Several proofs and generalizations of a fractional inequality with constraints

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Abstract: Ten different proofs are given for a fractional inequality with constraints. Finally, two generalized forms are established by introducing exponent parameters and additive terms.

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1 Introduction

The 2nd problem given at the 36th IMO held at Toronto (Canada) in 1995 was:

Problem 1. Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}. \tag{1}$$

In this paper, we show several different proofs and generalized forms of the inequality (1).

2 Several proofs for the inequality (1)

Proof 1. It follows from the condition $abc = 1$ that

$$\frac{1}{a^3(b+c)} = \frac{b^2c^2}{a(b+c)},\quad \frac{1}{b^3(c+a)} = \frac{c^2a^2}{b(c+a)},\quad \frac{1}{c^3(a+b)} = \frac{a^2b^2}{c(a+b)}.$$

Now, the inequality (1) is equivalent to

$$\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} \geq \frac{3}{2}.$$

By Cauchy-Schwarz inequality (see [1]), we have

$$\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^3\right)\left(\frac{b^2c^2}{\lambda_1^2} + \frac{c^2a^2}{\lambda_2^2} + \frac{a^2b^2}{\lambda_3^2}\right) \geq (bc + ca + ab)^2.$$

Using a substitution

$$\lambda_1^2 = a(b+c),\quad \lambda_2^2 = b(c+a),\quad \lambda_3^2 = c(a+b)$$

in the above inequality, and applying the arithmetic-geometric means inequality, we obtain

$$\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} \geq \frac{1}{2}(bc + ca + ab) \geq \frac{3}{2}\sqrt[3]{(abc)^2} = \frac{3}{2}.$$

Proof 2. We note that the inequality (1) is equivalent to

$$\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ba}+\frac{a^2b^2}{ca+cb}\geq\frac{3}{2}.$$

Let

$$K=\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ba}+\frac{a^2b^2}{ca+cb}.$$

Using Cauchy-Schwarz inequality and arithmetic-geometric means inequality gives

$$\begin{aligned}
&[(ab+ac)+(bc+ba)+(ca+cb)]\,K\\
\geq\ &\left(\sqrt{ab+ac}\cdot\frac{bc}{\sqrt{ab+ac}}+\sqrt{bc+ba}\cdot\frac{ca}{\sqrt{bc+ba}}+\sqrt{ca+cb}\cdot\frac{ab}{\sqrt{ca+cb}}\right)^2\\
=\ &(bc+ca+ab)^2\\
\geq\ &3(bc+ca+ab)\sqrt[3]{(bc)(ca)(ab)}\\
=\ &3(bc+ca+ab).
\end{aligned}$$

Hence $K\geq\frac{3}{2}$. The desired conclusion follows.

Proof 3. Note that for $a>0$,

$$a+\frac{1}{a}\geq 2\Longleftrightarrow a\geq 2-\frac{1}{a}.$$

We thus have

$$\frac{1}{a^3(b+c)}=\frac{1}{2a}\left[\frac{2}{a^2(b+c)}\right]\geq\frac{1}{2a}\left(2-\frac{a^2(b+c)}{2}\right)=\frac{1}{a}-\frac{ab+ac}{4}.$$

Similarly

$$\frac{1}{b^3(c+a)}\geq\frac{1}{b}-\frac{bc+ba}{4},$$

$$\frac{1}{c^3(a+b)}\geq\frac{1}{c}-\frac{ca+cb}{4}.$$

Adding the above inequalities yields

$$\begin{aligned}
&\frac{1}{a^3\,(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\\
&\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{2}(ab+bc+ca)\\
&=\frac{1}{2}(ab+bc+ca).
\end{aligned}$$

Finally, the arithmetic-geometric means inequality leads us to the required inequality.

