WEIGHTED $f-$GINI MEAN DIFFERENCE FOR CONVEX AND SYMMETRIC FUNCTIONS IN LINEAR SPACES

S.S. DRAGOMIR

Abstract. The concept of *weighted $f-$Gini mean difference* for convex and symmetric functions in linear spaces is introduced. Some fundamental inequalities and applications for norms are also provided.

1. Introduction

The *Gini mean difference* of the *sample* $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ is defined by

$$G(\mathbf{a})=\frac{1}{2n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}|a_i-a_j|=\frac{1}{n^2}\sum_{1\le i<j\le n}|a_i-a_j|$$

and

$$R(\mathbf{a})=\frac{1}{\bar{a}}G(\mathbf{a})$$

is the Gini index of $\mathbf{a}$, provided the sample mean $\bar{a}$ is not zero [7, p. 257].

The Gini index of $\mathbf{a}$ equals the Gini mean difference of the "scaled down" sample $\tilde{a}=\left(\frac{a_1}{\bar{a}},\dots,\frac{a_n}{\bar{a}}\right)$ $(\bar{a}\neq 0)$

$$R(a_1,\dots,a_n)=\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{a_i}{\bar{a}}-\frac{a_j}{\bar{a}}\right|.$$

The following elementary properties of the Gini index for an empirical distribution of nonnegative data hold [7, p. 257]:

(i) Let $(a_1,\dots,a_n)\in\mathbb{R}_+^n$ with $\sum_{i=1}^n a_i>0$. Then

$$0=R(\bar{a},\dots,\bar{a})\le R(a_1,\dots,a_n)\le R\left(0,\dots,0,\sum_{i=1}^n a_i\right)=1-\frac{1}{n}<1,$$
$$R(\beta a_1,\dots,\beta a_n)=R(a_1,\dots,a_n)\quad\text{for every}\quad\beta>0$$

and

$$R(a_1+\lambda,\dots,a_n+\lambda)=\frac{\bar{a}}{\bar{a}+\lambda}R(a_1,\dots,a_n)\quad\text{for }\lambda>0.$$

(ii) R is a continuous function on $\mathbb{R}_+^n$.

Date: March 04, 2009.

1991 *Mathematics Subject Classification.* 26D15; 94.

Key words and phrases. Convex functions, Jensen's inequality, Norms, Means, Weighted $f-$Gini mean difference.

These and other properties have been investigated in [7], [4] and [5].

For $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\dots,p_n)$ a probability sequence, meaning that $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$, we considered in [1] the *weighted Gini mean difference* defined by formula

$$G(\mathbf{p},\mathbf{a})=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}p_ip_j|a_i-a_j|=\sum_{1\leq i<j\leq n}p_ip_j|a_i-a_j|, \tag{1.1}$$

and proved that

$$\frac{1}{2}K(\mathbf{p},\mathbf{a})\leq G(\mathbf{p},\mathbf{a})\leq\inf_{\gamma\in\mathbb{R}}\left[\sum_{i=1}^{n}p_i|a_i-\gamma|\right]\leq K(\mathbf{p},\mathbf{a}), \tag{1.2}$$

where $K(\mathbf{p},\mathbf{a})$ is the *mean absolute deviation*, namely

$$K(\mathbf{p},\mathbf{a}):=\sum_{i=1}^{n}p_i\left|a_i-\sum_{j=1}^{n}p_ja_j\right|. \tag{1.3}$$

We have also shown that if more information on the sampling data $\mathbf{a}=(a_1,\dots,a_n)$ is available, i.e., there exists the real numbers a and A such that $a\leq a_i\leq A$ for each $i\in\{1,\dots,n\}$, then

$$G(\mathbf{p},\mathbf{a})\leq(A-a)\max_{J\subseteq\{1,\dots,n\}}[P_J(1-P_J)]\qquad\left(\leq\frac{1}{4}(A-a)\right), \tag{1.4}$$

where $P_J:=\sum_{j\in J}p_j$. Also, we have shown that

$$G(\mathbf{p},\mathbf{a})\leq\sum_{i=1}^{n}p_i\left|a_i-\frac{A+a}{2}\right|\qquad\left(\leq\frac{1}{2}(A-a)\right). \tag{1.5}$$

Notice that in general the bounds for the weighted Gini mean difference $G(\mathbf{p},\mathbf{a})$ provided by (1.4) and (1.5) cannot be compared to conclude that one is always better than the other [1].

