ON NEW NOTIONS OF ORTHOGONALITY IN NORMED SPACES VIA THE 2-HH-NORMS

EDER KIKIANTY AND SEVER S. DRAGOMIR

Abstract. Kikianty and Dragomir in 2008 introduced the p-HH-norms on the Cartesian product of two copies of a normed space, which are equivalent to the well-known p-norms. In this paper, notions of orthogonality in terms of 2-HH-norms are introduced. The main properties of these orthogonalities are discussed. Several characterizations of inner product spaces are established, as well as the characterization of strictly convex spaces.

1. Introduction

An real inner product space $\mathbf{X}$ is a real vector space equipped with a mapping $\langle \cdot, \cdot \rangle : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ which satisfies the following properties

(1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,
(3) $\langle x, y\rangle = \langle y, x\rangle$,
(4) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$,

for all $x, y, z \in \mathbf{X}$ and $\alpha \in \mathbb{R}$. A real normed space $\mathbf{X}$ is a real vector space equipped with a mapping $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ which satisfies the following properties:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$,
(2) $\|\alpha x\| = |\alpha|\|x\|$,
(3) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality),

for all $x, y \in \mathbf{X}$ and $\alpha \in \mathbb{R}$. Every inner product induces a norm by the following identity: $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ $(x \in \mathbf{X})$. This norm satisfies the *parallelogram law*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \text{for all} \quad x, y \in \mathbf{X}.$$

Every norm satisfying the parallelogram law is induced by the inner product

$$\langle x, y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2], \quad \text{for all} \quad x, y \in \mathbf{X}.$$

Therefore an inner product space is a normed space, but not conversely. For example, the space $l^p := \{(x_n) : x_n \in \mathbb{R} | \sum |x_n|^p < \infty\}$, for $p \neq 2$, is a normed space, but not an inner product space.

In an inner product space $(\mathbf{X}, \langle \cdot, \cdot \rangle)$, a vector $x \in \mathbf{X}$ is said to be orthogonal to $y \in \mathbf{X}$ (denoted by $x \perp y$) if the inner product $\langle x, y\rangle$ is zero. Since a normed space is not necessarily an inner product space, we cannot define orthogonality in any normed space, in the same manner to that of inner product space. Numerous notions of orthogonality in normed spaces have been introduced via equivalent propositions to the usual orthogonality in inner product spaces, e.g. orthogonal vectors satisfy the Pythagorean law. For more results on these notions of orthogonality, their main properties, and the implication as well as equivalent statements amongst them, we refer to the survey papers by Alonso and Benitez [1, 2].

1991 *Mathematics Subject Classification.* 46B20, 46C05.

Key words and phrases. orthogonality, Pythagorean orthogonality, Isosceles orthogonality, inner product space, strictly convex space, Cartesian product space.

Any pair of vectors in a normed space $(\mathbf{X}, \|\cdot\|)$ can be viewed as an element of the Cartesian product space $\mathbf{X}^2$. The space $\mathbf{X}^2$ is again a normed space, when it is equipped with any of the well known p-norms. In 2008, Kikianty and Dragomir [6] introduced the p-HH-norms $(1 \le p < \infty)$ on the vector space $\mathbf{X}^2$ of pairs of elements x and y in $\mathbf{X}$ as follows:

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{\frac{1}{p}}.$$

These norms are equivalent to the p-norms but, unlike the p-norms, they do not depend only on the norms of the two elements in the pair, but also reflect the relative position of the two elements within the original space $\mathbf{X}$.

In particular, when $\mathbf{X}$ is an inner product space, the 2-HH-norm is induced by an inner product in $\mathbf{X}^2$, and

$$\|(x,y)\|_{2-HH}^2 = \frac{1}{3}\left(\|x\|^2 + \langle x, y\rangle + \|y\|^2 \right). \tag{1.1}$$

Note that when x is orthogonal to y, the inner product $\langle x, y\rangle$ vanishes, and the right-hand side of (1.1) reduces to $\frac{1}{3}\left(\|x\|^2 + \|y\|^2\right)$. This motivates us to consider a notion of orthogonality in normed spaces, where x is said to be orthogonal to y, when $\|(x,y)\|_{2-HH}^2 = \frac{1}{3}\left(\|x\|^2 + \|y\|^2\right)$. We discuss the properties of this orthogonality in Section 3 and establish some characterizations of inner product spaces. However, we also note that this orthogonality is closely related to the Pythagorean orthogonality (for references see [4]). In the same manner, we consider another notion of orthogonality, which is closely related to James' Isosceles orthogonality. The definition and its main properties will be discussed in Section 4. We also establish some characterizations of inner product spaces, as well as strictly convex spaces.

2. Definitions, notation and preliminary results

All definitions, notation and preliminary results regarding the orthogonality in normed spaces are summarized in this section for references. Note that throughout the paper, all linear spaces are considered over the field of real numbers.