Proof 4. The inequality (1) is equivalent to

$$\frac{b^2c^2}{a(b+c)}+\frac{c^2a^2}{b(c+a)}+\frac{a^2b^2}{c(a+b)}\geq\frac{3}{2}.$$

On the other hand, we have for $\lambda>0$,

$$
\begin{aligned}
\frac{b^2c^2}{a(b+c)} + \lambda a(b+c) &\geq 2\sqrt{\lambda} bc,\\
\frac{c^2a^2}{b(c+a)} + \lambda b(c+a) &\geq 2\sqrt{\lambda} ca,\\
\frac{a^2b^2}{c(a+b)} + \lambda c(a+b) &\geq 2\sqrt{\lambda} ab.
\end{aligned}
$$

Adding the above inequalities yields

$$
\begin{aligned}
\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} &\geq (2\sqrt{\lambda} - 2\lambda)(ab+bc+ca)\\
&\geq 6(\sqrt{\lambda} - \lambda)\sqrt[3]{(abc)^2}\\
&= 6(\sqrt{\lambda} - \lambda).
\end{aligned}
$$

Choosing $\lambda = \frac{1}{4}$ gives

$$
\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} \geq \frac{3}{2},
$$

which is the required inequality.

Proof 5. We make the substitution $bc = x,\ ca = y,\ ab = z,\ x+y+z = s.$ Then

$$
\begin{aligned}
\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} &= \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\\
&= \frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}.
\end{aligned}
$$

We consider the probability distribution sequence of random variable ξ below:

$$
p\left(\xi = \frac{x}{s-x}\right) = \frac{s-x}{2s}, \quad p\left(\xi = \frac{y}{s-y}\right) = \frac{s-y}{2s}, \quad p\left(\xi = \frac{z}{s-z}\right) = \frac{s-z}{2s}.
$$

It follows that

$$
E\xi = \frac{x}{s-x}\cdot\frac{s-x}{2s} + \frac{y}{s-y}\cdot\frac{s-y}{2s} + \frac{z}{s-z}\cdot\frac{s-z}{2s} = \frac{x+y+z}{2s} = \frac{1}{2},
$$

$$
E\xi^2 = \left(\frac{x}{s-x}\right)^2 \frac{s-x}{2s} + (\ -z) = \frac{1}{2s}\left(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\right).
$$

According to $D(\xi) = E\xi^2 - (E\xi)^2 > 0$, we have

$$
\frac{1}{2s}\left(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\right) \geq \frac{1}{4},
$$

so

$$
\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z} \geq \frac{1}{2}s = \frac{1}{2}(x+y+z) \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}.
$$

Hence

$$
\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.
$$

Proof 6. Let $bc = x,\ ca = y,\ ab = z$. The inequality (1) is equivalent to

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

By symmetry, we may assume that $x \geq y \geq z$, then

$$\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}.$$

Using the rearrangement inequality (see [2]) gives

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq z \cdot \frac{x}{y+z} + x \cdot \frac{y}{x+z} + y \cdot \frac{z}{x+y},$$

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq y \cdot \frac{x}{y+z} + z \cdot \frac{y}{z+x} + x \cdot \frac{z}{x+y},$$

Adding the above inequalities yields

$$2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq x + y + z \geq 3\sqrt[3]{xyz} = 3,$$

The required inequality follows.

Proof 7. Apply the same substitution as in Proof 6. The inequality (1) is equivalent to

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Since (x^2, y^2, z^2) and $(\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y})$ are similarly sorted sequences, it follows from the rearrangement inequality that

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{1}{2}\left(\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y}\right).$$

By the power mean inequality, we have

$$\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y} \geq \frac{y+z}{2} + \frac{z+x}{2} + \frac{x+y}{2} \geq 6\left(\frac{y}{2} \cdot \frac{z}{2} \cdot \frac{z}{2} \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot \frac{y}{2}\right)^{\frac{1}{6}} = 3,$$

this yields

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Proof 8. Let $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = t\ (t > 0)$, then

$$\begin{aligned} &\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \\ =& \frac{a^{-2}}{b^{-1}+c^{-1}} + \frac{b^{-2}}{c^{-1}+a^{-1}} + \frac{c^{-2}}{a^{-1}+b^{-1}} \\ =& \frac{a^{-2}}{t-a^{-1}} + \frac{b^{-2}}{t-b^{-1}} + \frac{c^{-2}}{t-c^{-1}}. \end{aligned}$$

Consider the following function:

$$\mathrm{g}(x) = \frac{x^2}{t-x} \quad (0 < x < t).$$

Since

$$g''(x) = \frac{2t^2}{(t-x)^3} > 0 \quad (0 < x < t),$$

we conclude that the function g is convex on $(0,\ t)$.
Using Jensen's inequality gives

$$\begin{aligned}
g(a^{-1}) + g(b^{-1}) + g(c^{-1}) &\geq 3g(\frac{a^{-1}+b^{-1}+c^{-1}}{3}) \\
&= \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\
&\geq \frac{3}{2}\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}} \\
&= \frac{3}{2}.
\end{aligned}$$