For $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\dots,p_n)$ a probability sequence, meaning that $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$, define the *$r-$weighted Gini mean difference,* for $r\in[1,\infty)$, by the formula [1, p. 291]:

$$G_r(\mathbf{p},\mathbf{a}):=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}p_ip_j|a_i-a_j|^r=\sum_{1\leq i<j\leq n}p_ip_j|a_i-a_j|^r. \tag{1.6}$$

For $r=1$ we have the *weighted Gini mean difference* $G(\mathbf{p},\mathbf{a})$ of (1.1) which becomes, for the uniform probability distribution $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ the *Gini mean difference*

$$G(\mathbf{a}):=\frac{1}{2n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}|a_i-a_j|=\frac{1}{n^2}\sum_{1\leq i<j\leq n}|a_i-a_j|.$$

For the uniform probability distribution $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ we denote

$$G_r(\mathbf{a}):=G_r(\mathbf{p},\mathbf{a})=\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_i-a_j|^r=\frac{1}{n^2}\sum_{1\leq i<j\leq n}|a_i-a_j|^r.$$

Now, if we define $\Delta := \{(i,j) \,|\, i,j \in \{1,\ldots,n\}\}$, then we can simply write from (1.6)

$$G_r(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^r, \qquad r \geq 1. \tag{1.7}$$

The following result concerning upper and lower bounds for $G_r(\mathbf{p},\mathbf{a})$ may be stated (see [2]):

Theorem 1. *For any $p_i \in (0,1)$, $i \in \{1,\ldots,n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1,\ldots,n\}$, we have the inequalities*

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{\frac{p_i^r p_j^r + p_i p_j (1-p_i p_j)^{r-1}}{(1-p_i p_j)^{r-1}} |a_i - a_j|^r\right\} \leq G_r(\mathbf{p},\mathbf{a}) \leq \frac{1}{2}\max_{(i,j)\in\Delta}|a_i - a_j|^r, \tag{1.8}$$

where $r \in [1,\infty)$.

Remark 1. *The case $r = 2$ is of interest, since*

$$G_2(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i\right)^2,$$

for which we can obtain from Theorem 1 the following bounds:

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{\frac{p_i p_j}{1-p_i p_j}(a_i - a_j)^2\right\} \leq G_2(\mathbf{p},\mathbf{a}) \leq \frac{1}{2}\max_{(i,j)\in\Delta}(a_i - a_j)^2. \tag{1.9}$$

Remark 2. *Since the function*

$$h_r(t) := \frac{t^r + t(1-t)^{r-1}}{(1-t)^{r-1}} = t + t^r(1-t)^{1-r}$$

defined for $t \in [0,1)$ and $r > 1$ is strictly increasing on $[0,1)$ from Theorem 1 we can obtain a coarser but, perhaps, a more useful lower bound for the $r-$weighted Gini mean difference, namely (see [2]):

$$G_r(\mathbf{p},\mathbf{a}) \geq \frac{1}{2}\cdot\frac{p_m^{2r} + p_m^2(1-p_m^2)^{r-1}}{(1-p_m^2)^{r-1}}\cdot\max_{(i,j)\in\Delta}|a_i - a_j|^r, \tag{1.10}$$

where p_m is defined above.

For $r = 2$, we then have:

$$G_2(\mathbf{p},\mathbf{a}) \geq \frac{1}{2}\cdot\frac{p_m^2}{1-p_m^2}\cdot\max_{(i,j)\in\Delta}(a_i - a_j)^2. \tag{1.11}$$

For other results related to the above, see the recent paper [2]. For various inequalities concerning $G_2(\mathbf{p},\mathbf{a})$, see the book [3] and the references therein.

