JENSEN'S INEQUALITY FOR QUASICONVEX FUNCTIONS

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Abstract. Some inequalities of Jensen type and connected results are given for quasiconvex functions on convex sets in real linear spaces.
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1 Introduction

Throughout this paper X denotes a real linear space and $C \subseteq X$ a convex set, so that $x, y \in C$ with $\lambda \in [0,1]$ implies that $\lambda x + (1-\lambda)y \in C$.

Definition 1.1 A mapping $f : C \to I\!R$ is called *quasiconvex* on the convex set C if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \quad \text{for all } x, y \in C \text{ and } \lambda \in [0,1].$$

This class of functions strictly contains the class of convex functions defined on a convex set in a real linear space. See [8] and citations therein for an overview of this issue.

Some recent studies have shown that quasiconvex functions have quite close resemblances to convex functions – see, for example, [4], [6], [7], [10] for quasiconvex and even more general extensions of convex functions in the context of Hadamard's pair of inequalities. Apart from generalizations to theory, weakening the convexity condition can increase applicability. Thus in [9] use is made of quasiconvexity to obtain a global extremum with rather less effort than via convexity. In this article we pursue the concept further and derive a number of Jensen–type inequalities for quasiconvex functions. See also [5] for functions of Godunova–Levin type in the context of Jensen's inequality.

2 Preliminaries

For an arbitrary mapping $f : C \to I\!R$ and x, y two fixed elements in C, we can define the map $g_{x,y} : [0,1] \to I\!R$ by $g_{x,y}(t) = f(tx + (1-t)y)$. This provides a characterization of quasiconvexity.

Proposition 2.1 *The following statements are equivalent:*
(i) f *is quasiconvex on* C*;*
(ii) for every $x, y \in C$*, the mapping* $g_{x,y}$ *is quasiconvex on* $[0,1]$.

Proof. Suppose (i) holds. Let $t_1, t_2 \in [0,1]$ and $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$. Then

$$\begin{aligned} g_{x,y}\left(\sum_{i=1}^{2} \alpha_i t_i\right) &= f\left(\sum_{i=1}^{2} \alpha_i t_i x + \left[1 - \sum_{i=1}^{2} \alpha_i t_i\right] y\right) \\ &= f\left(\sum_{i=1}^{2} \alpha_i \left[t_i x + (1-t_i)y\right]\right) \leq \max_{i=1,2} \left[f\left(t_i x + (1-t_i)y\right)\right] = \max_{i=1,2} \left\{g_{x,y}(t_i)\right\}, \end{aligned}$$

which shows that the mapping $g_{x,y}$ is quasiconvex on $[0,1]$.

For the reverse implication, suppose (ii) holds. Then

$$f(tx + (1-t)y) = g_{x,y}(t) = g_{x,y}((1-t).0 + t.1) \leq \max\left\{g_{x,y}(0), g_{x,y}(1)\right\} = \max\left\{f(x), f(y)\right\},$$

which shows that f is quasiconvex on C. □

Proposition 2.2 *Suppose that ϕ_k is quasiconvex on $[0,1]$ for $k=1,\dots,n$. Then $\max_{1\le k\le n}\phi_k$ is quasiconvex on $[0,1]$.*

Proof. Let $t_1,t_2\in[0,1]$ and $\alpha_1,\alpha_2\ge 0$ with $\alpha_1+\alpha_2=1$. Put $\phi(t)=\max_{1\le k\le n}\phi_k(t)$. Then

$$\phi(\alpha_1t_1+\alpha_2t_2)=\max_{1\le k\le n}\phi_k(\alpha_1t_1+\alpha_2t_2)\le\max_{1\le k\le n}\max_{i=1,2}\phi_k(t_i)=\max_{i=1,2}\max_{1\le k\le n}\phi_k(t_i)=\max_{i=1,2}\phi(t_i),$$

establishing the quasiconvexity of ϕ. □

Lemma 2.3 *If ϕ is quasiconvex on $[0,1]$ and $\phi(t)=\phi(1-t)$ for all $t\in[0,1]$, then $\phi(t)\ge\phi(1/2)$ for all $t\in[0,1]$.*

Proof. From the given conditions, for each $t\in[0,1]$,

$$\phi(t)=\max\left[\phi(t),\phi(1-t)\right]\ge\phi\left((1/2)(t+(1-t))\right)=\phi(1/2).$$ □

For a given mapping $f:C\to I\!R$ we may also define a map $G_t:C^2\to I\!R$ by $G_t(x,y)=f(tx+(1-t)y)$ for fixed $t\in[0,1]$. Again we have a characterization of quasiconvexity.

Proposition 2.4 *We have the following:*
(i) if f is quasiconvex on C, then G_t is quasiconvex on C^2 for all $t\in[0,1]$;
(ii) if C is a cone in X and G_t is quasiconvex on C^2 $\forall t\in(0,1)$, then f is quasiconvex on C.

Proof. (i) Fix $t\in[0,1]$ and let $(x,y),(z,u)\in C^2$. Then for all $\lambda\in[0,1]$

$$\begin{aligned}G_t(\lambda(x,y)+(1-\lambda)(z,u))&=G_t(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)u)\\&=f(t[\lambda x+(1-\lambda)z]+(1-t)[\lambda y+(1-\lambda)u])\\&=f(\lambda(tx+(1-t)y)+(1-\lambda)(tz+(1-t)u))\\&\le\max\{f(tx+(1-t)y),f(tz+(1-t)u)\}=\max\{G_t(x,y),G_t(z,u)\},\end{aligned}$$

which shows that G_t is quasiconvex on C^2.

