NEW INEQUALITIES FOR THE HURWITZ ZETA FUNCTION

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Abstract. We establish various new inequalities for the Hurwitz zeta function. Our results generalize some known results for the polygamma functions to the Hurwitz zeta function.

1. Introduction

The Hurwitz zeta function $\zeta(s,x)$ is traditionally defined for any x, which is not a negative integer and zero, by the series

$$\zeta(s,x)=\sum\nolimits_{n=0}^{\infty}\frac{1}{(n+x)^s}$$

for all complex numbers s with $\mathbf{Re}s>1$. In this paper we restrict x to positive real numbers. We can analytically continue it to the whole complex s-plane (except for a simple pole at $s=1$) by means of the contour integral

$$\zeta(s,x)=\frac{\Gamma(1-s)}{2\pi i}\int_C\frac{z^{s-1}e^{xz}}{1-e^z}\,dz,$$

where C is a loop that starts from $-\infty$ along the lower side of the real axis, encircles the origin and then returns to $-\infty$ along the upper side of the real axis. For convenience, we will use a slightly non-standard notation here for this function: $H_s(x)=\zeta(s,x)$. Hurwitz zeta function occurs in a variety of disciplines. Most commonly, it occurs in analytic number theory. The Riemann zeta function defined for $\mathbf{Re}\ s>1$ by

$$\zeta(s)=\sum\nolimits_{n=1}^{\infty}\frac{1}{n^s}$$

and the polygamma functions defined for all positive integers m and positive real numbers x by

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$$\psi^{(m)}(x) = (-1)^{m+1} m! \sum\nolimits_{n=1}^{\infty} \frac{1}{(x+n)^{m+1}} \tag{1.1}$$

are special cases of the Hurwitz zeta function, namely $H_s(1) = \zeta(s)$, $H_s(1/2) = (2^s - 1)\zeta(s)$ and $\psi^{(m)}(x) = (-1)^{m+1} m! H_{m+1}(x)$. The Bernoully polynomials defined by the generating function

$$\frac{ze^{zx}}{e^z - 1} = \sum\nolimits_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n$$

are also special cases of the Hurwitz zeta function:

$$\zeta(-m, x) = -\frac{B_{m+1}(x)}{m+1} \quad (m = 0, 1, 2, \ldots,),$$

please see[6, Theorem 12.13]. It satisfies the following integral representation (see [6, Teorem 12.2]):

$$H_s(x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1} e^{-xu}}{1 - e^{-u}} du,$$

where $\Gamma(s)$ is Euler's gamma function, is valid for $\mathbf{Re} s > 1$. Among many places in which $H_s(x)$ appears we mention here two of them: The first is the evaluation by Kolbig [10] of integrals of the form

$$R_m(\mu, \upsilon) = \int_0^{\infty} e^{-\mu t} t^{\upsilon - 1} \log^m t \, dt,$$

an example of which is

$$R_2(\mu, \upsilon) = \mu^{-\upsilon} \Gamma(\upsilon)[(\psi(\upsilon) - \log \mu)^2 + H_2(\upsilon)],$$

with

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is logarithmic derivative of $\Gamma(x)$, also called the digamma function. The second one is Vardi's evaluation [11] of the integrals

$$\int_{\pi/4}^{\pi/2} \log\log(\tan x) dx = \frac{\pi}{2} \log\left(\frac{\Gamma(3/4)\sqrt{2\pi}}{\Gamma(1/4)}\right).$$

Its basic properties can be found in [5, 6]. Recently some authors studied this function and they obtained interesting inequalities, see

[1, 2, 3, 7]. The main purpose of this paper is to generalize the following inequalities for polygamma functions proved by the author [9] to Hurwitz zeta function: For all positive real number x and all positive integers n the following inequalities hold:

$$(n-1)!\exp\left(-n\psi(x+1/2)\right) < \left|\psi^{(n)}(x)\right| < (n-1)!\exp\left(-n\psi(x)\right),$$

$$\frac{(n-1)!}{(k-1)!}\left|\psi^{(k)}(x+1/2)\right|^{n/k} < \left|\psi^{(n)}(x)\right| < \frac{(n-1)!}{(k-1)!}\left|\psi^{(k)}(x)\right|^{n/k}$$