In this paper, motivated by the above results, we introduce the more general concept of *weighted $f-$Gini mean difference* for convex and symmetric functions in linear spaces. Moreover, we provide some fundamental inequalities for the new quantity and apply them for norms that naturally extend to vectors the results obtained for real numbers.

2. Weighted $f-$Gini Mean Difference

Consider $f : X \to \mathbb{R}$ be a convex function on the linear space X. Assume that $f(0) = 0$ and f is symmetric, i.e., $f(x) = f(-x)$ for any $x \in X$. In these circumstances we have

$$f(x) = \frac{f(x) + f(-x)}{2} \geq f\left(\frac{x - x}{2}\right) = f(0) = 0$$

showing that f is nonnegative on the entire space X.

For $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we define the *weighted $f-$Gini mean difference* of the n-tuple $\mathbf{x}$ with the probability distribution $\mathbf{p}$ the positive quantity

$$G_f(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j f(x_i - x_j) = \sum_{1 \leq i < j \leq n}^{n} p_i p_j f(x_i - x_j) \geq 0. \tag{2.1}$$

For the uniform distribution $\mathbf{u} = \left(\frac{1}{n}, ..., \frac{1}{n}\right) \in \mathbf{P}^n$ we have the $f-$*Gini mean difference* defined by

$$G_f(\mathbf{x}) := \frac{1}{2n^2} \sum_{i,j=1}^{n} f(x_i - x_j) = \frac{1}{n^2} \sum_{1 \leq i < j \leq n}^{n} f(x_i - x_j).$$

A natural example of such $f-$*Gini mean difference* can be provided by the convex function $f(x) := \|x\|^r$ with $r \geq 1$ defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$G_r(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r = \sum_{1 \leq i < j \leq n}^{n} p_i p_j \|x_i - x_j\|^r.$$

Further on, we need to consider another quantity that is naturally related with $f-$*Gini mean difference.* For a convex function $f : X \to \mathbb{R}$ defined on the linear space X with the properties that $f(0) = 0$ define the *mean f-deviation* of an n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ by the non-negative quantity

$$K_f(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^{n} p_i f\left(x_i - \sum_{k=1}^{n} p_k x_k\right). \tag{2.2}$$

The fact that $K_f(\mathbf{p}, \mathbf{x})$ is non-negative follows by Jensen's inequality, namely

$$K_f(\mathbf{p}, \mathbf{x}) \geq f\left(\sum_{i=1}^{n} p_i \left(x_i - \sum_{k=1}^{n} p_k x_k\right)\right) = f(0) = 0.$$

A natural example of such deviations can be provided by the convex function $f(x) := \|x\|^r$ with $r \geq 1$ defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$K_r(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^{n} p_i \left\|x_i - \sum_{k=1}^{n} p_k x_k\right\|^r \tag{2.3}$$

and call it the *mean r-absolute deviation* of the n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

The following connection between the $f-$Gini mean difference and the mean f-deviation holds true:

Theorem 2. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequalities*

$$G_f(\mathbf{p},\mathbf{x}) \geq \frac{1}{2} K_f(\mathbf{p},\mathbf{x}) \geq G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right). \tag{2.4}$$

Both inequalities in (2.4) are sharp and the constant $\frac{1}{2}$ best possible.

Proof. By the Jensen inequality we have

$$\begin{aligned} G_f(\mathbf{p},\mathbf{x}) &= \frac{1}{2}\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{n} p_j f(x_i - x_j)\right) \\ &\geq \frac{1}{2}\sum_{i=1}^{n} p_i f\left(\sum_{j=1}^{n} p_j (x_i - x_j)\right) = \frac{1}{2}\sum_{i=1}^{n} p_i f\left(x_i - \sum_{j=1}^{n} p_j x_j\right) \\ &= \frac{1}{2} K_f(\mathbf{p},\mathbf{x}) \end{aligned}$$

which proves the first part of (2.4).