(ii) Let $x,y\in C$ and $t\in(0,1)$. If C is a cone in X, that is, $C+C\subseteq C$ and $\alpha C\subseteq C$ for all $\alpha\ge0$, then $t^{-1}x,\ (1-t)^{-1}y\in C$ and $(t^{-1}x,0),\ (0,(1-t)^{-1}y)\in C^2$. On the other hand, since G_t is quasiconvex on C^2, we have

$$\begin{aligned}f(tx+(1-t)y)&=G_t(x,y)\\&=G_t\left(t\left(t^{-1}x,0\right)+(1-t)\left(0,(1-t)^{-1}y\right)\right)\\&\le\max\left\{G_t\left(t^{-1}x,0\right),G_t\left(0,(1-t)^{-1}y\right)\right\}=\max\{f(x),f(y)\}\end{aligned}$$

for all $t\in(0,1)$. The inequality holds also for $t=0,1$, so the proposition is proved. □

3 Jensen's inequality

Hereafter $x_i\in C$ $(i=1,\dots,n)$. We assume $p_i>0$ $(1\le i\le n)$ and define $P_n=\sum_{i=1}^n p_i$.

Theorem 3.1 *If f is quasiconvex, then*

$$\begin{aligned}f\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)&\le\max\left\{f\left(\frac{1}{P_{n-1}}\sum_{i=1}^{n-1}p_ix_i\right),f(x_n)\right\}\\&\le\max\left\{f\left(\frac{1}{P_{n-2}}\sum_{i=1}^{n-2}p_ix_i\right),f(x_{n-1}),f(x_n)\right\}\\&\le\dots\le\max\left\{f\left(\frac{p_1x_2+p_2x_2}{p_1+p_2}\right),f(x_3),\dots,f(x_n)\right\}\le\max_{1\le i\le n}f(x_i).\end{aligned}$$

Proof. We employ induction on n. The case $n = 1$ provides a trivial basis. Assume that the stated inequality holds for $n = 1, \ldots, k$ ($k \geq 1$). By quasiconvexity and the inductive assumption

$$\begin{aligned} f\left(\frac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i x_i\right) &= f\left(\frac{P_k}{P_{k+1}} \cdot \frac{1}{P_k} \sum_{i=1}^{k} p_i x_i + \frac{p_{k+1}}{P_{k+1}} x_{k+1}\right) \\ &\leq \max\left\{ f\left(\frac{1}{P_k} \sum_{i=1}^{k} p_i x_i\right), f(x_{k+1})\right\} \\ &\leq \max\left\{ \max\left\{ f\left(\frac{1}{P_{k-1}} \sum_{i=1}^{k-1} p_i x_i\right), f(x_k)\right\}, f(x_{k+1})\right\} \\ &\leq \ldots \leq \max\left\{ \max_{1\leq i\leq k}\{f(x_i)\}, f(x_{k+1})\right\}. \end{aligned}$$

This may be written as the result of the theorem with $n = k+1$, giving the inductive step and so completing the proof. □

Corollary 3.2 *For f quasiconvex*

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \leq \min\left\{ \max\left\{ f\left(\frac{p_{i_1} x_i + \ldots + p_{i_{n-1}} x_{i_{n-1}}}{p_{i_1} + \ldots + p_{i_{n-1}}}\right), f(x_{i_n})\right\}\right\},$$

where the minimum is over all distinct $i_1, \ldots, i_n \in \{1, \ldots, n\}$.

In particular, we have the following for the unweighted case.

Corollary 3.3 *For f quasiconvex*

$$\begin{aligned} f\left(\frac{x_1 + \ldots + x_n}{n}\right) &\leq \max\left\{ f\left(\frac{x_1 + \ldots + x_{n-1}}{n-1}\right), f(x_n)\right\} \\ &\leq \max\left\{ f\left(\frac{x_1 + \ldots + x_{n-2}}{n-2}\right), f(x_{n-1}), f(x_n)\right\} \\ &\leq \ldots \\ &\leq \max\left\{ f\left(\frac{x_1 + x_2}{2}\right), f(x_3), \ldots, f(x_n)\right\} \leq \max\{f(x_1), \ldots, f(x_n)\} \end{aligned}$$

and

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \min\left\{ \max\left\{ f\left(\frac{x_{i_1} + \ldots + x_{i_{n-1}}}{n-1}\right), f(x_{i_n})\right\}\right\},$$

where the minimum is over the same domain as in the previous corollary.

We now consider the mapping η given by $\eta(I, \mathbf{p}, \mathbf{x}, f) = \max_{i\in I}\{f(x_i)\} - f\left(\frac{1}{P_I} \sum_{i\in I} p_i x_i\right)$. Here $I \in \mathcal{P}_f(I\!N)$, the collection of finite sets of natural numbers, $\mathbf{p} = (p_i)_{i\in I}$ with each $p_i > 0$ and $P_I := \sum_{i\in I} p_i$, and $\mathbf{x} = (x_i)_{i\in I}$ with each $x_i \in C$.

Theorem 3.4 *For f quasiconvex,*
(i) the mapping $\eta(I, \cdot, \mathbf{x}, f)$ is quasi-superadditive;
(ii) the mapping $\eta(\cdot, \mathbf{p}, \mathbf{x}, f)$ is quasi-superadditive as an index set mapping on $\mathcal{P}_f(I\!N)$.

