LOWER AND UPPER BOUNDS OF THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. Lower and upper bounds of the Čebyšev functional for the Riemann-Stieltjes integral, in the monotonicity case of one function, are given. Applications in relation with the Steffensen generalisation of the Čebyšev inequality are provided.

1. Introduction

In [3], S.S. Dragomir introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t), \tag{1.1}$$

provided $u(b) \neq u(a)$ and the involved Riemann-Stieltjes integrals exist.

In order to bound the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of the integrals, as described in the definition of the Čebyšev functional (1.1), the first author obtained the inequality:

$$|T(f,g;u)| \leq \frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u), \tag{1.2}$$

provided u is of bounded variation, f,g are continuous on $[a,b]$ and $m \leq f(t) \leq M$ for any $t \in [a,b]$. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Moreover, if f,g are as above and u is monotonic nondecreasing on $[a,b]$, then

$$|T(f,g;u)| \leq \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t), \tag{1.3}$$

and the constant $\frac{1}{2}$ here is also sharp.

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Finally, if f and g are Riemann integrable and u is Lipschitzian with the constant $L>0$, then also

$$|T(f,g;u)| \leq \frac{1}{2}L(M-m)\frac{1}{|u(b)-u(a)|}\int_a^b \left| g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt, \tag{1.4}$$

provided $m \leq f(t) \leq M$, $t\in[a,b]$. The multiplicative constant $\frac{1}{2}$ is best possible in (1.4).

For results concerning bounds for the Čebyšev functional $T(f,g;u)$ see [4] and [5]. For other recent results on inequalities for the Riemann-Stieltjes integral, see [1], [2] and [6].

The main aim of this paper is to provide an upper and a lower bound for the functional $T(f,g;u)$ under the monotonicity assumption on the function f. An application for the Čebyšev inequality for Riemann-Stieltjes integrals that is related to Steffensen's result from [8] is given as well.

2. The Results

The following result providing upper and lower bounds for the quantity $[h(b)-h(a)]\,T(f,g,h;a,b)$ can be stated:

Theorem 1. *Let $f,g,h:[a,b]\to\mathbb{R}$ be such that $h(a)\neq h(b)$ and the Riemann-Stieltjes integrals $\int_a^b f(t)\,dh(t)$, $\int_a^b g(t)\,dh(t)$ and $\int_a^b f(t)g(t)\,dh(t)$ exist. If f is monotonic nondecreasing, then*

$$\begin{aligned}(2.1)\quad &[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}\\ &\leq \int_a^b f(t)g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\cdot\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)\\ &\leq [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}.\end{aligned}$$

If f is monotonic nonincreasing, then:

$$\begin{aligned}(2.2)\quad &[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}\\ &\leq \int_a^b f(t)g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\cdot\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)\\ &\leq [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}.\end{aligned}$$

The inequalities (2.1) and (2.2) are sharp.

Proof. We use the following Abel type inequality obtained by Mitrinović et al. in [7, p. 336]:

Let u be a nonnegative and monotonic nondecreasing function on $[a,b]$ and $v,w:[a,b]\to\mathbb{R}$ such that the Riemann-Stieltjes integrals $\int_a^b v(t)\,dw(t)$ and $\int_a^b u(t)v(t)\,dw(t)$

exist. Then

$$u(b)\inf_{t\in[a,b]}\left\{\int_t^b v(t)\,dw(t)\right\}\leq\int_a^b u(t)\,v(t)\,dw(t)\leq u(b)\sup_{t\in[a,b]}\left\{\int_t^b v(t)\,dw(t)\right\}. \tag{2.3}$$

We also use the representation (see [3])

$$T(f,g,h;a,b)=\frac{1}{h(b)-h(a)}\int_a^b[f(t)-\gamma]\left[g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right]dh(t), \tag{2.4}$$

which holds for any $\gamma\in\mathbb{R}$.