2.1. **Orthogonality in inner product spaces.** The following are the main properties of orthogonality in inner product space (for references see [1, 4, 7]). In the study of orthogonality in normed space, these properties are investigated to see how "close" the definition is to the usual one. In any inner product space $(\mathbf{X}, \langle\cdot,\cdot\rangle)$, let $x, y, z \in \mathbf{X}$. Then,

(1) If $x \perp x$, then $x = 0$ (*Nondegeneracy*);
(2) If $x \perp y$, then $\lambda x \perp \lambda y$ for all $\lambda \in \mathbb{R}$ (*Simplification*);
(3) If $(x_n), (y_n) \subset \mathbf{X}$ such that $x_n \perp y_n$ for every $n \in \mathbb{N}$, $x_n \to x$ and $y_n \to y$, then $x \perp y$ (*Continuity*);
(4) If $x \perp y$, then $\lambda x \perp \mu y$ for all $\lambda, \mu \in \mathbb{R}$ (*Homogeneity*);
(5) If $x \perp y$ then $y \perp x$ (*Symmetry*);
(6) If $x \perp y$ and $x \perp z$ then $x \perp (y + z)$ (*Additivity*);
(7) If $x \neq 0$, then there exists $\alpha \in \mathbb{R}$ such that $x \perp (\alpha x + y)$ (*Existence*);
(8) The above α is unique (*Uniqueness*);

Remark 1 (Existence and uniqueness)**.** Alonso and Benitez in [1, p. 2] defined the *existence* (and *uniqueness*) as follows: For every oriented plane P, every $x \in P\setminus\{0\}$ and every $\rho > 0$, there exists (a unique) $y \in P$ such that the pair $[x, y]$ is in the given orientation, $\|y\| = \rho$ and $x \perp y$. The definition for uniqueness as stated in this paper is due to James [4, p. 292] (in the paper by Partington [7], this property is referred to as *resolvability*). Alonso and Benitez noted that when the orthogonality is non homogeneous, the existence in James' sense is not equivalent to the one that they have stated. They also noted that, in the sense of their definition, the existence implies that for any nonzero vector x, the set $\{\alpha : x \perp \alpha x + y\}$ is a non empty compact interval. Therefore, in investigating the uniqueness, for non homogeneous orthogonalities, they refer to James' result as the α-uniqueness property, where the (above) interval is reduced to a point. However, in this paper, we will use James' result as our definition of existence, and we refer to the α-uniqueness as uniqueness as initially stated in this paper.

2.2. **Orthogonality in normed spaces.** In this section, we recall two definitions of orthogonality, namely the Pythagorean and James' Isosceles orthogonalities. For other definitions of orthogonality, we refer to the work by Alonso and Benitez [1].

Definition 1. Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$.

(1) Pythagorean (1945), [7]:

$$x \perp y \ (P) \quad \text{iff} \quad \|x\|^2 + \|y\|^2 = \|x+y\|^2;$$

(2) Isosceles (1945), [4]:

$$x \perp y \ (I) \quad \text{iff} \quad \|x+y\| = \|x-y\|.$$

Remark 2. Pythagorean orthogonality is initially defined as follows (see [4]):

$$x \perp y \ (P) \quad \text{iff} \quad \|x\|^2 + \|y\|^2 = \|x-y\|^2.$$

However, the results remain similar when we consider the definition as stated in Definition 1.

Pythagorean orthogonality satisfies the following properties (see [1, 4]):

(1) Nondegeneracy, simplification, continuity and symmetry.
(2) If x, y are elements of a normed space $\mathbf{X}$, where $x \neq 0$, then there exists a number α such that $x \perp \alpha x + y$ (P), i.e. P-orthogonality is existent;
(3) P-orthogonality is unique;
(4) If P-orthogonality is homogeneous (additive) in $\mathbf{X}$, then $\mathbf{X}$ is an inner product space;

Isosceles orthogonality satisfies the following properties (see [1, 4]):

(1) Nondegeneracy, simplification, continuity and symmetry.
(2) If x, y are elements of a normed space $\mathbf{X}$, where $x \neq 0$, then there exists a number α such that $x \perp \alpha x + y$ (I).
(3) Isosceles orthogonality is unique if and only if $\mathbf{X}$ is strictly convex;
(4) If I-orthogonality is homogeneous (additive), then $\mathbf{X}$ is an inner product space;

2.3. **The p-HH-norms on $\mathbf{X}^2$.** The Cartesian product space is defined by $\mathbf{X}^2 = \mathbf{X} \times \mathbf{X} := \{(x, y) : x, y \in \mathbf{X}\}$, where the addition and scalar multiplication are defined in the usual way. This space is also a normed space together with any of the well-known p-norms, which can be defined as follows:

$$\|(x,y)\|_p := \begin{cases} (\|x\|^p + \|y\|^p)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \max\{\|x\|, \|y\|\}, & p = \infty, \end{cases}$$

for any $(x, y) \in \mathbf{X}^2$.

Kikianty and Dragomir in [6] introduced a family of norms that can be defined in this space, which is referred to as the p-HH-norms.

Definition 2. The p-HH-norm on $\mathbf{X}^2$ is defined by

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{\frac{1}{p}}, \tag{2.1}$$

for any $(x, y) \in \mathbf{X}^2$ and $1 \le p < \infty$.

The p-HH-norm is symmetric, i.e., $\|(x,y)\|_{p-HH} = \|(y,x)\|_{p-HH}$. All the p-norms and the p-HH-norms are equivalent in $\mathbf{X}^2$.

Note that if the norm $\|\cdot\|$ on $\mathbf{X}$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then the 2-HH-norm has the explicit form as follows:

$$\|(x,y)\|_{2-HH}^2 = \int_0^1 \|(1-t)x+ty\|^2 dt = \frac{1}{3}\left(\|x\|^2 + \langle x,y\rangle + \|y\|^2\right). \tag{2.2}$$

For fundamental properties of this family of norms, see [6].