Proof (i) Let $\mathbf{p}, \mathbf{q} > 0$ with $P_I, Q_I > 0$ ($I \in \mathcal{P}_f(I\!N)$). Then

$$\begin{aligned} \eta(I, \mathbf{p} + \mathbf{q}, \mathbf{x}, f) &= \max_{i\in I}\{f(x_i)\} - f\left(\frac{1}{P_I + Q_I} \sum_{i\in I} (p_i + q_i) x_i\right) \\ &= \max_{i\in I}\{f(x_i)\} - f\left(\frac{P_I}{P_I + Q_I} \cdot \frac{1}{P_I} \sum_{i\in I} p_i x_i + \frac{Q_I}{P_I + Q_I} \cdot \frac{1}{Q_I} \sum_{i\in I} q_i x_i\right) \\ &\geq \max_{i\in I}\{f(x_i)\} - \max\left\{ f\left(\frac{1}{P_I} \sum_{i\in I} p_i x_i\right), f\left(\frac{1}{Q_I} \sum_{i\in I} q_i x_i\right)\right\}. \end{aligned} \tag{1}$$

Since $\max\{a,b\} = (1/2)[a+b+|a-b|]$ for $(a,b \in I\!R)$, we have from the definition of η that the last maximum in (1) can be written as

$$\frac{1}{2}\left[2\max_{i\in I}\{f(x_i)\} - \eta(I,\mathbf{p},\mathbf{x},f) - \eta(I,\mathbf{q},\mathbf{x},f) - |\eta(I,\mathbf{p},\mathbf{x},f) - \eta(I,\mathbf{q},\mathbf{x},f)|\right].$$

Because $\min\{a,b\} = (1/2)[a+b-|a-b|]$ for $a,b \in I\!R$, we thus have $\eta(I,\mathbf{p}+\mathbf{q},\mathbf{x},f) \geq \min\{\eta(I,\mathbf{p},\mathbf{x},f),\eta(I,\mathbf{q},\mathbf{x},f)\}$, which establishes part (i).

For (ii), let $I,J \in \mathcal{P}_f(I\!N)$ with $I\cap J=\emptyset$ and suppose $\mathbf{p}>0$ with $P_I,P_J>0$. Then

$$\begin{aligned}
&\eta(I\cup J,\mathbf{p},\mathbf{x},f)\\
&= \max_{i\in I\cup J}\{f(x_i)\} - f\left(\frac{1}{P_{I\cup J}}\sum_{i\in I\cup J}p_ix_i\right)\\
&= \max\left\{\max_{i\in I}\{f(x_i)\},\max_{j\in J}\{f(x_j)\}\right\} - f\left(\frac{P_I}{P_I+P_J}\cdot\frac{1}{P_I}\sum_{i\in I}p_ix_i + \frac{P_J}{P_I+P_J}\cdot\frac{1}{P_J}\sum_{j\in J}p_jx_j\right)\\
&\leq \tfrac{1}{2}\left[\max_{i\in I}\{f(x_i)\} + \max_{j\in J}\{f(x_j)\} + \left|\max_{i\in I}\{f(x_i)\} - \max_{j\in J}\{f(x_J)\}\right|\right]\\
&\qquad\qquad - \max\left\{f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right), f\left(\frac{1}{P_J}\sum_{j\in J}p_jx_j\right)\right\}\\
&= \tfrac{1}{2}\left[\max_{i\in I}\{f(x_i)\} + \max_{j\in J}\{f(x_j)\} + \left|\max_{i\in I}\{f(x_i)\} - \max_{j\in J}\{f(x_j)\}\right|\right]\\
&\quad - \tfrac{1}{2}\left[\max_{i\in I}\{f(x_i)\} + \max_{j\in J}\{f(x_j)\} - \eta(I,p,x,f) - \eta(J,p,x,f) - |\eta(I,p,x,f)-\eta(J,p,x,f)|\right]\\
&= \frac{1}{2}\Big[\eta(I,p,x,f)+\eta(J,p,x,f) - |\eta(I,p,x,f)-\eta(J,p,x,f)|\Big] + \tfrac{1}{2}\left|\max_{i\in I}\{f(x_i)\} - \max_{j\in I}\{f(x_j)\}\right|\\
&\geq \min\{\eta(I,p,x,f),\eta(J,p,x,f)\},
\end{aligned}$$

and we are done. □

4 Two mappings associated with Jensen's inequality

Suppose $x_i, y_j \in C$ for $i=1,\dots,n$ and $j=1,\dots,m$ and $\theta_{i,j}=\theta_{i,j}(t)=tx_i+(1-t)y_j$. In what follows, the mappings $\mathcal{H},\mathcal{F}:[0,1]\to I\!R$ are given by

$$\mathcal{H}(t) = \max_{1\leq i\leq n}\max_{1\leq j\leq m} f(\theta_{i,j}), \quad \mathcal{F}(t)=\max\{\mathcal{H}(t),\mathcal{H}(1-t)\}.$$

Theorem 4.1 *For f quasiconvex,*
(i) $\mathcal{H},\mathcal{F}$ are quasiconvex on $[0,1]$;
(ii) $\mathcal{F}(t)=\mathcal{F}(1-t)$ for $t\in[0,1]$;
(iii) $\mathcal{F}(1/2)\leq\mathcal{F}(t)\leq\mathcal{F}(0)=\mathcal{F}(1)$ for $t\in[0,1]$.