By the convexity and symmetry of f we also have

$$\begin{aligned} f\left(\frac{1}{2}x_i - \frac{1}{2}x_j\right) &= f\left(\frac{1}{2}x_i - \frac{1}{2}\sum_{k=1}^{n} p_k x_k + \frac{1}{2}\sum_{k=1}^{n} p_k x_k - \frac{1}{2}x_j\right) \\ &= f\left[\frac{1}{2}\left(x_i - \sum_{k=1}^{n} p_k x_k\right) + \frac{1}{2}\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\right] \\ &\leq \frac{1}{2}\left[f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) + f\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\right] \end{aligned} \tag{2.5}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying the inequality (2.5) by $p_i p_j$ and summing over i and j from 1 to n we get

$$\begin{aligned} 2G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right) &\leq \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left[f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) + f\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\right] \\ &= K_f(\mathbf{p},\mathbf{x}) \end{aligned}$$

which proves the last part of (2.4).

Now, if assume that $(X, \|\cdot\|)$ is a normed space and consider $f(x) = \|x\|$, then for $n = 2$ and $p_1, p_2 \in [0,1]$ with $p_1 + p_2 = 1$ we have

$$G_1(\mathbf{p},\mathbf{x}) = p_1 p_2 \|x_1 - x_2\| \text{ and } K_1(\mathbf{p},\mathbf{x}) = 2p_1 p_2 \|x_1 - x_2\|$$

which shows that the equality can be realized in (2.4) for a nonzero quantity when $x_1 \neq x_2$ and $p_1, p_2 \in (0,1)$. □

The following particular case for norms is of interest due to its natural generalization for the scalar case that is used in applications:

Corollary 1. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have*

$$G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} K_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2^r} G_r(\mathbf{p}, \mathbf{x}) \tag{2.6}$$

or, equivalently,

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^r \geq \frac{1}{2^{r-1}} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \tag{2.7}$$

for any $r \geq 1$.

3. A Lower Bound for $G_f(\mathbf{p}, \mathbf{x})$

As in the introduction, consider $\Delta := \{(i,j) \,| i, j \in \{1, \ldots, n\}\}$ and denote by Γ a nonempty subset of Δ, namely $\Gamma = I \times J$ where I and J are nonempty parts of $\{1, \ldots, n\}$. We denote by $\mathcal{G}_n$ the set of all nonempty Γ as above.

We define, for a given probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ and $\Gamma = I \times J$ the following quantities

$$P_\Gamma := \sum_{(i,j)\in\Gamma} p_i p_j = \sum_{i\in I} p_i \sum_{j \in J} p_j = P_I P_J, \text{ and } \bar{P}_\Gamma := P_{\bar{\Gamma}} \text{ where } \bar{\Gamma} := \Delta \setminus \Gamma.$$

Since $P_\Delta = \sum_{(i,j)\in\Delta} p_i p_j = \sum_{i,j=1}^{n} p_i p_j = 1$, then obviously $P_\Gamma \in [0,1]$ and $\bar{P}_\Gamma = 1 - P_\Gamma = 1 - P_I P_J$.

We can state the following result:

Theorem 3. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ with all terms nonzero we have the inequality*

$$G_f(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2} \max_{\mathbf{I}\times\mathbf{J}\in\mathcal{G}_n} L_f(\mathbf{p}, \mathbf{x}, \mathbf{I} \times \mathbf{J}) \, (\geq 0) \tag{3.1}$$

where

$$L_f(\mathbf{p}, \mathbf{x}, \mathbf{I} \times \mathbf{J}) := P_I P_J f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i - \frac{1}{P_J}\sum_{j\in J} p_j x_j\right) + (1 - P_I P_J) f\left[\frac{P_I P_J}{1 - P_I P_J}\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i - \frac{1}{P_J}\sum_{j\in J} p_j x_j\right)\right]. \tag{3.2}$$