3. HH-P-orthogonality

Let $(\mathbf{X},\langle\cdot,\cdot\rangle)$ be an inner product space. For any $x,y\in\mathbf{X}$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2,$$

and when x is orthogonal to y

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2. \tag{3.1}$$

This is the motivation to define orthogonality in any normed space, namely the *Pythagorean orthogonality* (see Subsection 2.2).

If we consider x and y as pair of vectors in the space $\mathbf{X}^2$ equipped with the 2-HH-norm, and if $\mathbf{X}$ is an inner product space, then 2-HH-norm of pair of vectors (x,y) in $\mathbf{X}^2$ is

$$\|(x,y)\|_{2-HH}^2 = \int_0^1 \|(1-t)x+ty\|^2\ dt = \frac{1}{3}(\|x\|^2 + \langle x,y\rangle + \|y\|^2).$$

If $x\perp y$, i.e., $\langle x,y\rangle = 0$, then

$$\int_0^1 \|(1-t)x+ty\|^2\ dt = \frac{1}{3}(\|x\|^2 + \|y\|^2), \tag{3.2}$$

Therefore, we can consider a notion of orthogonality as follows:

Definition 3. In any normed space $(\mathbf{X},\|\cdot\|)$, a vector $x\in\mathbf{X}$ is said to be *HH-P-orthogonal* to $y\in\mathbf{X}$ if and only if they satisfy (3.2); and we denote it by $x\perp_{HH-P} y$.

It is easy to check that the HH-P-orthogonality is equivalent to the usual orthogonality, if the space is equipped with an inner product.

The following lemma discusses some of the main properties of HH-P-orthogonality and will be used to prove the other properties.

Lemma 1. *The HH-P-orthogonality satisfies the nondegeneracy, simplification, continuity and symmetry properties.*

Proof. If $x\perp_{HH-P} x$, then $\|x\|^2 = \int_0^1 \|(1-t)x+tx\|^2 dt = \frac{1}{3}(\|x\|^2+\|x\|^2) = \frac{2}{3}\|x\|^2$, which implies that $\|x\| = 0$, i.e., $x = 0$, which proves the nondegeneracy property. If $x\perp_{HH-P} y$, then for any $\lambda\in\mathbb{R}$,

$$\begin{aligned}\int_0^1 \|(1-t)\lambda x + t\lambda y\|^2 dt &= |\lambda|^2 \int_0^1 \|(1-t)x+ty\|^2 dt \\ &= \frac{|\lambda|^2}{3}(\|x\|^2+\|y\|^2) = \frac{1}{3}(\|\lambda x\|^2 + \|\lambda y\|^2),\end{aligned}$$

i.e., $\lambda x\perp_{HH-P}\lambda y$, for any $\lambda\in\mathbb{R}$. If $x\perp_{HH-P} y$, then

$$\int_0^1 \|(1-t)y+tx\|^2 dt = \int_0^1 \|(1-t)x+ty\|^2 dt = \frac{1}{3}(\|x\|^2+\|y\|^2) = \frac{1}{3}(\|y\|^2+\|x\|^2),$$

(since 2-HH-norm is symmetric) i.e. the HH-P-orthogonality is symmetric. If $x_n \to x$, $y_n \to y$ and $x_n \perp_{HH-P} y_n$ for any $n \in \mathbb{N}$, then by the continuity of norm,

$$\begin{aligned}\int_0^1 \|(1-t)x+ty\|^2 dt &= \int_0^1 \lim_{n\to\infty} \|(1-t)x_n+ty_n\|^2 dt \\ &= \lim_{n\to\infty} \int_0^1 \|(1-t)x_n+ty_n\|^2 dt \\ &= \lim_{n\to\infty} \left(\frac{1}{3}(\|x_n\|^2+\|y_n\|^2)\right) = \frac{1}{3}(\|x\|^2+\|y\|^2),\end{aligned}$$

which shows the continuity. □

Remark 3. In general, the HH-P-orthogonality is not right-additive nor homogeneous. For example, choose $\mathbf{X} = \mathbb{R}^2$ equipped with l^1-norm, $x = (0, -1)$ and $y = (1, \sqrt[3]{2} - 1)$. Note that $x \perp_{HH-P} y$, but $x \not\perp_{HH-P} 2y$.

The next theorem shows that the HH-P-orthogonality is existent in any normed linear space. This property is the most important, since it would keep the concept of orthogonality from being vacuous [4, p. 292].

Theorem 1. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, the HH-P-orthogonality is existent, i.e. for any $x, y \in \mathbf{X}$, there exists an $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp_{HH-P} x$.*

Proof. We will prove this theorem by the similar continuity argument and the intermediate value theorem, which was used by James in [4, p. 299–300]. Fix $x, y \in \mathbf{X}$, where $x \neq 0$ (as the proof is trivial for the case of $x = 0$) and let $f : \mathbb{R} \times (0, 1) \to \mathbb{R}$ be a function defined by

$$f(\alpha,t) := \|(1-t)(\alpha x+y)\|^2 + \|tx\|^2 - \|(1-t)(\alpha x+y)+tx\|^2, \tag{3.3}$$

and F be a function on $\mathbb{R}$ defined by $F(\alpha) := \int_0^1 f(\alpha, t)\, dt$. We will show that the continuous function F has the value of zero for some α by using the intermediate value theorem.