Proof Part (i) follows from Propositions 2.1 and 2.2 and part (ii) from the definition of $\mathcal{F}$. The first inequality in (iii) derives from part (ii) and Lemma 2.3. The remainder of (iii) is a consequence of (ii) and the quasiconvexity of $\mathcal{F}$. □

Put $\mu := (1/P_n)\sum_{i=i}^n p_ix_i$. In the special case $m=1$ and $y_1=\mu$ we write $\mathcal{H}=H_0$ and $\mathcal{F}=F_0$. In the special case $m=n$ and $y_i=x_i$ $(i=1,\dots,n)$, we write $\mathcal{H}=H_1$ and $\mathcal{F}=F_1$. These mappings were introduced by Dragomir in the case of f convex but have more general applicability. For notational convenience we rebadge the corresponding forms of $\theta_{i,j}$ as

$$\psi_i(t)=tx_i+(1-t)\mu \quad\text{and}\quad \psi_{i,j}(t)=tx_i+(1-t)x_j.$$

Theorem 4.2 *For f quasiconvex and $t\in[0,1]$, we have*
(a) $H_0(0)\leq H_0(t)\leq H_0(1)$;

(b) $f(\mu) \le F_1(1/2) \le F_1(t) \le F_1(1) = \max_{1\le i\le n} f(x_i)$*;*
(c) $F_1(t) \ge F_0(t)$*.*

Proof The outermost inequality of Theorem 3.1 may be written $f(\mu) \le \max_{1\le i\le n} f(x_i)$, so by the definition of H_0 and the quasiconvexity of f

$$H_0(t) \le \max_{1\le i\le n} \max\{f(x_i), f(\mu)\} = \max\left\{\max_{1\le i\le n}\{f(x_i)\}, f(\mu)\right\} = \max_{1\le i\le n} f(x_i),$$

whence we deduce the second inequality in (a).

The outermost inequality of Theorem 3.1 gives also that

$$H_0(t) = \max_i f(\psi_i) \ge f\left(\frac{1}{P_n}\sum_{i=1}^n p_i\psi_i\right).$$

Since $\sum_{i=1}^n p_i\psi_i = \sum_{i=1}^n p_i x_i$, this provides $H_0(t) \ge f(\mu)$, whence the first inequality in (a).

From Theorem 3.1, we have successively

$$\begin{aligned} F_1(1/2) &= \max_{1\le j\le n}\max_{1\le i\le n} f\left(\frac{x_i+x_j}{2}\right) \\ &\ge \max_{1\le j\le n} f\left(\frac{1}{P_n}\sum_{i=1}^n p_i\left(\frac{x_i+x_j}{2}\right)\right) \\ &= \max_{1\le j\le n} f\left(\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i x_i + x_j\right]\right) \\ &\ge f\left(\frac{1}{2P_n}\sum_{j=1}^n p_j\left[\frac{1}{P_n}\sum_{i=1}^n p_i x_i + x_j\right]\right) = f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \end{aligned}$$

giving the first inequality in (b). Further, $F_1(t) \le \max_{1\le i,j\le n}\max\{f(x_i), f(x_j)\} = \max_{1\le i\le n} f(x_i)$ for all $t \in [0,1]$, from which we have the rest of (b). Again by Theorem 3.1,

$$H_0(t) = \max_{1\le i\le n} f(\psi_i) = \max_{1\le i\le n} f\left(\frac{1}{P_n}\sum_{j=1}^n p_j\psi_{i,j}\right) \le \max_{1\le i\le n}\max_{1\le j\le n} f(\psi_{i,j}) = F_1(t)$$

for all $t \in [0,1]$. Since $F_1(t) = F_1(1-t)$, we have also $H_0(1-t) \le F_1(t)$, so (c) holds. □

5 Further related maps

Some further maps on $[0,1]$ intimately related to H_0, F_1 are $K(t) := (1/P_n)\sum_{i=1}^n p_i f(\psi_i)$,

$$L(t) := \frac{1}{P_n^2}\sum_{i,j=1}^n p_i p_j f(\psi_{i,j}), \quad T(t) := \frac{1}{P_n}\sum_{i=1}^n p_i \max_{1\le j\le n} f(\psi_{i,j}), \quad W(t) := \max_{1\le j\le n}\frac{1}{P_n}\sum_{i=1}^n p_i f(\psi_{i,j}).$$

The mappings K and L were introduced (with different notation) in [1] and their properties studied in the case where f is convex. See also [2, 3]. These mappings provided useful interpolations of Jensen's discrete inequality. Their behaviour in the convex context is similar to that of H_0 and F_1 respectively of the previous section. The present context is more subtle in that a sum of quasiconvex functions need not be quasiconvex.

Remark 5.1 *We have from the definitions that for all* $t \in [0,1]$

$$W(t) \le \frac{1}{P_n}\sum_{i=1}^n p_i \max_{1\le j\le n} f(\psi_{i,j}) = T(t).$$

Proposition 5.2 *For f quasiconvex*

$$K(t) \leq \min\{H(t), T(t)\} \quad \text{for all } t \in [0,1] \text{ and} \tag{2}$$

$$\begin{aligned} K(t) &\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + f(\mu)\right] + \frac{1}{2P_n}\sum_{i=1}^{n} p_i \left|f(x_i) - f(\mu)\right| \\ &\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \max_{1\leq j\leq n} f(x_j)\right] + \frac{1}{2P_n}\sum_{i=1}^{n} p_i \left|f(x_i) - \max_{1\leq j\leq n} f(x_j)\right| \leq \max_{1\leq j\leq n} f(x_j). \end{aligned} \tag{3}$$

Proof. From Theorem 3.1 we have for $t \in [0,1]$ that $f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \psi_{i,j}\right) \leq \max_{1\leq j\leq n} f(\psi_{i,j})$, so that

$$K(t) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \psi_{i,j}\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \max_{1\leq j\leq n} f(\psi_{i,j}) = T(t).$$

On the other hand, from its definition, $K(t) \leq \max_{1\leq i\leq n} f(y_i) = H(t)$ for all $t \in [0,1]$. Taken together, these two results yield (2).