for all $k = 1, 2, ..., n-1$,

$$\alpha < \left((-1)^{n-1}\psi^{(n)}(x+1)\right)^{-1/n} - \left((-1)^{n-1}\psi^{(n)}(x)\right)^{-1/n} < \beta,$$

where the constants $\alpha = (n!\zeta(n+1))^{-1/n}$ and $\beta = ((n-1)!)^{-1/n}$ are best possible, and

$$(n!)^{\frac{1}{n+1}}\left[x - \left(x^{-1/n}+\alpha\right)^{-n}\right]^{-\frac{1}{n+1}} < \left(\left(-1\right)^{n-1}\psi^{(n)}\right)^{-1}(x)$$
$$< (n!)^{\frac{1}{n+1}}\left[x - \left(x^{-1/n}+\beta\right)^{-n}\right]^{-\frac{1}{n+1}},$$

where the constants $\alpha = ((n-1)!)^{-1/n}$ and $\beta = (n!\zeta(n+1))^{-1/n}$ are best possible. Our results lead to some new lower and upper bounds for the Riemann zeta function.

The proofs of our main results are based on the following lemmas.

2. Lemmas

Lemma 2.1. *For a fixed real number $x > 0$ let*

$$h_x(s) = \left((s-1)H_s(x)\right)^{1/(s-1)}. \tag{2.1}$$

Then h_x is strictly decreasing for $s > 1$; (see [1]*).*

Lemma 2.2. *For $s > 2$ and $x > 0$ we have*

$$\frac{s^2-1}{s^2} < \frac{(H_{s+1}(x))^2}{H_s(x)H_{s+2}(x)} < 1. \tag{2.2}$$

Proof. The proof follows from slight modifications in the proof of Theorem 2.1 and Corollary 2.3 of [4]. ∎

Lemma 2.3. *Let x and s be positive real numbers with $s > 1$ and*

$$\Delta(s,x) = H_{s+1}^{-1}\left(\frac{1}{sx^s}\right) - x, \tag{2.3}$$

where $H_q^{-1}(t)$ is the inverse of the function $t \to H_q(t)$. Then we have

(a) $\frac{\partial\Delta(s,x)}{\partial x} > 0$.

(b) $0 < \Delta(s,x) < 1/2$.

(c) $\frac{\partial\Delta(s,x)}{\partial s} < 0$.

(d) $\frac{\partial^2\Delta(s,x)}{\partial^2 x} < 0$.

Proof. We define for $s > 0$ and a fixed real number $x > 0$

$$\phi(x,s) = \frac{1}{h_x(s+1)},$$

where h_x is defined by (2.1). Using (2.3), we get for all $x > 0$, and $s > 1$

$$\Delta(s,\phi(x,s)) = x - \phi(x,s). \tag{2.4}$$

Differentiation with respect to x gives for $x > 0$ and $s > 1$

$$\frac{\partial\Delta(s,\phi(x,s))}{\partial x} = \frac{1}{\partial\phi(x,s)/\partial x} - 1. \tag{2.5}$$

Since

$$\frac{\partial\phi(x,s)}{\partial x} = \left[\frac{((s+1)H_{s+2}(x))^{1/(s+1)}}{(sH_{s+1}(x))^{1/s}}\right]^{s+1} = \left[\frac{h_x(s+2)}{h_x(s+1)}\right]^{s+1} \tag{2.6}$$

and h_x is strictly decreasing for $x > 0$ and $s > 1$ by Lemma 2.1, (2.5) implies that $\frac{\partial\Delta(s,\phi(x,s))}{\partial x} > 0$ for all $x > 0$ and $s > 1$. But since the mapping $x \to \phi(x,s)$ is strictly increasing and continuous on $(0,\infty)$ it is bijective, leading to $\frac{\partial\Delta(s,x)}{\partial x} > 0$ for all $x > 0$ and $s > 1$. This proves (a). From (2.3) we get for $x > 0$ and $s > 1$

$$H_{s+1}(x + \Delta(s,x)) = \frac{1}{sx^s}. \tag{2.7}$$

Also, from the definition of $H_s(x)$ we have $H_s'(x) = -sH_{s+1}(x)$ and

$$H_s(x+1) - H_s(x) = -\frac{1}{x^s}. \tag{2.8}$$

If we apply the mean value theorem for differentiation to $H_s(t)$ on the interval $[x, x+1]$ and use the relation (2.8), there exists an ε, depending on x and s, such that