Since $t \neq 1$, we have the following identity

$$1 = -\frac{2t(1-t)\alpha+t^2}{(1-t)^2\alpha^2} + \left(1+\frac{t}{(1-t)\alpha}\right)^2.$$

Therefore, we may write

$$\begin{aligned}f(\alpha,t) &= t^2\|x\|^2 - \frac{2t(1-t)\alpha+t^2}{(1-t)^2\alpha^2}\|(1-t)(\alpha x+y)\|^2 \\ &\quad + \left(1+\frac{t}{(1-t)\alpha}\right)^2 \|(1-t)(\alpha x+y)\|^2 - \|[(1-t)\alpha+t]x+(1-t)y\|^2 \\ &= t^2\|x\|^2 - \left[2t(1-t)\alpha+t^2\right]\left\|x+\frac{y}{\alpha}\right\|^2 \\ &\quad + \left\|[(1-t)\alpha+t]x+\left[(1-t)+\frac{t}{\alpha}\right]y\right\|^2 - \|[(1-t)\alpha+t]x+(1-t)y\|^2 \\ &= t^2\|x\|^2 - \left[2t(1-t)\alpha+t^2\right]\left\|x+\frac{y}{\alpha}\right\|^2 \\ &\quad + \left[\left\|[(1-t)\alpha+t]x+\left[(1-t)+\frac{t}{\alpha}\right]y\right\| - \|[(1-t)\alpha+t]x+(1-t)y\|\right] \\ &\quad \times \left[\left\|[(1-t)\alpha+t]x+\left[(1-t)+\frac{t}{\alpha}\right]y\right\| + \|[(1-t)\alpha+t]x+(1-t)y\|\right]\end{aligned}$$

$$\begin{aligned}
&\le t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left\|x + \frac{y}{\alpha}\right\|^2 + \left|\frac{t}{\alpha}\right|\|y\| \\
&\quad \times \left[\left\|[(1-t)\alpha + t]x + \left[(1-t) + \frac{t}{\alpha}\right]y\right\| + \|[(1-t)\alpha + t]x + (1-t)y\|\right] \\
&\le t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left\|x + \frac{y}{\alpha}\right\|^2 \\
&\quad +2\left|\frac{t}{\alpha}\right||(1-t)\alpha + t|\|x\|\|y\| + \left|\frac{t}{\alpha}\right|\left[\left|(1-t) + \frac{t}{\alpha}\right| + (1-t)\right]\|y\|^2.
\end{aligned}$$

Assume $\alpha > 0$, then $\frac{t}{\alpha} > 0$, $(1-t)\alpha + t > 0$, $(1-t) + \frac{t}{\alpha} > 0$, and $2t(1-t)\alpha + t^2 > 0$ and therefore

$$\begin{aligned}
f(\alpha, t) &\le t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left\|x + \frac{y}{\alpha}\right\|^2 \\
&\quad + \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&\le t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left(\|x\| - \left\|\frac{y}{\alpha}\right\|\right)^2 \\
&\quad + \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&= t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\|x\|^2 \\
&\quad +2\left[2t(1-t)\alpha + t^2\right]\|x\|\left\|\frac{y}{\alpha}\right\| - \left[2t(1-t)\alpha + t^2\right]\left\|\frac{y}{\alpha}\right\|^2 \\
&\quad + \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&= -2t(1-t)\alpha\|x\|^2 + \left[4t(1-t) + \frac{2t^2}{\alpha}\right]\|x\|\|y\| - \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&\quad + \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&= -2t(1-t)\alpha\|x\|^2 + \left[6t(1-t) + \frac{4t^2}{\alpha}\right]\|x\|\|y\|.
\end{aligned}$$

Integrate the last inequality with respect to t on $(0,1)$ to get

$$\begin{aligned}
F(\alpha) = \int_0^1 f(\alpha,t)dt &\le \int_0^1 \left\{-2t(1-t)\alpha\|x\|^2 + \left[6t(1-t) + \frac{4t^2}{\alpha}\right]\|x\|\|y\|\right\}dt \\
&= -\frac{1}{3}\alpha\|x\|^2 + \left[1 + \frac{4}{3\alpha}\right]\|x\|\|y\|.
\end{aligned}$$

By taking α sufficiently large, and since x is nonzero, we have

$$F(\alpha) = \int_0^1 f(\alpha,t)dt < 0.$$

Now, consider

$$\begin{aligned}
f(-\alpha, t) &= \|(1-t)(-\alpha x + y)\|^2 + \|tx\|^2 - \|(1-t)(-\alpha x + y) + tx\|^2 \\
&= \|(1-t)(\alpha x - y)\|^2 + \|tx\|^2 - \|(1-t)(\alpha x - y) - tx\|^2.
\end{aligned}$$

Since $t \ne 1$, we have the following identity

$$1 = \frac{2t(1-t)\alpha - t^2}{(1-t)^2\alpha^2} + \left(1 - \frac{t}{(1-t)\alpha}\right)^2.$$