Also, by the definitions of quasiconvexity and $K(t)$,

$$\begin{aligned} K(t) &\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \max\{f(x_i), f(\mu)\} \\ &= \frac{1}{P_n}\sum_{i=1}^{n} p_i \cdot \frac{1}{2}\left[f(x_i) + f(\mu) + |f(x_i) - f(\mu)|\,\right] \\ &= \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + f(\mu)\right] + \frac{1}{2P_n}\sum_{i=1}^{n} p_i \left|f(x_i) - f(\mu)\right|, \end{aligned}$$

which provides the first inequality in (3). For the remainder of (3), Theorem 3.1 provides

$$\begin{aligned} \frac{1}{P_n}\sum_{i=1}^{n} p_i \max\{f(x_i), f(\mu)\} &\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \max\left\{f(x_i), \max_{1\leq j\leq n} f(x_j)\right\} \\ &= \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \max_{1\leq j\leq n} f(x_j)\right] + \frac{1}{2P_n}\sum_{i=1}^{n} p_i \left|f(x_i) - \max_{1\leq j\leq n} f(x_j)\right| \\ &\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \max_{1\leq j\leq n} f(x_j) = \max_{1\leq j\leq n} f(x_j). \end{aligned}$$

□

Proposition 5.3 *For all $t \in [0,1]$, we have $L(t) \leq W(t)$ and*

$$L(t) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \frac{1}{P_n^2}\sum_{1\leq i<j\leq n} p_i p_j |f(x_i) - f(x_j)| \leq \max_{1\leq i\leq n} f(x_i). \tag{4}$$

Proof. The first inequality is provided by

$$L(t) = \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j f(\psi_{i,j}) = \frac{1}{P_n}\sum_{j=1}^{n} p_j \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f(\psi_{i,j})\right) \leq \max_{1\leq j\leq n}\left\{\frac{1}{P_n}\sum_{i=1}^{n} p_i f(\psi_{i,j})\right\}.$$

Quasiconvexity yields $f(\psi_{i,j}) \leq \max\{f(x_i), f(x_j)\}$ for all $i, j \in \{1, \ldots, n\}$ and $t \in [0,1]$. Multiplying by $p_i p_j$ and summation over i, j yields

$$\begin{aligned} L(t) &\leq \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \max\{f(x_i), f(x_j)\} \\ &= \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \frac{f(x_i) + f(x_j) + |f(x_i) - f(x_j)|}{2} \\ &= \frac{1}{2}\left[\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j [f(x_i) + f(x_j)] + \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j |f(x_i) - f(x_j)|\right], \end{aligned}$$

which equals the right–hand side of the first inequality in (4).

Since $\max\{f(x_i), f(x_j)\} \le \max_{1\le k\le n} f(x_k)$ for all $i, j \in \{1, 2, \ldots, n\}$, we have

$$\frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \max\{f(x_i), f(x_j)\} \le \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \max_{1\le k\le n} f(x_k),$$

which equals the right–hand side of the second inequality in (4). The proposition is proved. □

Proposition 5.4 *For all $t \in [0, 1]$, we have $T(t) \le F(t)$ and*

$$\begin{aligned} T(t) &\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \max_{1\le i\le n} f(x_i)\right] + \frac{1}{2}\sum_{i=1}^{n} p_i \max_{1\le j\le n} |f(x_i) - f(x_j)|, \\ W(t) &\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + \max_{1\le i\le n} f(x_i)\right] + \frac{1}{2}\max_{1\le j\le n}\left\{\frac{1}{P_n}\sum_{i=1}^{n} p_i |f(x_i) - f(x_j)|\right\}. \end{aligned}$$

Proof. From the definition of T, we have for $t \in [0, 1]$ that

$$T(t) \le \max_{1\le i\le n}\max_{1\le j\le n} f(\psi_{i,j}) = \max_{1\le i,j\le n} f(\psi_{i,j}),$$

whence the first inequality follows. Again by quasiconvexity

$$\begin{aligned} \max_{1\le j\le n} f(\psi_{i,j}) &\le \max_{1\le j\le n}\left\{\frac{1}{2}[f(x_i) + f(x_j) + |f(x_i) - f(x_j)|\right\} \\ &\le \frac{1}{2}\left[f(x_i) + \max_{1\le j\le n} f(x_j) + \max_{1\le j\le n} |f(x_i) - f(x_j)|\right]. \end{aligned}$$

Multiplying by p_i and summation over i yields the second inequality. Similarly quasiconvexity supplies for all $j \in \{1, \ldots, n\}$ and $t \in [0, 1]$ that

$$\begin{aligned} \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\psi_{i,j}) &\le \frac{1}{P_n}\sum_{i=1}^{n} p_i \,\frac{1}{2}[f(x_i + f(x_j) + |f(x_i) - f(x_j)|] \\ &\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + f(x_j) + \frac{1}{P_n}\sum_{i=1}^{n} p_i |f(x_i) - f(x_j)|\right]. \end{aligned}$$

Taking the maximum over j provides the third and final inequality. □

6 Refinements of Jensen's inequality for quasiconvex functions

We begin by extending Theorem 3.1 to multisums. The following elementary lemma is useful.

