In this paper we shall establish some generalizations and analogous forms of the Nesbitt's inequality.

2 Generalizations of the Nesbitt's inequality

Theorem 1. *Let $x,\ y,\ z,\ k$ be positive real numbers. Then*

$$\frac{x}{ky+z}+\frac{y}{kz+x}+\frac{z}{kx+y}\geq\frac{3}{1+k}. \tag{2}$$

Proof. By using the Cauchy-Schwarz inequality (see [3]), we have

$$(kxy+zx+kyz+xy+kxz+yz)\left(\frac{x^2}{kxy+zx}+\frac{y^2}{kyz+xy}+\frac{z^2}{kxz+yz}\right)$$
$$\geq (x+y+z)^2.$$

Hence

$$\begin{aligned}\frac{x}{ky+z}+\frac{y}{kz+x}+\frac{z}{kx+y} &\geq \frac{(x+y+z)^2}{(1+k)(xy+yz+zx)}\\ &= \frac{x^2+y^2+z^2+2xy+2yz+2zx}{(1+k)(xy+yz+zx)}\\ &\geq \frac{3}{1+k}.\end{aligned}$$

The Theorem 1 is proved.

Theorem 2. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \geq 2$. Then*

$$\frac{x_1}{x_2+x_3+\cdots+x_n}+\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}+\cdots+\frac{x_n}{x_1+x_2+\cdots+x_{n-1}} \geq \frac{n}{n-1}. \tag{3}$$

Proof. Let $s = x_1 + x_2 + \cdots + x_n$, one has

$$\frac{x_1}{x_2+x_3+\cdots+x_n}+\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}+\cdots+\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}$$
$$=\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+\cdots+\frac{x_n}{s-x_n}.$$

By symmetry, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$, then

$$s-x_1 \leq s-x_2 \leq \cdots \leq s-x_n, \quad \frac{x_1}{s-x_1} \geq \frac{x_2}{s-x_2} \geq \cdots \geq \frac{x_n}{s-x_n}.$$

Using the Chebyshev's inequality (see [3]) gives

$$\frac{x_1}{s-x_1}(s-x_1)+\frac{x_2}{s-x_2}(s-x_2)+\cdots+\frac{x_n}{s-x_n}(s-x_n)$$
$$\leq \frac{1}{n}\left(\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+\cdots+\frac{x_n}{s-x_n}\right)[(s-x_1)+(s-x_2)+\cdots+(s-x_n)],$$

or equivalently

$$\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+\cdots+\frac{x_n}{s-x_n} \geq \frac{n}{n-1},$$

this is exactly the required inequality.

Theorem 3. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \geq 2$, $k \geq 1$. Then*

$$\left(\frac{x_1}{x_2+x_3+\cdots+x_n}\right)^k+\left(\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}\right)^k+\cdots+\left(\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}\right)^k \geq \frac{n}{(n-1)^k}. \tag{4}$$

Proof. Using the power mean inequality and the inequality (3), we have

$$\left(\frac{x_1}{x_2+x_3+\cdots+x_n}\right)^k+\left(\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}\right)^k+\cdots+\left(\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}\right)^k$$
$$\geq n^{1-k}\left(\sum_{i=1}^{n}\frac{x_i}{s-x_i}\right)^k$$
$$\geq \frac{n}{(n-1)^k}.$$

This completes the proof.

Theorem 4. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let $\lambda \geq 1$, $r \geq s > 0$, $\sum_{i=1}^{n} x_i^s = p$. Then*

$$\sum_{i=1}^{n}\left(\frac{x_i^r}{p-x_i^s}\right)^\lambda \geq n^{1-\lambda}\left(\frac{n}{n-1}\right)^\lambda\left(\frac{p}{n}\right)^{\lambda\left(\frac{r}{s}-1\right)}. \tag{5}$$

Proof. Using the power mean inequality (see [3]), we have

$$\sum_{i=1}^{n}\left(\frac{x_i^r}{p-x_i^s}\right)^\lambda \geq n^{1-\lambda}\left(\sum_{i=1}^{n}\frac{x_i^r}{p-x_i^s}\right)^\lambda.$$

On the other hand, by symmetry, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$, then

$$x_1^s \geq x_2^s \geq \cdots \geq x_n^s > 0, \quad p - x_n^s \geq p - x_{n-1}^s \geq \cdots \geq p - x_1^s > 0.$$

Applying the generalized Radon's inequality (see [4-7])

$$\sum_{i=1}^{n} \frac{a_i^{\alpha}}{b_i} \geq n^{2-\alpha} (\sum_{i=1}^{n} a_i)^{\alpha} / (\sum_{i=1}^{n} b_i)$$

($a_1 \geq a_2 \geq \cdots \geq a_n > 0$, $b_n \geq b_{n-1} \geq \cdots \geq b_1 > 0$, $\alpha \geq 1$), we deduce that

$$\sum_{i=1}^{n} \frac{x_i^r}{p - x_i^s} = \sum_{i=1}^{n} \frac{(x_i^s)^{\frac{r}{s}}}{p - x_i^s} \geq n^{2-\frac{r}{s}} \cdot \frac{(\sum\limits_{i=1}^{n} x_i^s)^{\frac{r}{s}}}{\sum\limits_{i=1}^{n} (p - x_i^s)} = \frac{n}{n-1} \left(\frac{p}{n}\right)^{\frac{r}{s} - 1},$$

Therefore

$$\sum_{i=1}^{n} \left(\frac{x_i^r}{p - x_i^s} \right)^{\lambda} \geq n^{1-\lambda} \left(\sum_{i=1}^{n} \frac{x_i^r}{p - x_i^s} \right)^{\lambda} \geq n^{1-\lambda} \left(\frac{n}{n-1} \right)^{\lambda} \left(\frac{p}{n} \right)^{\lambda(\frac{r}{s} - 1)}.$$

The proof of Theorem 4 is complete.