Therefore,

$$\begin{aligned}
f(-\alpha,t) &= \|tx\|^2 + \frac{2t(1-t)\alpha - t^2}{(1-t)^2\alpha^2}\|(1-t)(\alpha x - y)\|^2 \\
&\quad + \left(1 - \frac{t}{(1-t)\alpha}\right)^2 \|(1-t)(\alpha x - y)\|^2 - \|[(1-t)\alpha - t]x - (1-t)y\|^2 \\
&= \|tx\|^2 + [2t(1-t)\alpha - t^2]\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad + \left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\|^2 - \|[(1-t)\alpha - t]x - (1-t)y\|^2 \\
&= \|tx\|^2 + [2t(1-t)\alpha - t^2]\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad + \left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| - \|[(1-t)\alpha - t]x - (1-t)y\|\right] \\
&\quad \times \left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| + \|[(1-t)\alpha - t]x - (1-t)y\|\right] \\
&\geq \|tx\|^2 + [2t(1-t)\alpha - t^2]\left\|x - \frac{y}{\alpha}\right\|^2 - \left|\frac{t}{\alpha}\right|\|y\| \\
&\quad \times \left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| + \|[(1-t)\alpha - t]x - (1-t)y\|\right] \\
&\geq \|tx\|^2 + [2t(1-t)\alpha - t^2]\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad - \left|\frac{t}{\alpha}\right|\|y\|\left[2|(1-t)\alpha - t|\ \|x\| + \left[\left|1 - t - \frac{t}{\alpha}\right| + 1 - t\right]\|y\|\right] \\
&= \|tx\|^2 + 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - t^2\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad - \left|\frac{t}{\alpha}\right|\|y\|\left[2|(1-t)\alpha - t|\ \|x\| + \left[\left|1 - t - \frac{t}{\alpha}\right| + 1 - t\right]\|y\|\right].
\end{aligned}$$

Since $t > 0$, $-t^2 < 0$ and by triangle inequality we get

$$\begin{aligned}
f(\alpha,t) &\geq \|tx\|^2 + 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - t^2\left(\|x\| + \left\|\frac{y}{\alpha}\right\|\right)^2 \\
&\quad -2\left|(1-t)t - \frac{t^2}{\alpha}\right|\ \|x\|\,\|y\| - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \left|\frac{(1-t)t}{\alpha}\right|\right]\|y\|^2 \\
&= \|tx\|^2 + 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - t^2\|x\|^2 - 2t^2\|x\|\left\|\frac{y}{\alpha}\right\| - t^2\left\|\frac{y}{\alpha}\right\|^2 \\
&\quad -2\left|(1-t)t - \frac{t^2}{\alpha}\right|\ \|x\|\,\|y\| - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \left|\frac{(1-t)t}{\alpha}\right|\right]\|y\|^2 \\
&= 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{|\alpha|}\right]\ \|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \left|\frac{(1-t)t}{\alpha}\right| + \frac{t^2}{\alpha^2}\right]\|y\|^2.
\end{aligned}$$

Assume $\alpha > 0$, then

$$\begin{aligned}
f(-\alpha,t) &\geq 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{\alpha}\right]\ \|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \frac{(1-t)t}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2
\end{aligned}$$

$$
\begin{aligned}
&\geq 2t(1-t)\alpha\|x\|^2 - 4t(1-t)\alpha\|x\|\left\|\frac{y}{\alpha}\right\| + 2t(1-t)\alpha\left\|\frac{y}{\alpha}\right\|^2 \\
&\quad - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{\alpha}\right]\|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \frac{(1-t)t}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&= 2t(1-t)\alpha\|x\|^2 - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{\alpha} + 4t(1-t)\right]\|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| - \frac{(1-t)t}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2.
\end{aligned}
$$

Integrate on $(0,1)$, to get

$$
\begin{aligned}
F(-\alpha) = \int_0^1 f(-\alpha,t)dt \quad &\geq \frac{1}{3}\alpha\|x\|^2 - \left[\frac{\alpha^3+3\alpha+2}{3\alpha(\alpha+1)^2} + \frac{2}{3\alpha} + \frac{2}{3}\right]\|x\|\|y\| \\
&\quad - \left[\frac{\alpha^3+3\alpha+2}{6\alpha^2(\alpha+1)^2} - \frac{1}{6\alpha} + \frac{1}{3\alpha^2}\right]\|y\|^2.
\end{aligned}
$$

For α sufficiently large, and since x is nonzero, we have

$$
F(-\alpha) = \int_0^1 f(-\alpha,t)dt > 0.
$$

We have shown that there exist $\alpha_1 > 0$ such that $F(\alpha_1) < 0$ and $\alpha_2 < 0$ such that $F(\alpha_2) > 0$. Since F is a continuous function in α, it follows that there must be an $\alpha_0 \in \mathbb{R}$ such that $F(\alpha_0) = 0$, i.e.,

$$
\begin{aligned}
0 = F(\alpha_0) &= \int_0^1 \|(1-t)(\alpha_0 x + y)\|^2 + \|tx\|^2 - \|(1-t)(\alpha_0 x + y) + tx\|^2 dt \\
&= \frac{1}{3}\left(\|(\alpha_0 x + y)\|^2 + \|x\|^2\right) - \int_0^1 \|(1-t)(\alpha_0 x + y) + tx\|^2 dt,
\end{aligned}
$$

or equivalently,

$$
\int_0^1 \|(1-t)(\alpha_0 x + y) + tx\|^2 dt = \frac{1}{3}\left(\|\alpha_0 x + y\|^2 + \|x\|^2\right),
$$

i.e., $(\alpha_0 x + y) \perp_{HH-P} x$. □

The following lemma will be used to prove the uniqueness property of the HH-P-orthogonality.