Lemma 6.1 *Let K be a positive integer and $\sigma_1, \sigma_2, \cdots, \sigma_K$ real numbers. Real numbers $\rho_1, \ldots, \rho_K$ are defined by $\rho_i = r_1\sigma_i + r_2\sigma_{i+1} + \cdots + r_K\sigma_{i+K}$, where we interpret $\sigma_{\ell+K} = \sigma_\ell$. If $\sum_{\ell=1}^{K} r_\ell = 1$, then $\sum_{\ell=1}^{K} \rho_\ell = \sum_{\ell=1}^{K} \sigma_\ell$.*

Lemma 6.2 *Suppose $x_{i_1,\ldots,i_k} \in C$, $p_{i_1,\ldots,i_k} > 0$ for $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$. For f is quasiconvex*

$$f\left(\frac{\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1,\ldots,i_k} x_{i_1,\ldots,i_k}}{\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1,\ldots,i_k}}\right) \le \max_{1\le i_1,\ldots,i_k\le n} f(x_{i_1,\ldots,i_k}).$$

Proof. The vectors in C may be relabelled by positive integers via

$$x_1 = x_{1,1,\ldots,1,1},\ x_2 = x_{2,1,\ldots,1,1}, \ldots, x_{n^k-1} = x_{n,n,\ldots,n,n-1},\quad x_{n^k} = x_{n,n,\ldots,n,n}$$

with a similar relabelling for $p_{i_1,\ldots,i_k}$. The relation in the enunciation then becomes

$$f\left(\frac{\sum_{\ell=1}^{n^k} p_\ell x_\ell}{\sum_{\ell=1}^{n^k} p_\ell}\right) \leq \max_{1\leq \ell \leq n^k} f(x_\ell),$$

which holds by virtue of Theorem 3.1. □

Theorem 6.3 *Suppose f is quasiconvex. Let*

$$y_{1,k} = y_{1,k}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = (1/k)[x_{i_1} + x_{i_2} + \ldots + x_{i_k}] \quad \textit{and} \quad a_k = \max_{1\leq i_1, i_2, \ldots, i_k \leq n} f(y_{1,k}).$$

Then the sequence $(a_k)_{k\geq 1}$ is nonincreasing and bounded below by $f(\mu)$.

Proof. Take $x_{i_1,i_2,\ldots,i_k} = y_{1,k}$ in Lemma 6.2. The convexity of C ensures that $x_{i_1,i_2,\ldots,i_k} \in C$. Then for each $k \geq 1$, Lemma 6.2 gives

$$f\left(\frac{\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} y_{1,k}}{\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}}\right) \leq a_k. \tag{5}$$

Easy inductions on k provide

$$\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} = P_n^k \quad \text{and} \quad \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} y_{1,k} = P_n^{k-1}\sum_{i=1}^{n} p_i x_i,$$

so the left–hand side of (5) reduces to the required lower bound.

Put $\sigma_\ell = x_{i_\ell}$ $(1 \leq \ell \leq k+1)$ in Lemma 6.1 with $K = k+1$ and $r_i = 1/k$ for $1 \leq i \leq k$ and $r_{k+1} = 0$. We may extend the definition of $y_{1,k}$ to $y_{\ell,k}$ for $1 \leq \ell \leq k+1$ by setting $y_{\ell,k} = \rho_\ell$. The condition $\sum_{\ell=1}^{K} r_\ell = 1$ holds, so $\sum_{\ell=1}^{k+1} y_{\ell,k} = \sum_{\ell=1}^{k+1} x_{i_\ell}$ and by Theorem 3.1

$$f(y_{1,k+1}) = f\left((k+1)^{-1}[y_{1,k} + \ldots + y_{k+1,k}]\right) \leq \max_{1\leq \ell\leq k+1} f(y_{\ell,k}).$$

Taking the maximum yields

$$a_{k+1} \leq \max_{1\leq i_1,\ldots,i_{k+1}\leq n}\left\{\max_{1\leq \ell\leq k+1} f(y_{\ell,k})\right\} = \max_{1\leq \ell\leq k+1}\left\{\max_{1\leq i_1,\ldots,i_{k+1}\leq n} f(y_{\ell,k})\right\}.$$

By symmetry, each of the inner maxima takes the value $\max_{1\leq i_1,\ldots,i_k\leq n}\{f(y_{1,k})\} = a_k$, so we have $a_{k+1} \leq a_k$, and we are done. □

We may also derive a weighted refinement of Jensen's inequality for quasiconvex mappings.

Theorem 6.4 *Suppose f is quasiconvex and $q_j \geq 0$ $(1 \leq j \leq k)$ with $Q_k = \sum_{j=1}^{k} q_j > 0$. Define*

$$z_{1,k} = (1/Q_k)\left(q_1 x_{i_1} + \ldots + q_k x_{i_k}\right) \quad \textit{and} \quad b_k = \max_{1\leq i_1,i_2,\ldots,i_k\leq n} f(z_{1,k}).$$

Then $f(\mu) \leq a_k \leq b_k \leq \max_{1\leq i\leq n} f(x_i)$.