In Theorem 4, choosing $\lambda = 1$, $s = 1$, $n = 3$, $x_1 = x$, $x_2 = y$, $x_3 = z$, we get

Theorem 5. *Let x, y, z be positive real numbers, and let $x + y + z = p$, $r \geq 1$. Then*

$$\frac{x^r}{y+z} + \frac{y^r}{z+x} + \frac{z^r}{x+y} \geq \frac{3}{2} \left(\frac{p}{3} \right)^{r-1}. \tag{6}$$

In particular, when $r = 1$, the inequality (6) becomes the Nesbitt's inequality (1).

3 Analogous forms of the Nesbitt's inequality

Theorem 6. *Let x, y, z be positive real numbers, Then*

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \leq \frac{3\sqrt{2}}{2} \tag{7}$$

Proof. Note that

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}}$$

$$= \sqrt{z+x}\sqrt{\frac{x}{(x+y)(z+x)}} + \sqrt{x+y}\sqrt{\frac{y}{(y+z)(x+y)}} + \sqrt{y+z}\sqrt{\frac{z}{(z+x)(y+z)}}.$$

By using the Cauchy-Schwarz inequality, we have

$$\left(\sqrt{z+x}\sqrt{\frac{x}{(x+y)(z+x)}} + \sqrt{x+y}\sqrt{\frac{y}{(y+z)(x+y)}} + \sqrt{y+z}\sqrt{\frac{z}{(z+x)(y+z)}} \right)^2$$

$$\leq (z + x + x + y + y + z) \left[\frac{x}{(x+y)(z+x)} + \frac{y}{(y+z)(x+y)} + \frac{z}{(z+x)(y+z)} \right].$$

Thus, to prove the inequality (7), it suffices to show that

$$(x+y+z) \left[\frac{x}{(x+y)(z+x)} + \frac{y}{(y+z)(x+y)} + \frac{z}{(z+x)(y+z)} \right] \leq \frac{9}{4}.$$

Direct computation gives

$$
\begin{aligned}
&(x+y+z)\left[\frac{x}{(x+y)(z+x)}+\frac{y}{(y+z)(x+y)}+\frac{z}{(z+x)(y+z)}\right]-\frac{9}{4}\\
=&\ \frac{(x+y+z)[x(y+z)+y(z+x)+z(x+y)]}{(x+y)(y+z)(z+x)}-\frac{9}{4}\\
=&\ \frac{4(x+y+z)[x(y+z)+y(z+x)+z(x+y)]-9(x+y)(y+z)(z+x)}{4(x+y)(y+z)(z+x)}\\
=&\ \frac{8(x+y+z)(xy+yz+zx)-9(x+y)(y+z)(z+x)}{4(x+y)(y+z)(z+x)}\\
=&\ \frac{6xyz-x^2y-x^2z-xy^2-y^2z-xz^2-yz^2}{4(x+y)(y+z)(z+x)}\\
\le&\ 0,
\end{aligned}
$$

where the inequality sign is due to the arithmetic-geometric means inequality. The Theorem 6 is thus proved.

Theorem 7. *Let x, y, z be positive real numbers, $\alpha \le 1/2$, Then*

$$\left(\frac{x}{x+y}\right)^{\alpha}+\left(\frac{y}{y+z}\right)^{\alpha}+\left(\frac{z}{z+x}\right)^{\alpha}\le\frac{3}{2^{\alpha}}. \tag{8}$$

Proof. It follows from the power mean inequality that

$$
\begin{aligned}
&\left(\frac{x}{x+y}\right)^{\alpha}+\left(\frac{y}{y+z}\right)^{\alpha}+\left(\frac{z}{z+x}\right)^{\alpha}\\
\le&\ 3^{1-2\alpha}\left(\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}\right)^{2\alpha}\\
\le&\ 3^{1-2\alpha}\left(\frac{3}{\sqrt{2}}\right)^{2\alpha}\\
=&\ \frac{3}{2^{\alpha}}.
\end{aligned}
$$

The inequality (8) is proved.

Remark. The inequality (8) is the exponential generalization of inequality (7). As a further generalization of inequality (7), we put forward the following the following conjecture.

Conjecture. *Let $x_1, x_2, \dots, x_n$ be positive real numbers, $n \ge 2$, $\alpha \le 1/2$. Then*

$$\left(\frac{x_1}{x_1+x_2}\right)^{\alpha}+\left(\frac{x_2}{x_2+x_3}\right)^{\alpha}+\cdots+\left(\frac{x_{n-1}}{x_{n-1}+x_n}\right)^{\alpha}+\left(\frac{x_n}{x_n+x_1}\right)^{\alpha}\le\frac{n}{2^{\alpha}}. \tag{9}$$

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