Lemma 2. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. Let g be a function on $\mathbb{R}$ defined by*

$$
g(k) := \int_0^1 \|(1-t)y + k(tx)\|^2 \, dt.
$$

Then, g is a convex function on $\mathbb{R}$, and furthermore, for any $s \in (0,1)$ and $k_1, k_2 \in \mathbb{R}$ where $g(k_1) \neq g(k_2)$, we have

$$
g[sk_1 + (1-s)k_2] < sg(k_1) + (1-s)g(k_2).
$$

Proof. Let $s \in (0,1)$ and $k_1, k_2 \in \mathbb{R}$, where $k_1 \neq k_2$. Then,

$$
\begin{aligned}
&g[sk_1 + (1-s)k_2] \\
&= \int_0^1 \|(1-t)y + [sk_1 + (1-s)k_2](tx)\|^2 \, dt \\
&= \int_0^1 \|s[(1-t)y + k_1 tx] + (1-s)[(1-t)y + k_2 tx]\|^2 \, dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq s^2\int_0^1 \|(1-t)y+k_1tx\|^2\,dt+(1-s)^2\int_0^1 \|(1-t)y+k_2tx\|^2\,dt\\
&\quad +2s(1-s)\int_0^1 \|(1-t)y+k_1tx\|\|(1-t)y+k_2tx\|\,dt\\
&= s\int_0^1 \|(1-t)y+k_1tx\|^2\,dt+(1-s)\int_0^1 \|(1-t)y+k_2tx\|^2\,dt\\
&\quad +(s^2-s)\left(\int_0^1 \|(1-t)y+k_1tx\|^2\,dt+\int_0^1 \|(1-t)y+k_2tx\|^2\,dt\right)\\
&\quad +2s(1-s)\int_0^1 \|(1-t)y+k_1tx\|\|(1-t)y+k_2tx\|\,dt\\
&= s\int_0^1 \|(1-t)y+k_1tx\|^2\,dt+(1-s)\int_0^1 \|(1-t)y+k_2tx\|^2\,dt \qquad (3.4)\\
&\quad -s(1-s)\left(\int_0^1 |\|(1-t)y+k_1tx\|-\|(1-t)y+k_2tx\||^2\,dt\right)\\
&\leq s\int_0^1 \|(1-t)y+k_1tx\|^2\,dt+(1-s)\int_0^1 \|(1-t)y+k_2tx\|^2\,dt\\
&= sg(k_1)+(1-s)g(k_2).
\end{aligned}
$$

Note that when equality holds, we conclude from (3.4) that then the term

$$\int_0^1 |\|(1-t)y+k_1tx\|-\|(1-t)y+k_2tx\||^2\,dt=0,$$

i.e. $\|(1-t)y+k_1tx\|-\|(1-t)y+k_2tx\|=0$ almost everywhere on $(0,1)$. It implies that

$$g(k_1)=\int_0^1 |\|(1-t)y+k_1tx\|^2dt=\int_0^1 \|(1-t)y+k_2tx\|^2dt=g(k_2).$$

Therefore, if $g(k_1)\neq g(k_2)$, then the inequality is strict. □

Theorem 2. *Let $(\mathbf{X},\|\cdot\|)$ be a normed space. Then, HH-P-orthogonality is unique on $\mathbf{X}$.*

Proof. The proof is inspired by that of Kapoor and Prasad in [5, p. 406]. Suppose that HH-P-orthogonality is not unique. Then there exist $x,y\in\mathbf{X}$, $x\neq 0$ and $\alpha>0$, such that

$$y\perp_{HH-P} x, \qquad (3.5)$$

and

$$\alpha x+y\perp_{HH-P} x. \qquad (3.6)$$

Recall the convex function g as defined in Lemma 2, and observe that

$$g(1)=\int_0^1 \|(1-t)y+tx\|^2\,dt=\frac{\|y\|^2}{3}+\frac{\|x\|^2}{3}=g(0)+\frac{\|x\|^2}{3}, \qquad (3.7)$$

by (3.5). Set $\alpha'(t)=\frac{(1-t)\alpha}{t}$, and observe that

$$
\begin{aligned}
g(\alpha'(t)+1) &= \int_0^1 \|(1-t)y+(\alpha'(t)+1)tx\|^2\,dt\\
&= \int_0^1 \|(1-t)(\alpha x+y)+tx\|^2\,dt\\
&= \frac{\|\alpha x+y\|^2}{3}+\frac{\|x\|^2}{3}=g(\alpha'(t))+\frac{\|x\|^2}{3}, \qquad (3.8)
\end{aligned}
$$

by (3.6). Now, suppose that $0 < \alpha'(t) < 1$, and note that $g(1) \neq g(0)$ (since $x \neq 0$) and Lemma 2 gives us

$$g(\alpha'(t)) < \alpha'(t)\ g(1) + (1-\alpha'(t))\ g(0). \tag{3.9}$$

Also, $g(\alpha'(t)+1) \neq g(\alpha'(t))$ (since $x \neq 0$), and Lemma 2 gives us

$$\begin{aligned} g(1) &< \alpha'(t)\ g(\alpha'(t)) + (1-\alpha'(t))\ g(\alpha'(t)+1) \\ &= \alpha'(t)\ g(\alpha'(t)) + (1-\alpha'(t))\ \left[g(\alpha'(t)) + \frac{\|x\|^2}{3}\right] \\ &= \alpha'(t)\ g(\alpha'(t)) + (1-\alpha'(t))\ \left[g(\alpha'(t)) + g(1) - g(0)\right], \end{aligned}$$

by (3.7) and (3.8). Therefore (by rearranging the last inequality), we have

$$\alpha'(t)g(1) + (1-\alpha'(t))g(0) < g(\alpha'(t)),$$

which contradicts (3.9).