Proof. We have just established the first inequality. For the second, take $K = k$ in Lemma 6.1 with $\sigma_\ell = x_{i_\ell}$ and define $r_\ell = q_\ell/Q_k$. We extend the definition of $z_{1,k}$ to $z_{\ell,k}$ for $1 \leq \ell \leq k$ by $z_{\ell,k} = \rho_\ell$. Then $\sum_{\ell=1}^{K} r_\ell = 1$ and so $y_{1,k} = (1/k)\sum_{\ell=1}^{k} x_{i_\ell} = (1/k)\sum_{\ell=1}^{k} z_{\ell,k}$. Thus

$$f(y_{1,k}) = f((1/k)\sum_{\ell=1}^{k} z_{\ell,k}) \leq \max_{1\leq \ell\leq k} f(z_{\ell,k}).$$

Taking maxima provides

$$a_k \leq \max_{1\leq i_1,\ldots,i_k\leq n}\left\{\max_{1\leq \ell\leq k} f(z_{\ell,k}),\right\} = \max_{1\leq i_1,\ldots,i_k\leq n} f(z_{1,k}) = b_k,$$

by symmetry, and we have the second inequality.

Finally, by quasiconvexity $f(z_{1,k}) \leq \max\{f(x_{i_1}), \ldots, f(x_{i_k})\}$. Taking maxima yields $b_k \leq \max_{1\leq i_1,\ldots,i_k\leq n}\{\max_{1\leq \ell\leq k} f(x_{i_\ell})\} = \max_{1\leq i\leq k} f(x_i)$ and we are done. □

7 Associated sequences of mappings

We introduce a sequence of mappings $L_n^{[k+1]} : [0,1] \to I\!R$ defined by

$$L_n^{[k+1]}(t) = \max_{1\le i_1,\ldots,i_{k+1}\le n} \left\{ f(ty_{1,k} + (1-t)x_{i_{k+1}}) \right\}.$$

Theorem 7.1 *For f quasiconvex, $L_n^{[k+1]}$ is quasiconvex on $[0,1]$ with*

$$a_{k+1} \le L_n^{[k+1]}(t) \le \max_{1\le i\le n} f(x_i), \tag{6}$$

$$a_{k+1} = L_n^{k+1}\left(k/(k+1)\right) \le L_n^{[k+1]}(t) \le L_n^{[k+1]}(0) = \max_{1\le i\le n} f(x_i).$$

Proof. Quasiconvexity is immediate from Proposition 2.2. Now put

$$\omega_{1,k} = ty_{1,k} + (1-t)x_{i_{k+1}} \quad \text{and} \quad \omega_{\ell,k} = ty_{\ell,k} + (1-t)x_{i_{\ell-1}} \quad \text{for } 2 \le \ell \le k+1.$$

Then $\sum_{\ell=1}^{k+1} \omega_{\ell,k} = \sum_{\ell=1}^{k+1} x_{i_\ell} = y_{1,k+1}$, while by Theorem 3.1

$$f\left((k+1)^{-1}[\omega_{1,k} + \omega_{2,k} + \ldots + \omega_{k+1,k}]\right) \le \max_{1\le\ell\le k+1} f(\omega_{\ell,k}).$$

Hence $f(y_{1,k+1}) \le \max_{1\le\ell\le k+1} f(\omega_{\ell,k})$ for all $t \in [0,1]$. Taking maxima yields

$$a_{1,k+1} \le \max_{1\le i_1,\ldots,i_{k+1}\le n} \left\{ \max_{1\le j\le k+1} f(\omega_{j,k}) \right\} = \max_{1\le i_1,\ldots,i_{k+1}\le n} f(\omega_{1,k}),$$

by symmetry. This gives the first inequality in (6).

The quasiconvexity of f provides $f(\omega_{1,k}) \le \max\{f(y_{1,k}), f(x_{i_{k+1}})\}$. Taking maxima gives

$$\begin{aligned} L_n^{[k+1]}(t) &\le \max_{1\le i_1,\ldots,i_{k+1}\le n} \left\{ \max\left\{ f(y_{1,k}, f(x_{i_{k+1}}) \right\} \right\} \\ &= \max\left\{ \max_{1\le i_1,\ldots,i_{k+1}\le n} f(y_{1,k}), \max_{1\le i_1,\ldots,i_{k+1}\le n} f(x_{i_{k+1}}) \right\} \\ &= \max\left\{ \max_{1\le i_1,\ldots,i_k\le n} f(y_{1,k}), \max_{1\le i\le n} f(x_i) \right\}. \end{aligned}$$

By Theorem 6.3, $f(y_{1,k}) \le \max_{1\le i\le n} f(x_i)$, whence we derive the second inequality in (6). The remaining inequalities follow directly. □

Remark 7.2. *For $k = 1$ we recover the mapping $L_n^{[2]}(t) = \max_{1\le i_1,i_2\le n} f(\psi_{i_1,i_2}) = \max_{1\le i,j\le n} f(\psi_{i,j}) = F_1(t)$ studied in Section 4.*

Define $u_k = u_k(x_{i_{k+1}}, \ldots, x_{i_{2k}})$ by $u_k = (x_{i_{k+1}} + \ldots + x_{i_{2k}})/k$. A further sequence of mappings $F_n^{[2k]} : [0,1] \to I\!R$ $(k \ge 1)$ associated with a quasiconvex f is given by

$$F_n^{[2k]}(t) = \max_{1\le i_1,\ldots,i_{2k}\le n} f\left(ty_{1,k} + (1-t)u_k\right)$$

where again each $x_i \in C$ $(1 \le i \le n)$ and $k \ge 1$.

