Hadamard-Type Inequalities For Twice Differentiable Functions

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Abstract. In this paper we give an estimate, from below and from above, of the mean value of the function $f : [a, b] \to \mathbb{R}$ such that f is continuous on $[a, b]$ and twice differentiable on (a, b).

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1 Introduction and main results

Throughout this note, we write I and I' for the intervals $[a, b]$ and (a, b) respectively. A function f is said to be convex on I if and only if $\lambda f(x) + (1 - \lambda)f(y) \geqslant f(\lambda x + (1 - \lambda)y)$ for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on I. A function f that is continuous on I and twice differentiable on I' is convex on I if and only if $f''(x) \geqslant 0$ for all $x \in I$. (f is concave if the inequality is flipped).

The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function $f : I \to \mathbb{R}$ which was first published in [5]:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

An account on the history of this inequality can be found in [6]. Surveys on various generalizations and developments can be found in [7] and [3]. The description of best possible inequalities of Hadamard-Hermite type are due to Fink [4]. A generalization to higher-order convex functions can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to a two dimensional linear space of continuous functions.

In [3] , S.S. Dragomir, C.E.M. Pearce have studied this type of inequalities for twice differential function with bounded second derivative and have obtained the following:

Theorem A [3, Theorem 30, p. 38] . *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I*, twice differentiable on* I' *and there exist* k, K such that $k \leqslant f''(x) \leqslant K$ on I. *Then*

$$\frac{k}{3}\left(\frac{b-a}{2}\right)^2 \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{K}{3}\left(\frac{b-a}{2}\right)^2. \tag{1.2}$$

In this paper, we give an estimate, from below and from above, of the mean value of $f : I \to \mathbb{R}$ such that f is continuous on I, twice differentiable on I' and there exist $m = \inf\limits_{x\in I'} f''(x)$, or $M = \sup\limits_{x\in I'} f''(x)$ and we obtain the following results:

Theorem 1.1 *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I *twice differentiable on* I' *and there exist* $m = \inf\limits_{x\in I'} f''(x)$. *Then we have*

$$f\left(\frac{a+b}{2}\right) + \frac{m}{6}\left(\frac{b-a}{2}\right)^2 \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} - \frac{m}{3}\left(\frac{b-a}{2}\right)^2, \tag{1.3}$$

and equality in (1.3) *holds if* $f(x) = \alpha x^2 + \beta x + \gamma,\ \ \alpha, \beta, \gamma \in \mathbb{R}$.

Theorem 1.2 *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I *twice differentiable on* I' *and there exist* $M = \sup\limits_{x\in I'} f''(x)$. *Then we have*

$$\frac{f(a)+f(b)}{2}-\frac{M}{3}\left(\frac{b-a}{2}\right)^2 \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant f\left(\frac{a+b}{2}\right)+\frac{M}{6}\left(\frac{b-a}{2}\right)^2, \tag{1.4}$$

and equality in (1.4) *holds if* $f(x)=\alpha x^2+\beta x+\gamma,\ \ \alpha,\beta,\gamma\in\mathbb{R}$.

From Theorem 1.1 and Theorem 1.2, we obtain the following Corollaries.

Corollary 1.1 *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I, *twice differentiable on* $I^{'}$ *and there exist* $m = \inf\limits_{x\in I^{'}} f^{''}(x)$ *and* $M = \sup\limits_{x\in I^{'}} f^{''}(x)$. *Then we have*

$$\frac{m}{3}\left(\frac{b-a}{2}\right)^2 \leqslant \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{M}{3}\left(\frac{b-a}{2}\right)^2 \tag{1.5}$$

and

$$\frac{m}{6}\left(\frac{b-a}{2}\right)^2 \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \leqslant \frac{M}{6}\left(\frac{b-a}{2}\right)^2, \tag{1.6}$$

and equality in (1.5) and (1.6) *holds if* $f(x)=\alpha x^2+\beta x+\gamma,\ \ \alpha,\beta,\gamma\in\mathbb{R}$.

Corollary 1.2 *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I, *twice differentiable on* $I^{'}$ *and there exist* $m = \inf\limits_{x\in I^{'}} f^{''}(x)$ *and* $M = \sup\limits_{x\in I^{'}} f^{''}(x)$. *Then we have*

$$\frac{1}{2}\left(\frac{m}{6}-\frac{M}{3}\right)\left(\frac{b-a}{2}\right)^2 \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx -$$

$$\frac{1}{2}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right) \leqslant \frac{1}{2}\left(\frac{M}{6}-\frac{m}{3}\right)\left(\frac{b-a}{2}\right)^2 \tag{1.7}$$

and

$$\frac{m}{2}\left(\frac{b-a}{2}\right)^2 \leqslant \frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right) \leqslant \frac{M}{2}\left(\frac{b-a}{2}\right)^2, \tag{1.8}$$

and equality in (1.7) *holds if* $f(x)=\alpha x^2+\beta x+\gamma \quad \alpha,\beta,\gamma\in\mathbb{R}$.

In the following corollary, if $f:I\to\mathbb{R}$ is continuous on I, convex, or concave and twice differentiable on $I^{'}$, then we obtain estimation better than (1.1) in [5].