Now, if we choose $\gamma=f(a)$, then we observe that the function $u(t)=f(t)-f(a)$ is nonnegative and monotonic nondecreasing on $[a,b]$ and applying (2.3) for $w(t)=h(t)$ and $v(t)=g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,dh(s)$ we deduce:

$$\begin{aligned}[f(b)-f(a)]&\inf_{t\in[a,b]}\left\{\int_t^b\left[g(s)-\frac{1}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right]dh(s)\right\}\\ &\leq[h(b)-h(a)]\,T(f,g,h;a,b)\\ &\leq[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b\left[g(s)-\frac{1}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right]dh(s)\right\},\end{aligned} \tag{2.5}$$

which is equivalent with the desired inequality (2.1).

For the second inequality, we use (2.4) with $\gamma=f(b)$ and the following Abel type result for functions u which are monotonic nonincreasing and nonnegative(see [7, p. 336]):

$$u(a)\inf_{t\in[a,b]}\left\{\int_a^t v(t)\,dw(t)\right\}\leq\int_a^b u(t)\,v(t)\,dw(t)\leq u(a)\sup_{t\in[a,b]}\left\{\int_a^t v(t)\,dw(t)\right\}. \tag{2.6}$$

The details are omitted.

Let us prove for instance the sharpness of the second inequality in (2.1).

If we choose $h(t)=t$ and $g(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$ then we have to show that the inequality:

$$\int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\leq[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b\operatorname{sgn}\left(s-\frac{a+b}{2}\right)ds\right\} \tag{2.7}$$

is sharp provided f is monotonic nondecreasing on $[a,b]$.

Notice that

$$\lambda(t):=\int_t^b\operatorname{sgn}\left(s-\frac{a+b}{2}\right)ds=\begin{cases}t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right]\\ b-t & \text{if } t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

and then $\sup_{t\in[a,b]} \lambda(t) = \frac{b-a}{2}$.

Therefore (2.7) becomes

$$\int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt \le [f(b)-f(a)]\cdot\frac{b-a}{2}. \tag{2.8}$$

Now, if in this inequality we choose $f(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, which is monotonic nondecreasing on $[a,b]$, we get in both sides of (2.8) the same quantity $b-a$.

The sharpness of the other inequalities can be shown in a similar way. The details are omitted. ∎

Remark 1. *We observe that*

$$\begin{aligned}
&\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau) \\
&= \int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\cdot\left[\int_a^t g(s)\,dh(s) + \int_t^b g(s)\,dh(s)\right] \\
&= \frac{h(t)-h(a)}{h(b)-h(a)}\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\int_a^t g(s)\,dh(s) \\
&= \frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)} \\
&\qquad\times\left[\frac{1}{h(b)-h(t)}\int_t^b g(s)\,dh(s) - \frac{1}{h(t)-h(a)}\int_a^t g(s)\,dh(s)\right].
\end{aligned}$$

Therefore, if we denote by $\Delta(g,h;t,a,b)$ the difference

$$\frac{1}{h(b)-h(t)}\int_t^b g(s)\,dh(s) - \frac{1}{h(t)-h(a)}\int_a^t g(s)\,dh(s),$$

provided $h(t)\neq h(a), h(b)$ for $t\in[a,b]$, then from (2.1) we get

$$\begin{aligned}
&[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\} \\
&\le \int_a^b f(t)g(t)\,dh(t) - \frac{1}{h(b)-h(a)}\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t) \\
&\le [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},
\end{aligned} \tag{2.9}$$

provided f is monotonic nondecreasing on $[a,b]$.

A similar result can be stated from (2.2) on noticing that

$$\begin{aligned}
&\int_a^t g(s)\,dh(s) - \frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau) \\
&\qquad\qquad = -\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b).
\end{aligned}$$

Indeed, since

$$\begin{aligned}
&\inf_{t\in[a,b]}\left(\sup_{t\in[a,b]}\right)\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(s)\,dh(s)\right\}\\
&=\inf_{t\in[a,b]}\left(\sup_{t\in[a,b]}\right)\left\{-\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}\\
&=-\sup_{t\in[a,b]}\left(\inf_{t\in[a,b]}\right)\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},
\end{aligned}$$

then from (2.2) we get

$$\begin{aligned}
&[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},\\
&\le\frac{1}{h(b)-h(a)}\int_a^b f(t)\,dh(t)\int_a^b g(t)\,dh(t)-\int_a^b f(t)\,g(t)\,dh(t)\\
&\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}
\end{aligned}$$

provided that f is monotonic nonincreasing on $[a,b]$.