Theorem 7.3 *For f quasiconvex*
(i) $F_n^{[2k]}$ is quasiconvex on $[0,1]$;
(ii) $F_n^{[2k]}(1-t) = F_n^{[2k]}(t)$ for all $t \in [0,1]$;
(iii) we have for all $t \in [0,1]$ the bounds

$$F_n^{[2k]}(t) \le F_n^{[2k]}(0) = F_n^{[2k]}(1) = a_k \quad \text{and} \quad F_n^{[2k]}(t) \ge F_n^{[2k]}(1/2) = a_{2k}.$$

Proof. The proof follows familiar lines. We address only the pair of inequalities $a_k \ge F_n^{[2k]}(t) \ge a_{2k}$. By quasiconvexity, $f(ty_{1,k} + (1-t)u_k) \le \max\left\{f(y_{1,k}), f(u_k)\right\}$. Taking maxima yields

$$\begin{aligned} \max_{1\le i_1,\ldots,i_{2k}\le n} f(ty_{1,k} + (1-t)u_k) &\le \max_{1\le i_1,\ldots,i_{2k}\le n} \left\{\max\left\{f(y_{1,k}), f(u_k)\right\}\right\} \\ &= \max\left\{ \max_{1\le i_1,\ldots,i_{2k}\le n} \left\{f(y_{1,k})\right\}, \max_{1\le i_1,\ldots,i_{2k}\le n} \left\{f(u_k)\right\} \right\} \\ &= \max_{1\le i_1,\ldots,i_k\le n} f(y_{1,k}) = a_k, \end{aligned}$$

which proves the first inequality. By the symmetry and quasiconvexity of $F_n^{[2k]}$,

$$F_n^{[2k]}(t) = \max\left\{F_n^{[2k]}(t), F_n^{[2k]}(1-t)\right\} \geq F_n^{[2k]}\left((1/2)[t+(1-t)]\right) = F^{[2k]}\left(1/2\right) = a_{2k}$$

and the second is proved. □

Finally, we consider for $t \in [0,1]$ the mappings $H_n^{[k]}(t) := \max_{1 \leq i_1, \ldots, i_k \leq n} f(ty_{1,k} + (1-t)u_k)$.

Theorem 7.4 *The mapping* $H_n^{[k]}$ *is quasiconvex on* $[0,1]$ *with*

$$f(\mu) = H_n^{[k]}(0) \leq H_n^{[k]}(t) \leq \max_{1 \leq i_1, \ldots, i_{k+1} \leq n} f(ty_{1,k} + (1-t)x_{i_{k+1}}) = L_n^{[k+1]}(t), \tag{7}$$

$$a_k = H_n^{[k]}(1) \geq H_n^{[k]}(t). \tag{8}$$

Proof. Quasiconvexity is immediate. By Lemma 6.2,

$$H_n^{[k]}(t) \geq f\left[\frac{1}{P_n^k} \sum_{1 \leq i_1, \ldots, i_k \leq n} p_{i_1} \ldots p_{i_k}\left(ty_{1,k} + (1-t)\mu\right)\right] = f(\mu)$$

and the first two relations in (7) are established. For the rest, observe that

$$\begin{aligned} H_n^{[k]}(t) &= \max_{1 \leq i_1, \ldots, i_k \leq n}\left\{f\left[\frac{1}{P_n}\sum_{i_{k+1}=1}^{n} p_{i_{k+1}}\left(ty_{1,k} + (1-t)x_{i_{k+1}}\right)\right]\right\} \\ &\leq \max_{1 \leq i_1, \ldots, i_k \leq n}\left\{\max_{1 \leq i_{k+1} \leq n}\left\{f\left(ty_{1,k} + (1-t)x_{i_{k+1}}\right)\right\}\right\} = L_n^{[k+1]}(t). \end{aligned}$$

By the quasiconvexity of f, we have $f\left(ty_{1,k} + (1-t)\mu\right) \leq \max\left\{f(y_{1,k}), f\left(\mu\right)\right\}$, so

$$\begin{aligned} H_n^{[k]}(t) &\leq \max_{1 \leq i_1, \ldots, i_k \leq n}\left\{\max\left\{f(y_{1,k}), f\left(\mu\right)\right\}\right\} \\ &= \max\left\{\max_{1 \leq i_1, \ldots, i_k \leq n} f(y_{1,k}), f(\mu)\right\} = \max_{1 \leq i_1, \ldots, i_k \leq n} f(y_{1,k}) = a_k \end{aligned}$$

and (8) is proved. □

References

[1] S.S. Dragomir, Two mappings associated with Jensen's inequality, *Extracta Math.* **8** (1993), 102–105.

[2] S.S. Dragomir and D.M. Milošević, Two mappings in connection to Jensen's inequality, *Zb. Rad. (Krajujevac)* **15** (1994), 65–73.

[3] S.S. Dragomir and D.M. Milošević, Two mappings in connection to Jensen's inequality, *Math. Balkanica (N.S.)* **9** (1995), 3–9.

[4] S.S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin, *Indian J. Math.* **39** (1997), 1–9.

[5] S.S. Dragomir and C.E.M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin, *Periodica Math. Hungar.* **33 33** (1996), 93–100.

[6] S.S. Dragomir and C.E.M. Pearce, Quasi–convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377–385.

[7] S.S. Dragomir, J.E. Pečarić and L.E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* **21** (1995), 335–341.

[8] A. Eberhard and C.E.M. Pearce, Class–inclusion properties for convex functions, *Progress in Optimization*, Eds X. Yang *et al.*, Kluwer Academic Publishers, Dordrecht (2000), 129–133.

[9] C.E.M. Pearce, Quasiconvexity, fractional programming and extremal traffic congestion, in *Frontiers in Global Optimization*, Kluwer, Dordrecht, *Nonlinear Optimization and its Applications* **74** (2004), 403–409.

[10] C.E.M. Pearce and A.M. Rubinov, P–functions, quasi–convex functions and Hadamard–type inequalities, *J. Math. Anal. & Applic.* **240** (1999), 92–104.