Proof. On applying Jensen's inequality for multiple sums we have for $\Gamma = I \times J$ that

$$G_f(\mathbf{p},\mathbf{x}) = \frac{1}{2}\sum_{(i,j)\in\Delta} p_i p_j f(x_i - x_j) \tag{3.3}$$
$$= \frac{1}{2}\sum_{(i,j)\in\Gamma} p_i p_j f(x_i - x_j) + \frac{1}{2}\sum_{(i,j)\in\bar{\Gamma}} p_i p_j f(x_i - x_j)$$
$$\geq \frac{1}{2}\sum_{(i,j)\in\Gamma} p_i p_j f\left[\frac{\sum_{(i,j)\in\Gamma} p_i p_j (x_i - x_j)}{\sum_{(i,j)\in\Gamma} p_i p_j}\right]$$
$$+ \frac{1}{2}\sum_{(i,j)\in\bar{\Gamma}} p_i p_j f\left[\frac{\sum_{(i,j)\in\bar{\Gamma}} p_i p_j (x_i - x_j)}{\sum_{(i,j)\in\bar{\Gamma}} p_i p_j}\right].$$

However

$$\sum_{(i,j)\in\Gamma} p_i p_j = P_I P_J, \quad \sum_{(i,j)\in\bar{\Gamma}} p_i p_j = 1 - P_I P_J$$

$$\sum_{(i,j)\in\Gamma} p_i p_j (x_i - x_j) = \sum_{(i,j)\in I\times J} p_i p_j (x_i - x_j) = P_J \sum_{i\in I} p_i x_i - P_I \sum_{j\in J} p_j x_j$$

and

$$\sum_{(i,j)\in\bar{\Gamma}} p_i p_j (x_i - x_j) = \sum_{(i,j)\in\Delta\setminus\Gamma} p_i p_j (x_i - x_j)$$
$$= \sum_{(i,j)\in\Delta} p_i p_j (x_i - x_j) - \sum_{(i,j)\in\Gamma} p_i p_j (x_i - x_j)$$
$$= -\sum_{(i,j)\in\Gamma} p_i p_j (x_i - x_j) = -\left(P_J \sum_{i\in I} p_i x_i - P_I \sum_{j\in J} p_j x_j\right).$$

Utilising the inequality (3.3) and the symmetry of the function f we get

$$G_f(\mathbf{p},\mathbf{x}) \geq \frac{1}{2} P_I P_J f\left(\frac{P_J \sum_{i\in I} p_i x_i - P_I \sum_{j\in J} p_j x_j}{P_I P_J}\right)$$
$$+ \frac{1}{2}(1 - P_I P_J) f\left(\frac{P_J \sum_{i\in I} p_i x_i - P_I \sum_{j\in J} p_j x_j}{1 - P_I P_J}\right).$$

which is equivalent with the desired inequality (3.2). □

A particular case of interest is for $I = \{i\}$ and $J = \{j\}$ giving the following

Corollary 2. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then we have the inequality*

$$G_f(\mathbf{p},\mathbf{x}) \geq \frac{1}{2}\max_{(i,j)\in\Delta} L_f(\mathbf{p},\mathbf{x},i,j) \ (\geq 0) \tag{3.4}$$

where

$$L_f(\mathbf{p},\mathbf{x},i,j) := p_i p_j f(x_i - x_j) + (1 - p_i p_j) f\left[\frac{p_i p_j}{1 - p_i p_j}(x_i - x_j)\right]. \tag{3.5}$$

Remark 3. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ with all terms nonzero we have from (3.1) that*

$$G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2}\left[P_I P_J + (1 - P_I P_J)^{1-r} P_I^r P_J^r\right] \left\| \frac{1}{P_I} \sum_{i \in I} p_i x_i - \frac{1}{P_J} \sum_{j \in J} p_j x_j \right\|^r \tag{3.6}$$

for any $I \times J \in \mathcal{G}_n$.

In particular, we have

$$G_r(\mathbf{p}, \mathbf{x}) \geq \frac{1}{2}\left[p_i p_j + (1 - p_i p_j)^{1-r} p_i^r p_j^r\right] \|x_i - x_j\|^r \tag{3.7}$$

for any $(i, j) \in \Delta$, which extends and improves the scalar case obtained in [2].