$$sH_{s+1}(x + \varepsilon(x,s)) = \frac{1}{x^s}, \tag{2.9}$$

with $0 < \varepsilon = \varepsilon(x) < 1$, for all $x > 0$ and $s > 1$. From (2.7) and (2.9) we conclude $\varepsilon(x,s) = \Delta(s,x)$ that gives $0 < \Delta(s,x) < 1$ for all real numbers $x > 0$ and $s > 1$. Since the function $x \to \Delta(s,x)$ is bounded and monotonic increasing, it has a limit as x tends to infinity. In order to compute this limit we replace x by $x+1$ in (2.7) and use (2.8) to get

$$\frac{1}{(x + \Delta(s, x+1))^{s+1}} = H_{s+1}(x + \Delta(s, x+1)) - \frac{1}{s(x+1)^s}$$

or

$$\Delta(s, x+1) = \frac{1}{\left(H_{s+1}\left(x + \Delta(s,x+1)\right) - \frac{1}{s(x+1)^s}\right)^{1/(s+1)}} - x.$$

Since $\lim_{x\to\infty} \Delta(s, x+1) = \lim_{x\to\infty} \Delta(s,x)$, using (2.7) this becomes

$$\begin{aligned} \lim_{x\to\infty} \Delta(s,x) &= \lim_{x\to\infty} \left[\left(H_{s+1}\left(x + \Delta(s,x)\right) - \frac{1}{s(x+1)^s}\right)^{-1/(s+1)} - x\right] \\ &= \lim_{x\to\infty} \left[\left(\frac{sx^s(x+1)^s}{(x+1)^s - x^s}\right)^{1/(s+1)} - x\right]. \end{aligned}$$

Using L'Hospital's rule it is easy to see that the value of this limit is $1/2$. From (2.3) we also have $\Delta_s(0) = 0$, so that we get from (a) for $x > 0$ and $s > 1$ that

$$0 = \Delta(s,0) < \Delta(s,x) < \lim_{x\to\infty} \Delta(s,x) = 1/2.$$

This proves (b). For our convenience we let $y = \phi(x,s)$. Differentiating both sides of (2.4) with respect to s yields

$$\frac{\partial \Delta(s,y)}{\partial s} + \left(1 + \frac{\partial \Delta(s,y)}{\partial y}\right) \frac{\partial y}{\partial s} = 0. \tag{2.10}$$

Since

$$\frac{\partial y}{\partial s} = \frac{\partial \phi(x,s)}{\partial s} > 0$$

by Lemma 2.1 and

$$\frac{\partial \Delta(s,y)}{\partial y} > 0$$

by (a), we conclude from (2.10) that

$$\frac{\partial \Delta(s,x)}{\partial s} < 0$$

for all $x > 0$ and $s > 1$. This proves (c). Now we are ready to prove (d). Differentiation of both sides of (2.5) with respect to x gives

$$\frac{\partial^2 \Delta(s,\phi(x,s))}{\partial x^2} = -\frac{\partial^2 \phi(x,s)/\partial x^2}{\left(\partial \phi(x,s)/\partial x\right)^2},$$

so that to prove (d) it suffices to see

$$\frac{\partial^2 \phi(x,s)}{\partial x^2} > 0.$$

If we differentiate both sides of (2.6) with respect to x, we obtain for all $s > 1$ and $x > 0$

$$\begin{aligned}\frac{\partial^2 \phi(x,s)}{\partial x^2} &= (s+1)^3 \left(sH_{s+1}(x)\right)^{2+1/s} \\ &\times \left[(H_{s+2}(x))^2 - \frac{(s+1)^2 - 1}{(s+1)^2} H_{s+1}(x) H_{s+3}(x)\right].\end{aligned}$$

Applying Lemma 2.2 this reveals that

$$\frac{\partial^2 \phi(x,s)}{\partial x^2} > 0. \tag{2.11}$$

□

Now we are ready to establish our main results.