Now, consider the case that $\alpha'(t) > 1$. We have,

$$g(1) \leq \frac{\alpha'(t)-1}{\alpha'(t)} g(0) + \frac{1}{\alpha'(t)} g(\alpha'(t)),$$

i.e.,

$$\frac{\|x\|^2}{3} = g(1) - g(0) \leq \frac{1}{\alpha'(t)}[g(\alpha'(t)) - g(0)].$$

Since $x \neq 0$, then, $[g(\alpha'(t)) - g(0)] \neq 0$ and therefore, $g(\alpha'(t)) \neq g(0)$. Thus, Lemma 2 gives us

$$g(1) < \frac{\alpha'(t)-1}{\alpha'(t)} g(0) + \frac{1}{\alpha'(t)} g(\alpha'(t)). \tag{3.10}$$

Also, $g(1) \neq g(\alpha'(t)+1)$ (since $g(\alpha'(t)) \neq g(0)$), and Lemma 2 gives us

$$\begin{aligned} g(\alpha'(t)) &< \frac{1}{\alpha'(t)}\ g(1) + \frac{\alpha'(t)-1}{\alpha'(t)}\ g(\alpha'(t)+1) \\ &= \frac{1}{\alpha'(t)} g(1) + \frac{\alpha'(t)-1}{\alpha'(t)} \left[g(\alpha'(t)) + \frac{\|x\|^2}{3}\right] \\ &= \frac{1}{\alpha'(t)} g(1) + \frac{\alpha'(t)-1}{\alpha'(t)} [g(\alpha'(t)) + g(1) - g(0)], \end{aligned}$$

by (3.7) and (3.8). Therefore (by rearranging the last inequality), we have

$$\frac{1}{\alpha'(t)} g(\alpha'(t)) + \frac{\alpha'(t)-1}{\alpha'(t)}\ g(0) < g(1),$$

which contradicts (3.10). For the case where $\alpha'(t) = 1$, we have

$$g(\alpha'(t)+1) = g(2) = \frac{\|x\|^2}{3} + g(1) = g(0) + \frac{2\|x\|^2}{3}.$$

Again, note that $g(0) \neq g(2)$ since $x \neq 0$, and Lemma 2 gives us

$$g(1) < \frac{1}{2}\ g(0) + \frac{1}{2} g(2) = g(0) + \frac{1}{3}\|x\|^2$$

which contradicts (3.7). Therefore, HH-P-orthogonality must be unique. □

The following theorems give some characterisations of inner product spaces.

Theorem 3. *Let* $\mathbf{X}$ *be a normed space. Then,* $\mathbf{X}$ *is an inner product space if and only if HH-P-orthogonality is homogeneous.*

Proof. We use a similar argument to that of James in [4, p. 301]. If $\mathbf{X}$ is an inner product space, then HH-P-orthogonality is equivalent to the usual orthogonality, and therefore it is homogeneous. Conversely, assume that the homogeneity property of HH-P-orthogonality holds, and let $x, y \in \mathbf{X}$. By existence, there exists an $\alpha \in \mathbf{X}$, such that

$$\int_0^1 \|(1-t)(\alpha x + y) + tx\|^2 dt = \frac{1}{3}\left(\|\alpha x + y\|^2 + \|x\|^2\right).$$

Since the homogeneity property holds, we have the following for any $k \in \mathbb{R}$

$$\int_0^1 \|(1-t)(\alpha x + y) + tkx\|^2 dt = \frac{1}{3}\left(\|\alpha x + y\|^2 + \|kx\|^2\right). \tag{3.11}$$

Assuming $t \in (0,1)$, we set $k = \frac{(1-t)(1-\alpha)}{t}$, and the left-hand side of (3.11) becomes

$$\int_0^1 \|(1-t)(\alpha x + y) + (1-t)(1-\alpha)x\|^2 dt = \|x+y\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3}\|x+y\|^2.$$

The right-hand side of (3.11) becomes

$$\frac{1}{3}\left(\|\alpha x + y\|^2 + \left\|\frac{(1-t)(1-\alpha)}{t}x\right\|^2\right) = \frac{1}{3}\left(\|\alpha x + y\|^2 + \frac{(1-t)^2(1-\alpha)^2}{t^2}\|x\|^2\right).$$

Thus, by (3.11), we have

$$\|x+y\|^2 = \|(\alpha x + y)\|^2 + \frac{(1-t)^2(1-\alpha)^2}{t^2}\|x\|^2.$$

Since $t \neq 0$, we have

$$t^2\|x+y\|^2 = t^2\|\alpha x + y\|^2 + (1-t)^2(1-\alpha)^2\|x\|^2.$$

Integrate with respect to t over $(0,1)$ to get

$$\|x+y\|^2 = \|\alpha x + y\|^2 + (1-\alpha)^2\|x\|^2. \tag{3.12}$$

Analogously, we set $k = \frac{-(1-t)(1+\alpha)}{t}$ for any $t \in (0,1)$, and the left-hand side of (3.11) becomes

$$\int_0^1 \|(1-t)(\alpha x + y) - (1-t)(1+\alpha)x\|^2 dt = \|x-y\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3}\|x-y\|^2.$$