The following corollary gives a particular result of interest for Riemann weighted integrals.

Corollary 1. *Let $f,g,w:[a,b]\to\mathbb{R}$ be such that the Riemann integrals $\int_a^b f(t)\,w(t)\,dt$, $\int_a^b g(t)\,w(t)\,dt$, $\int_a^b f(t)\,g(t)\,w(t)\,dt$ and $\int_a^b w(t)\,dt$ exist, and $\int_a^b w(t)\,dt\neq 0$.*

If f is monotonic nondecreasing, then

(2.10)

$$\begin{aligned}
&[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,w(s)\,ds-\frac{\int_t^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}\\
&\le\int_a^b f(t)\,g(t)\,w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\
&\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,w(s)\,ds-\frac{\int_t^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}.
\end{aligned}$$

If f is monotonic nonincreasing, then

(2.11)

$$\begin{aligned}
&[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,w(s)\,ds-\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}\\
&\le\int_a^b f(t)\,g(t)\,w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\
&\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,w(s)\,ds-\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}.
\end{aligned}$$

Remark 2. *If we define*

$$\tilde{\Delta}(g,w;t,a,b) := \frac{1}{\int_t^b w(s)\,ds}\int_t^b g(s)\,w(s)\,ds - \frac{1}{\int_a^t w(s)\,ds}\int_a^t g(s)\,w(s)\,ds,$$

provided $\int_a^t w(s)\,ds,\ \int_t^b w(s)\,ds \neq 0$, *then, under the assumptions of Corollary 1, we have:*

$$\begin{aligned}(2.12)\qquad & [f(b)-f(a)]\inf_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\\ &\leq \int_a^b f(t)\,g(t)\,w(t)\,dt - \frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\ &\leq [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\},\end{aligned}$$

provided f is monotonic nondecreasing on $[a,b]$, *and*

$$\begin{aligned}(2.13)\qquad & [f(a)-f(b)]\inf_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\},\\ &\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt - \int_a^b f(t)\,g(t)\,w(t)\,dt\\ &\leq [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\end{aligned}$$

if f is monotonic nonincreasing on $[a,b]$.

Remark 3. *In the particular case where* $w(t)=1$, $t\in[a,b]$, *we get the simpler inequalities:*

$$\begin{aligned}(2.14)\qquad & [f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,ds - \frac{b-t}{b-a}\int_a^b g(\tau)\,d\tau\right\}\\ &\leq \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\ &\leq [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,ds - \frac{b-t}{b-a}\int_a^b g(\tau)\,d\tau\right\}\end{aligned}$$

in the case where f is monotonic nondecreasing on $[a,b]$.

If f is monotonic nonincreasing on $[a,b]$, *then*

$$\begin{aligned}(2.15)\qquad & [f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(\tau)\,d\tau\right\}\\ &\leq \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\ &\leq [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(\tau)\,d\tau\right\}.\end{aligned}$$

If we denote

$$\bar{\Delta}(g;t,a,b) := \frac{1}{b-t}\int_t^b g(s)\,ds - \frac{1}{t-a}\int_a^t g(s)\,ds,$$

then we get from (2.14)

$$\begin{aligned}
&\frac{[f(b)-f(a)]}{b-a}\inf_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\}\\
&\le \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\
&\le \frac{[f(b)-f(a)]}{b-a}\sup_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\},
\end{aligned}$$

provided f is monotonic nondecreasing and from (2.15)

$$\begin{aligned}
&\frac{[f(a)-f(b)]}{b-a}\inf_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\}\\
&\le \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt\\
&\le \frac{[f(a)-f(b)]}{b-a}\sup_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\}
\end{aligned}$$

if f is monotonic nonincreasing on $[a,b]$.