3. Main Results

Theorem 3.1. *For all real numbers $x > 0$ and $s > 1$ we have*

$$\frac{1}{s}\exp\left(-s\psi(x+1/2)\right) < H_{s+1}(x) < \frac{1}{s}\exp(-s\psi(x)), \tag{3.1}$$

where ψ is the logarithmic derivative of the classical gamma function, known as digamma function.

Proof. Since $s \to \Delta(s,x)$ is strictly decreasing by Lemma 2.3 (c), we have for $s > 1$ and $x > 0$ that

$$\Delta(s,x) < \Delta(1,x) = (\psi')^{-1}(1/x) - x. \tag{3.2}$$

In [9], the author proved that

$$(\psi')^{-1}(1/x) - x < \psi^{-1}(\log x) - x.$$

Thus, by virtue of (2.3), (3.2) gives for $s > 1$ and $x > 0$

$$H_{s+1}^{-1}\left(\frac{1}{sx^s}\right) < \psi^{-1}(\log x).$$

Replacing x by $e^{\psi(x)}$ here and using the fact that the function $x \to H_s^{-1}(x)$ is decreasing for $s > 1$, we obtain

$$H_{s+1}(x) < \frac{1}{s}e^{-s\psi(x)}.$$

for all $x > 0$ and $s > 1$. This proves the right hand side of (3.1). Since $\Delta(s,x) < 1/2$ by Lemma 2.3.b and

$$\psi^{-1}(\log x) - x < 1/2$$

by [8], we obtain

$$\psi^{-1}(\log x) - x - \Delta(s,x) < 1/2$$

or using (2.3)

$$\psi^{-1}(\log x) - H_{s+1}^{-1}\left(\frac{1}{sx^s}\right) < 1/2.$$

Replacing x by $e^{\psi(x)}$ here again, we get

$$x-\frac{1}{2}<H_{s+1}^{-1}\left(\frac{1}{s}\exp\left(-s\psi(x)\right)\right).$$

Therefore for $x>1/2$ we find that

$$H_{s+1}\left(x-1/2\right)>\frac{1}{s}\exp\left(-s\psi(x)\right).$$

Replacing x by $x+\frac{1}{2}$ here this becomes

$$H_{s+1}(x)>\frac{1}{s}\exp\left(-s\psi(x+1/2)\right).$$

□

This completes the proof of Theorem 3.1.

Theorem 3.2. *Let x, p and q be positive real numbers with $1<p<q$. Then we have*

$$\left(pH_{p+1}(x+1/2)\right)^{1/p}<\left(qH_{q+1}(x)\right)^{1/q}<\left(pH_{p+1}(x)\right)^{1/p}. \tag{3.3}$$

Proof. The right inequality follows from Lemma 2.1. Applying Lemma 2.3 (b) we get for $1<p<q$

$$\Delta(p,x)-\Delta(q,x)<1/2.$$

Using (2.3) this can be rewritten as

$$H_{p+1}^{-1}\left(\frac{1}{px^p}\right)-\frac{1}{2}<H_{q+1}^{-1}\left(\frac{1}{qx^q}\right).$$

Replacing x by

$$\frac{1}{\left(pH_{p+1}(x)\right)^{1/p}}$$

here we get for $x>1/2$

$$x-\frac{1}{2}<H_{q+1}^{-1}\left(\frac{1}{q}\left(pH_{p+1}(x)\right)^{q/p}\right).$$

This is equivalent to

$$H_{q+1}(x-1/2)>\frac{1}{q}\left(pH_{p+1}(x)\right)^{q/p}.$$

Replacing x by $x+1/2$ here finishes the proof of Theorem 3. 2. □

Theorem 3.3. *For all real numbers $x > 0$ and $s > 2$ we have*

$$(\zeta(s))^{-1/(s-1)} < (H_s(x+1))^{-1/(s-1)} - (H_s(x))^{-1/(s-1)} < (s-1)^{1/(s-1)},$$

where $\zeta(s)$ is the Riemann zeta function. Both of the bounds are best possible.