Proof 9. Consider the following function:

$$\begin{aligned}
f(x) &= \left(\frac{bc}{\sqrt{ab+ac}}x - \sqrt{ab+ac}\right)^2 + \left(\frac{ca}{\sqrt{bc+ab}}x - \sqrt{bc+ab}\right)^2 + \left(\frac{ab}{\sqrt{ac+bc}}x - \sqrt{ac+bc}\right)^2 \\
&= \left(\frac{b^2c^2}{ab+ac} + \frac{c^2a^2}{bc+ab} + \frac{a^2b^2}{ac+bc}\right)x^2 - 2(ab+bc+ca)x + 2(ab+bc+ca).
\end{aligned}$$

Since $f(x) \geq 0$ for $x \in \mathbb{R}$, we have the discriminant $\Delta \leq 0$, that is

$$4(ab+bc+ac)^2 - 8\left(\frac{b^2c^2}{ab+ac} + \frac{c^2a^2}{bc+ab} + \frac{a^2b^2}{ac+bc}\right)(ab+bc+ac) \leq 0.$$

Thus

$$\frac{b^2c^2}{ab+ac} + \frac{c^2a^2}{bc+ab} + \frac{a^2b^2}{ac+bc} \geq \frac{1}{2}(ab+bc+ac) \geq \frac{3}{2}\sqrt[3]{a^2b^2c^2} = \frac{3}{2},$$

which leads to

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Proof 10. Construct the following vectors:

$$\begin{aligned}
\overrightarrow{OA} &= \left(\sqrt{a(b+c)},\ \sqrt{b(c+a)},\ \sqrt{c(a+b)}\right), \\
\overrightarrow{OB} &= \left(\frac{bc}{\sqrt{a(b+c)}},\ \frac{ca}{\sqrt{b(c+a)}},\ \frac{ab}{\sqrt{c(a+b)}}\right).
\end{aligned}$$

We denote by $\theta\,(0 \leq \theta \leq \pi)$ the angle of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
Since

$$\left|\overrightarrow{OA}\right| = \sqrt{2(ab+bc+ca)},$$

$$\left|\overrightarrow{OB}\right| = \sqrt{\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)}},$$

we have

$$\begin{aligned}\overrightarrow{OA}\cdot\overrightarrow{OB} &= \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\theta\\
&= \sqrt{2\left(ab+bc+ca\right)}\sqrt{\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)}}\cos\theta\\
&\le \sqrt{2\left(ab+bc+ca\right)}\sqrt{\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)}}.\end{aligned}$$

On the other hand, we have

$$\overrightarrow{OA}\cdot\overrightarrow{OB} = ab+bc+ca.$$

Thus

$$\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)} \ge \frac{1}{2}\left(ab+bc+ca\right) \ge \frac{3}{2}\sqrt[3]{a^2b^2c^2} = \frac{3}{2},$$

which leads us to the inequality (1).

3 Generalizations of the inequality (1)

Theorem 1. *Let a, b, c be positive real numbers such that $abc = 1$, and let $\lambda \ge 2$. Then*

$$\frac{1}{a^{\lambda}(b+c)}+\frac{1}{b^{\lambda}(c+a)}+\frac{1}{c^{\lambda}(a+b)} \ge \frac{3}{2}. \tag{2}$$

Proof. Let $a = \frac{1}{x},\ b = \frac{1}{y},\ c = \frac{1}{z}$. Then

$$\frac{1}{a^{\lambda}(b+c)}+\frac{1}{b^{\lambda}(c+a)}+\frac{1}{c^{\lambda}(a+b)} = \frac{x^{\lambda-1}}{y+z}+\frac{y^{\lambda-1}}{z+x}+\frac{z^{\lambda-1}}{x+y},$$

By symmetry, we may assume that $x \ge y \ge z$, then

$$x^{\lambda-2} \ge y^{\lambda-2} \ge z^{\lambda-2}, \frac{1}{y+z} \ge \frac{1}{z+x} \ge \frac{1}{x+y}$$

and

$$\frac{x^{\lambda-2}}{y+z} \ge \frac{y^{\lambda-2}}{z+x} \ge \frac{z^{\lambda-2}}{x+y}.$$