3. Applications for the Čebyšev Inequality

Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p:[a,b]\to[0,\infty)$ be an integrable function, then [7, p. 239]:

$$\int_a^b p(x)\,dx\int_a^b p(x)f(x)g(x)\,dx \ge \int_a^b p(x)f(x)\,dx\int_a^b p(x)g(x)\,dx. \tag{3.1}$$

This inequality in known in the literature as Čebyšev's inequality.

For various other results related to this classical fact, see Chapter IX of the book [7].

Proposition 1. *Let $f,g,h:[a,b]\to\mathbb{R}$ be such that the Riemann-Stieltjes integrals $\int_a^b f(t)\,dh(t)$, $\int_a^b g(t)\,dh(t)$ and $\int_a^b f(t)g(t)\,dh(t)$ exist. If $h(b)>h(a)$, f is monotonic nondecreasing (nonincreasing) and*

$$[h(b)-h(a)]\int_a^t g(s)\,dh(s) \ge [h(b)-h(t)]\int_a^b g(s)\,dh(s) \tag{3.2}$$

for any $t\in[a,b]$, then

$$[h(b)-h(a)]\int_a^b f(t)g(t)\,dh(t) \ge (\le)\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t). \tag{3.3}$$

The proof follows by Theorem 1 on utilising

$$\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\int_a^b g(s)\,dh(s) = -\left[\int_a^t g(s)\,dh(s) - \frac{h(t)-h(a)}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right].$$

Remark 4. *The above proposition implies the following Čebyšev type inequality for weighted integrals (with not necessarily positive weights). Let* $f, g, w : [a,b] \to \mathbb{R}$ *be such that the Riemann integrals,* $\int_a^b w(t)\,dt$, $\int_a^b f(t)\,w(t)\,dt$, $\int_a^b g(t)\,w(t)\,dt$ *and* $\int_a^b f(t)\,g(t)\,w(t)\,dt$ *exist.*

If $\int_a^b w(t)\,dt > 0$, f *is monotonic nondecreasing (nonincreasing) and*

$$\int_a^b w(s)\,ds \int_t^b g(s)\,w(s)\,ds \geq \int_t^b w(s)\,ds \int_a^b g(s)\,w(s)\,ds \tag{3.4}$$

for any $t \in [a,b]$, *then*

$$\int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt \geq (\leq) \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt. \tag{3.5}$$

In particular (i.e., if $w(s) = 1$*), if* f *is monotonic nondecreasing (nonincreasing) and if*

$$(b-a)\int_a^t g(s)\,ds \geq (b-t)\int_a^b g(s)\,ds \tag{3.6}$$

for any $t \in [a,b]$, *then*

$$(b-a)\int_a^b f(t)\,g(t)\,dt \geq (\leq) \int_a^b f(t)\,dt \int_a^b g(t)\,dt. \tag{3.7}$$

Remark 5. *Notice that, the weighted inequality (3.5), as pointed out in* [7, p. 246], *can be also obtained from the Steffensen result* [8] *which states that: if* F, G, H *are integrable functions on* $[a,b]$ *such that for all* $x \in [a,b]$

$$\frac{\int_a^x G(t)\,dt}{\int_a^b G(t)\,dt} \leq \frac{\int_a^x H(t)\,dt}{\int_a^b H(t)\,dt},$$

then

$$\frac{\int_a^b F(t)\,G(t)\,dt}{\int_a^b G(t)\,dt} \geq \frac{\int_a^b F(t)\,H(t)\,dt}{\int_a^b H(t)\,dt}, \tag{3.8}$$

provided F *is monotonic nondecreasing on* $[a,b]$.

The choice $F(t) \equiv f(t)$, $H(t) = w(t)$, *and* $G(t) = g(t)\,w(t)$ *in (3.8) produces (3.5) under the condition that (3.4) holds and* f *is monotonic nondecreasing.*

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