Proof. For a fixed $s > 0$ and a positive real number x let

$$\varphi(x) = \Delta\left(s, \phi(x+1, s)\right) - \Delta\left(s, \phi(x, s)\right).$$

Then if we use (2.4) we find that

$$\varphi(x) = u(x) - u(x+1) + 1, \tag{3.4}$$

where $u(x) = \phi(x, s)$. By (2.11) we have $u''(x) > 0$ so that u' is strictly increasing on $(0, \infty)$. Therefore (3.4) implies that φ is strictly decreasing. This allows us to write $\varphi(\infty) < \varphi(x) < \varphi(0)$ for all $x > 0$. Since

$$\varphi(0) = 1 - \phi(1, s) = 1 - (sH_{s+1}(1))^{-1/s} = 1 - (s\zeta(s+1))^{-1/s}$$

and $\varphi(\infty) = \lim_{x\to\infty} \varphi(x) = 1/2 - 1/2 = 0$, we obtain that

$$0 < 1 - \phi(x+1, s) + \phi(x, s) < 1 - (s\zeta(s+1))^{-1/s}.$$

Replacing the value of $\phi(x, s)$ here and then rearranging these inequalities, we complete the proof of Theorem 3.3. $\square$

The Hurwitz zeta function has been investigated by many authors from many different directions and there have been a lot of literature about it, but the inverse of it has not been investigated sufficiently, and we know a little concerning it. In the following theorem, which provides a beautiful application of Theorem 3.3, we establish sharp upper and lower bounds for the inverse of the Hurwitz zeta function: $x \to H_s^{-1}(x)$.

Theorem 3.4. *For all $s > 2$ and $x > 0$ the following double inequality holds:*

$$\left[x - \left(x^{1/(1-s)} + a\right)^{1-s}\right]^{-1/s} < H_s^{-1}(x) < \left[x - \left(x^{1/(1-s)} + b\right)^{1-s}\right]^{-1/s},$$

where $a = (s-1)^{1/(s-1)}$ *and* $b = (\zeta(s))^{1/(1-s)}$ *are best possible constants.*

Proof. If we use (2.8) and replace x by $H_s^{-1}(x)$ in Theorem 3.3 and then rearrange the resulting inequality we finish the proof. $\square$

4. Remarks

Remark 4.1. In [7, Proposition 3] it was obtained that the inequality

$$\zeta(s) < \frac{e^{(s-1)\gamma}}{s-1} \tag{4.1}$$

holds for all $s > 1$. Now if we set $x = 1/2$ and $x = 1$ in (3.1),respectively, we get the following new bounds for the Riemann zeta function:

$$\frac{e^{(s-1)\gamma}}{(s-1)(2^s-1)} < \zeta(s) < \frac{4^{s-1}e^{-\gamma s}}{(s-1)(2^{s-1}-1)}$$

and

$$\frac{e^{-(2-\gamma)(s-1)}}{s-1} < \zeta(s) < \frac{e^{(s-1)\gamma}}{s-1}$$

for all $s > 2$. Here γ is the Euler's constant. Thus, Theorem 3.1 gives a generalization and a converse of (4.1).

Remark 4.2. If we set $x = 1$ and $x = 1/2$ in Theorem 3.3, we find the following new upper bounds for the Riemann zeta function:

$$\zeta(s) < \frac{1}{1-2^{1-s}}$$

and

$$\zeta(s) < \frac{2^s}{2^s - 1 - \left(1 + (2^s-1)^{1/(1-s)}\right)^{1-s}},$$

respectively.

Remark 4.3. Theorems 3.1 and 3.2 don't cover the cases $0 < s \leq 1$ and $0 < p \leq 1$, respectively but we believe that they are also valid for these values of s and p. Similarly, Theorems 3.3 and 3.4 do not include the case $1 < s \leq 2$. We believe that they are also valid for these s.

Remark 4.4. Numerical computations carried by the computer program Mathematica indicate that the constant $1/2$ in the left inequalities of (3.1) and (3.3) can be replaced by a much smaller number.

Remark 4.5. We conjecture that the function

$$\theta(s) = \left(1 - \frac{1}{\zeta(s+1)}\right)^{1/s}$$

is strictly increasing for $s > 0$.

Remark 4.6. We also conjecture that the function $x \to \frac{\partial \Delta(s,x)}{\partial x}$ is completely monotonic for all $s > 1$ and $x > 0$. Recall that a function f is completely monotonic on an interval I if f has derivatives of all order on I such that $(-1)^n f^{(n)}(x) \geq 0$ on I, for all non-negative integers n.

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