4. Another Lower Bound

Before we provide another lower bound for the weighted $f-$Gini mean difference we need to introduce some notations.

For the vector $\mathbf{e} = (1, 2, ..., n) \in \mathbb{R}^n$ and the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we define the weighted arithmetic mean

$$A_j(\mathbf{p}, \mathbf{e}) := \sum_{k=1}^{j} k p_k$$

and, similarly, for the n-tuple $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we define

$$A_j(\mathbf{p}, \mathbf{x}) := \sum_{k=1}^{j} p_k x_k.$$

Theorem 4. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequality*

$$G_f(\mathbf{p}, \mathbf{x}) \geq G(\mathbf{p}, \mathbf{e}) f\left(\frac{G(\mathbf{p}, \mathbf{x})}{G(\mathbf{p}, \mathbf{e})}\right) (\geq 0) \tag{4.1}$$

where

$$G(\mathbf{p}, \mathbf{e}) := \sum_{j=1}^{n} p_j \left[j P_j - A_j(\mathbf{p}, \mathbf{e})\right] > 0$$

and

$$G(\mathbf{p}, \mathbf{x}) := \sum_{j=1}^{n} p_j \left[P_j x_j - A_j(\mathbf{p}, \mathbf{x})\right].$$

Proof. By the convexity of f and the fact that $f(0) = 0$ we have that

$$f(tx) = f[(1-t) \cdot 0 + t \cdot x] \leq (1-t) f(0) + t f(x) = t f(x)$$

for any $t \in [0, 1]$ and $x \in X$.

Now, if $1 \leq i < j \leq n$ then $t := \frac{1}{j-i} \in [0, 1]$ and writing the above inequality for this t and for $x = x_j - x_i$ we get

$$f(x_j - x_i) \geq (j - i) f\left(\frac{x_j - x_i}{j - i}\right) \tag{4.2}$$

for any $1 \le i < j \le n$.

Multiplying (4.2) with $p_i p_j \ge 0$, summing over i,j with $1 \le i < j \le n$ and applying the Jensen inequality for multiple sums we get successively

$$G_f(\mathbf{p},\mathbf{x}) = \sum_{1\le i<j\le n}^{n} p_i p_j f(x_i - x_j) \ge \sum_{1\le i<j\le n}^{n} p_i p_j (j-i) f\left(\frac{x_i - x_j}{j-i}\right) \tag{4.3}$$
$$\ge \sum_{1\le i<j\le n}^{n} p_i p_j (j-i) f\left[\sum_{1\le i<j\le n}^{n} p_i p_j (j-i)\left(\frac{x_i - x_j}{j-i}\right)\right]$$
$$= \sum_{1\le i<j\le n}^{n} p_i p_j (j-i) f\left[\sum_{1\le i<j\le n}^{n} p_i p_j (x_i - x_j)\right].$$

However

$$(0<)\sum_{1\le i<j\le n}^{n} p_i p_j (j-i) = \sum_{j=1}^{n} p_j \sum_{i=1}^{j} p_i (j-i)$$
$$= \sum_{j=1}^{n} p_j \left(P_j - \sum_{i=1}^{j} i p_i\right) = G(\mathbf{p},\mathbf{e})$$

and

$$\sum_{1\le i<j\le n}^{n} p_i p_j (x_i - x_j) = \sum_{j=1}^{n} p_j \sum_{i=1}^{j} p_i (x_j - x_i)$$
$$= \sum_{j=1}^{n} p_j \left(P_j x_j - \sum_{i=1}^{j} i x_i\right) = G(\mathbf{p},\mathbf{x})$$

which together with (4.3) produces the desired result (4.1). □

The case of normed linear spaces is of interest. We have:

Corollary 3. *Let $(X,\|\cdot\|)$ be a normed space. Then for any n-tuple of vectors $\mathbf{x} = (x_1,...,x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1,...,p_n) \in \mathbf{P}^n$ we have*

$$G_r(\mathbf{p},\mathbf{x}) \ge G^{1-r}(\mathbf{p},\mathbf{e}) \|G(\mathbf{p},\mathbf{x})\|^r \ (\ge 0) \tag{4.4}$$

or, equivalently,

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \tag{4.5}$$
$$\ge \left(\sum_{j=1}^{n} p_j [jP_j - A_j(\mathbf{p},\mathbf{e})]\right)^{1-r} \left\|\sum_{j=1}^{n} p_j [P_j x_j - A_j(\mathbf{p},\mathbf{x})]\right\|^r$$

for any $r \ge 1$.