Using the rearrangement inequality gives

$$\frac{x^{\lambda-1}}{y+z}+\frac{y^{\lambda-1}}{z+x}+\frac{z^{\lambda-1}}{x+y} \ge z\cdot\frac{x^{\lambda-2}}{y+z}+x\cdot\frac{y^{\lambda-2}}{z+x}+y\cdot\frac{z^{\lambda-2}}{x+y},$$

$$\frac{x^{\lambda-1}}{y+z}+\frac{y^{\lambda-1}}{z+x}+\frac{z^{\lambda-1}}{x+y} \ge y\cdot\frac{x^{\lambda-2}}{y+z}+z\cdot\frac{y^{\lambda-2}}{z+x}+x\cdot\frac{z^{\lambda-2}}{x+y}.$$

Adding the above inequalities yields

$$\begin{aligned}\frac{x^{\lambda-1}}{y+z}+\frac{y^{\lambda-1}}{z+x}+\frac{z^{\lambda-1}}{x+y} &\ge \frac{1}{2}(x^{\lambda-2}+y^{\lambda-2}+z^{\lambda-2})\\
&\ge \frac{3}{2}\sqrt[3]{x^{\lambda-2}y^{\lambda-2}z^{\lambda-2}}\\
&= \frac{3}{2}.\end{aligned}$$

The inequality (2) is proved.

Theorem 2. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$, and let $n \geq 3,\ \lambda \geq 3$. Then*

$$\sum_{1\leq k<l\leq n} \frac{1}{(\prod\limits_{1\leq i\leq n,\ i\neq k,l} x_i)^{\lambda-1}(\sum\limits_{1\leq i<j\leq n,\ i\neq k,l} x_i x_j)} \geq \frac{n(n-1)}{(n+1)(n-2)}. \tag{3}$$

Proof. Applying the generalized Radon's inequality (see [3-6]):

$$\sum_{i=1}^{n} \frac{a_i^p}{b_i} \geq n^{2-p}(\sum_{i=1}^{n} a_i)^p/(\sum_{i=1}^{n} b_i)$$

$(a_i > 0,\ b_i > 0,\ i = 1, 2, \ldots, n,\ p \geq 2$ or $p \leq 0)$, we deduce that

$$\sum_{1\leq k<l\leq n} \frac{1}{(\prod\limits_{1\leq i\leq n,\ i\neq k,l} x_i)^{\lambda-1}(\sum\limits_{1\leq i<j\leq n,\ i\neq k,l} x_i x_j)}$$

$$= \sum_{1\leq k<l\leq n} \frac{1}{[(\prod\limits_{1\leq i\leq n} x_i)/x_k x_l)]^{\lambda-1}[(\sum\limits_{1\leq i<j\leq n} x_i x_j) - x_k x_l)]}$$

$$= \sum_{1\leq k<l\leq n} \frac{(x_k x_l)^{\lambda-1}}{(\sum\limits_{1\leq i<j\leq n} x_i x_j) - x_k x_l}$$

$$\geq \left[\frac{n(n-1)}{2}\right]^{3-\lambda} \frac{(\sum\limits_{1\leq k<l\leq n} x_k x_l)^{\lambda-1}}{\sum\limits_{1\leq k<l\leq n}[(\sum\limits_{1\leq i<j\leq n} x_i x_j) - x_k x_l)]}$$

$$= \left[\frac{n(n-1)}{2}\right]^{3-\lambda} \left[\frac{n(n-1)}{2} - 1\right]^{-1} (\sum_{1\leq k<l\leq n} x_k x_l)^{\lambda-2}$$

$$\geq \frac{2}{n^2-n-2} \left[\frac{n(n-1)}{2}\right]^{3-\lambda} \left[\frac{n(n-1)}{2}(\prod_{1\leq k\leq n} x_k)^{\frac{2}{n}}\right]^{\lambda-2}$$

$$= \frac{n(n-1)}{(n+1)(n-2)}.$$

This completes the proof of Theorem 2.

Remark. In a special case when $n = 3,\ \lambda = 3,\ x_1 = a,\ x_2 = b,\ x_3 = c$, the inequality (3) would reduce to the inequality (1).

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