Corollary 1.3 *Assume that* $f:I\to\mathbb{R}$ *is continuous on* I*, twice differentiable on* $I^{'}$.
(i) *If there exists* $m=\inf\limits_{x\in I^{'}} f^{''}(x)$ *and* f *convex on* I*. Then, we have*

$$f\left(\frac{a+b}{2}\right)\leqslant l\leqslant \frac{1}{b-a}\int_a^b f(x)\,dx\leqslant L\leqslant \frac{f(a)+f(b)}{2}, \tag{1.9}$$

where $l=f\left(\frac{a+b}{2}\right)+\frac{m}{6}\left(\frac{b-a}{2}\right)^2$, $L=\frac{f(a)+f(b)}{2}-\frac{m}{3}\left(\frac{b-a}{2}\right)^2$.
(ii) *If there exists* $M=\inf\limits_{x\in I^{'}} f^{''}(x)$ *and* f *concave on* I*. Then, we have*

$$\frac{f(a)+f(b)}{2}\leqslant \lambda\leqslant \frac{1}{b-a}\int_a^b f(x)\,dx\leqslant \mu\leqslant f\left(\frac{a+b}{2}\right), \tag{1.10}$$

where $\lambda=\frac{f(a)+f(b)}{2}-\frac{M}{3}\left(\frac{b-a}{2}\right)^2$, $\mu=f\left(\frac{a+b}{2}\right)+\frac{M}{6}\left(\frac{b-a}{2}\right)^2$.

Corollary 1.4 *Assume that* $f:I\to\mathbb{R}$ *is continuous on* I*, twice differentiable on* $I^{'}$ *and there exist* $m=\inf\limits_{x\in I^{'}} f^{''}(x)$ *and* $M=\sup\limits_{x\in I^{'}} f^{''}(x)$. *Then, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|\leqslant \frac{1}{3}\left(\frac{b-a}{2}\right)^2\max\{|m|,|M|\} \tag{1.11}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leqslant \frac{1}{6}\left(\frac{b-a}{2}\right)^2\max\{|m|,|M|\}. \tag{1.12}$$

Remark 1.1 In the above if $f\in C^2([a,b])$, then we can replace inf and sup by min and max respectively.

2 Proof of Theorems and Corollaries

Proof of Theorem 1.1 Let $f : I \to \mathbb{R}$ be twice differentiable on $I^{'}$. Set $g(x) = f(x) - \frac{m}{2}x^2$. Differentiating twice times both sides of g we get $g^{''}(x) = f^{''}(x) - m \geqslant 0$, then g is a convex function on I. By formula (1.1), we have

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b g(x)\,dx \leqslant \frac{g(a)+g(b)}{2}. \tag{2.1}$$

Substituting $g(x) = f(x) - \frac{m}{2}x^2$ into (2.1), we get

$$f\left(\frac{a+b}{2}\right) - \frac{m}{2}\left(\frac{b+a}{2}\right)^2 + \frac{1}{b-a}\int_a^b \frac{m}{2}x^2 dx \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\leqslant \frac{f(a)+f(b)}{2} - m\frac{a^2+b^2}{4} + \frac{1}{b-a}\int_a^b \frac{m}{2}x^2 dx. \tag{2.2}$$

By simple calculus from (2.2), we get (1.3).

Proof of Theorem 1.2 Let $f : I \to \mathbb{R}$ be twice differentiable on $I^{'}$. Set $h(x) = -f(x) + \frac{M}{2}x^2$. Differentiating twice times both sides of h we get $h^{''}(x) = -f^{''}(x) + M \geqslant 0$, then h is a convex function. By formula (1.1), we have

$$h\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b h(x)\,dx \leqslant \frac{h(a)+h(b)}{2}. \tag{2.3}$$

Substituting $g(x) = -f(x) + \frac{M}{2}x^2$ into (2.1) we get

$$-f\left(\frac{a+b}{2}\right) + \frac{M}{2}\left(\frac{b+a}{2}\right)^2 - \frac{1}{b-a}\int_a^b \frac{M}{2}x^2 dx \leqslant -\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\leqslant -\frac{f(a)+f(b)}{2} + M\frac{a^2+b^2}{4} - \frac{1}{b-a}\int_a^b \frac{M}{2}x^2 dx. \tag{2.4}$$

By simple calculus from (2.4), we get (1.4).

Proof of corollary 1.1 This can be concluded by using Theorem 1.1 and Theorem 1.2.

Proof of corollary 1.2 By Corollary 1.1, we have

$$\frac{m}{3}\left(\frac{b-a}{2}\right)^{2}\leqslant\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\leqslant\frac{M}{3}\left(\frac{b-a}{2}\right)^{2}\tag{2.5}$$

and

$$-\frac{M}{6}\left(\frac{b-a}{2}\right)^{2}\leqslant-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx+f\left(\frac{a+b}{2}\right)\leqslant-\frac{m}{6}\left(\frac{b-a}{2}\right)^{2}.\tag{2.6}$$

By addition from (2.5) and (2.6) , we get (1.7) . Now we prove (1.8) . By using corollary 1.1, we have

$$\frac{m}{3}\left(\frac{b-a}{2}\right)^{2}\leqslant\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\leqslant\frac{M}{3}\left(\frac{b-a}{2}\right)^{2}\tag{2.7}$$

and

$$\frac{m}{6}\left(\frac{b-a}{2}\right)^{2}\leqslant\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-f\left(\frac{a+b}{2}\right)\leqslant\frac{M}{6}\left(\frac{b-a}{2}\right)^{2}.\tag{2.8}$$

By addition from (2.7) and (2.8) , we get (1.8) .

Proof of corollary 1.3 (i) By f is convex function, we have $m\geqslant 0$. Then by (1.3) , we get (1.9). (ii) Using f is concave function we obtain $M\leqslant 0$. Then by (1.4) , we get (1.10).

Proof of corollary 1.4 This can be concluded by using Corollary 1.1.

Open question If f is only convex function on I, does there exist a real numbers l,L such that

$$f\left(\frac{a+b}{2}\right)\leqslant l\leqslant\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\leqslant L\leqslant\frac{f\left(a\right)+f\left(b\right)}{2}\ ?$$

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