5. Some Upper Bounds

From the definition of the weighted $f-$Gini mean difference we have

$$G_f(\mathbf{p},\mathbf{x}) = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j f(x_i - x_j) \leq \max_{1\leq i<j\leq n} f(x_i - x_j)\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j = \frac{1}{2}\max_{1\leq i<j\leq n} f(x_i - x_j) \tag{5.1}$$

and

$$G_f(\mathbf{p},\mathbf{x}) = \sum_{1\leq i<j\leq n} p_i p_j f(x_i - x_j) \leq \max_{1\leq i<j\leq n} f(x_i - x_j)\sum_{1\leq i<j\leq n} p_i p_j = \frac{1}{2}\max_{1\leq i<j\leq n} f(x_i - x_j)\left(1-\sum_{i=1}^{n} p_i^2\right) = \frac{1}{2}\max_{1\leq i<j\leq n} f(x_i - x_j)\sum_{i=1}^{n} p_i(1-p_i) \tag{5.2}$$

since, obviously,

$$\sum_{1\leq i<j\leq n} p_i p_j = \frac{1}{2}\left(\sum_{i,j=1}^{n} p_i p_j - \sum_{i=1}^{n} p_i^2\right) = \frac{1}{2}\left(1-\sum_{i=1}^{n} p_i^2\right) = \frac{1}{2}\sum_{i=1}^{n} p_i(1-p_i).$$

Observe that the second approach provides a better inequality, therefore we can state the following result:

Proposition 1. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequality*

$$\frac{1}{2}\max_{1\leq i<j\leq n} f(x_i - x_j)\sum_{i=1}^{n} p_i(1-p_i) \geq G_f(\mathbf{p},\mathbf{x}). \tag{5.3}$$

The following particular case for norms holds true:

Corollary 4. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have*

$$\frac{1}{2}\max_{1\leq i<j\leq n} \|x_i - x_j\|^r \sum_{i=1}^{n} p_i(1-p_i) \geq G_r(\mathbf{p},\mathbf{x})$$

for any $r \geq 1$.

For two vectors $x, y \in X$ we define the segment $[x, y]$ by $\{(1-t)x + ty, t \in [0,1]\}$. If $0 \in [x, y]$, then there exists a unique t with $t \in [0,1]$ such that $(1-t)x + ty = 0$.

The following result may be stated as well:

Theorem 5. *Assume that $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If x and y are two vectors and $t \in [0,1]$ with $(1-t)x + ty = 0$ then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the property that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, ..., n\}$ we have the inequality*

$$\frac{1}{2}[(1-t)f(x) + tf(y)] \geq G_f(\mathbf{p},\mathbf{x}) \tag{5.4}$$

for any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

Proof. Since $x_i - x_j \in [x, y]$ for $i, j \in \{1, ..., n\}$, then there exists the numbers $t_{ij} \in [0, 1]$ such that $x_i - x_j = (1 - t_{ij}) x + t_{ij} y$ for $i, j \in \{1, ..., n\}$.

Let $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$. Then by the above equality we get that

$$p_i p_j (x_i - x_j) = (1 - t_{ij}) p_i p_j x + t_{ij} p_i p_j y$$

for any $i, j \in \{1, ..., n\}$. If we sum over i, j from 1 to n, then we get

$$\begin{aligned} 0 = \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) &= \sum_{i,j=1}^{n} [(1 - t_{ij}) p_i p_j x + t_{ij} p_i p_j y] \\ &= \left(1 - \sum_{i,j=1}^{n} t_{ij} p_i p_j\right) x + \left(\sum_{i,j=1}^{n} t_{ij} p_i p_j\right) y. \end{aligned} \tag{5.5}$$

Now, due to the fact that $(1 - t) x + ty = 0$ and the representation is unique, we get that $t = \sum_{i,j=1}^{n} t_{ij} p_i p_j$.

On the other hand, due to the convexity of the function f we have that

$$\begin{aligned} G_f(\mathbf{p}, \mathbf{x}) &= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j f(x_i - x_j) \le \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j f[(1 - t_{ij}) x + t_{ij} y] \\ &\le \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j [(1 - t_{ij}) f(x) + t_{ij} f(y)] \\ &= \frac{1}{2} \left[\left(1 - \sum_{i,j=1}^{n} t_{ij} p_i p_j\right) f(x) + \left(\sum_{i,j=1}^{n} t_{ij} p_i p_j\right) f(y)\right] \\ &= \frac{1}{2} [(1 - t) f(x) + t f(y)] \end{aligned}$$

and the theorem is proved. □

In applications one may be able to provide the "smallest" symmetric interval $[-z, z]$ containing all the differences $x_i - x_j$. In that situation we can state the following particular case of interest:

Corollary 5. *Assume that $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If z is a nonzero vector in X then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the property that $x_i - x_j \in [-z, z]$ for all $i, j \in \{1, ..., n\}$ we have the inequality*

$$\frac{1}{2} f(z) \ge G_f(\mathbf{p}, \mathbf{x}) \tag{5.6}$$

for any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

Remark 4. *If X is a normed linear space and $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and z satisfy the condition from Corollary 5, then we have the inequality*

$$\frac{1}{2} \|z\|^r \ge G_r(\mathbf{p}, \mathbf{x}) \tag{5.7}$$

for each $r \ge 1$.

For an n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we consider the condition

$$x_i - \sum_{j=1}^{n} p_j x_j \in [x, y] \text{ for any } i \in \{1, ..., n\}. \tag{5.8}$$

Since the segment $[x, y]$ is a convex set then $0 = \sum p_j \left(x_i - \sum_{j=1}^{n} p_j x_j \right) \in [x, y]$. Moreover, the fact that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, ..., n\}$ also imply that the condition (5.8) holds true.

We can state the following result that provides an upper bound for the mean f-deviation $K_f(\mathbf{p}, \mathbf{x})$:

Theorem 6. *Let $f : X \to \mathbb{R}$ be a convex function defined on the linear space X with the properties that $f(0) = 0$. If, for an n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the condition (5.8), then*

$$(1 - t) f(x) + t f(y) \geq K_f(\mathbf{p}, \mathbf{x}), \tag{5.9}$$

where t is the unique real number for which we have $(1 - t) x + ty = 0$.

The argument is similar to that in the proof of Theorem 5 and the details are omitted.

References

[1] P. CERONE and S.S. DRAGOMIR, Bounds for the Gini mean difference of an empirical distribution, *Applied Math. Letters,* **19** (2006), 283-293.

[2] P. CERONE and S.S. DRAGOMIR, Bounds for the $r-$weighted Gini mean difference of an empirical distribution,

[3] S.S. DRAGOMIR, *Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type*, Nova Science Publishers, N.Y. 2004.

[4] G.M. GIORGI, Bibliographic portrait of the Grüss concentration ratio, *Metron,* **48** (1990), 183-221.

[5] G.M. GIORGI, Il rapperto di concentrazionne di Gini, Liberia Editrice Ticci, Sienna, 1992.

[6] S. IZUMINO and E. PEČARIĆ, Some extensions of Grüss' inequality and its applications, *Nihonkai Math. J.,* **13** (2002), 159-166.

[7] G.A. KOSHEVOY and K. MOSLER, Multivariate Gini indices, *J. Multivariate Analysis*, **60** (1997), 252-276.

Research Group in Mathematical Inequalities & Applications, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://www.staff.vu.edu.au/rgmia/dragomir/