The right-hand side of (3.11) becomes

$$\frac{1}{3}\left(\|\alpha x + y\|^2 + \frac{(1-t)^2(1+\alpha)^2}{t^2}\|x\|^2\right),$$

and thus

$$\|x-y\|^2 = \|\alpha x + y\|^2 + \frac{(1-t)^2(1+\alpha)^2}{t^2}\|x\|^2.$$

Since $t \neq 0$, we have

$$t^2\|x-y\|^2 = t^2\|\alpha x + y\|^2 + (1-t)^2(1+\alpha)^2\|x\|^2.$$

Integrate with respect to t over $(0,1)$ to get

$$\|x-y\|^2 = \|\alpha x + y\|^2 + (1+\alpha)^2\|x\|^2. \tag{3.13}$$

By adding (3.12) and (3.13), we get

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= 2\|\alpha x + y\|^2 + [(1-\alpha)^2 + (1+\alpha)^2]\|x\|^2 \\ &= 2\|\alpha x + y\|^2 + (2 + 2\alpha^2)\|x\|^2. \end{aligned} \tag{3.14}$$

Now, we note that by homogeneity, we also have $\alpha x + y \perp_{HH-P} \frac{(1-t)}{t}\alpha x$ for all $t \in (0,1)$, i.e.

$$\begin{aligned}\int_0^1 \|(1-t)(\alpha x + y) + (1-t)\alpha x\|^2 dt &= \|y\|^2 \int_0^1 (1-t)^2 dt \\ &= \frac{1}{3}\|y\|^2 \\ &= \frac{1}{3}\left(\|(\alpha x + y)\|^2 + \frac{(1-t)^2\alpha^2}{t^2}\|x\|^2\right).\end{aligned}$$

Since $t \neq 0$, we have

$$t^2\|y\|^2 = t^2\|\alpha x + y\|^2 + (1-t)^2\alpha^2\|x\|^2,$$

and integrate it over $(0,1)$ to get

$$\|y\|^2 = \|\alpha x + y\|^2 + \alpha^2\|x\|^2,$$

or equivalently,

$$\|\alpha x + y\|^2 = \|y\|^2 - \alpha^2\|x\|^2.$$

Therefore, (3.14) gives us

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|y\|^2 - \alpha^2\|x\|^2) + (2 + 2\alpha^2)\|x\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

and the proof is completed. □

Theorem 4. *The property of homogeneity and additivity of HH-P-orthogonality are equivalent.*

Proof. The proof is inspired by that of James [4, p. 301–302]. If the HH-P-orthogonality is homogeneous, then the underlying space is an inner product space and therefore is additive. Assume that the additivity property holds, and that $x \perp_{HH-P} y$. Consider x and $-y$, the existing property gives us an $\alpha \in \mathbb{R}$ such that $x \perp_{HH-P} \alpha x - y$. By additivity, we conclude that $x \perp \alpha x$. Therefore, $\alpha = 0$ when $x \neq 0$. Thus, $x \perp -y$. By symmetry and additivity, we conclude that $nx \perp my$ for all integers n and m. In particular, when $n \neq 0$,

$$\int_0^1 \left\|(1-t)x + t\left(\frac{m}{n}\right)y\right\|^2 dt = \frac{1}{3}\left(\|x\|^2 + \frac{m^2}{n^2}\|y\|^2\right),$$

which implies that $x \perp_{HH-P} ky$ for any $k \in \mathbb{Q}$. By the continuity of norm, $x \perp_{HH-P} ky$ for any $k \in \mathbb{R}$, and the proof is completed by the symmetry of HH-P-orthogonality. □

Corollary 1. *If HH-P-orthogonality is additive in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space.*

Remark 4.
1. If $x, y \in \mathbf{X}$ such that $(1-t)x \perp ty$ (P) for almost every $t \in [0,1]$, then, $x \perp_{HH-P} y$.
2. Note also that if $x, y \in \mathbf{X}$ such that $(1-t)x \perp ty$ (P) for almost every $t \in [0,1]$, then by the continuity of P-orthogonality, $(1-t)x \perp ty$ (P) for every $t \in [0,1]$; and furthermore, $\alpha x \perp \beta y$ (P) for any $\alpha, \beta \in \mathbb{R}$.
3. If $x \perp y$ (P) implies that $(1-t)x \perp ty$ (P), then the P-orthogonality is homogeneous, and therefore the underlying space is an inner product space. Thus, $x \perp_{HH-P} y$.
4. Note that $x \perp y$ (P) does not imply $x \perp_{HH-P} y$. For example, in $\mathbb{R}^2$ equipped with l^1-norm, $x = (-3,6)$ is P-orthogonal to $y = (8,4)$, but $x \not\perp_{HH-P} y$.

(5) Note that $x \perp_{HH-P} y$ does not imply $x \perp y$ (P) . For example, in $\mathbb{R}^2$ equipped with l^1-norm, $x=(2,1)$ is HH-P-orthogonal to $y=(\frac{11}{2}-\frac{\sqrt{145}}{2},1)$, but $x \not\perp y$ (P).

4. HH-I-orthogonality

Note that in any normed space $(\mathbf{X},\|\cdot\|)$, if $x,y\in\mathbf{X}$ such that $(1-t)x\perp ty$ (P) for almost every $t\in[0,1]$, then, $x\perp_{HH-P} y$. Using the same idea, we investigate that if $x,y\in\mathbf{X}$ such that $(1-t)x\perp ty$ (I) for almost every $t\in[0,1]$, i.e.,

$$\|(1-t)x+ty\|=\|(1-t)x-ty\|, \quad \text{a.e. } [0,1],$$

then

$$\int_0^1 \|(1-t)x+ty\|^2dt=\int_0^1 \|(1-t)x-ty\|^2dt. \tag{4.1}$$

This gives us a motivation to define a type of isosceles orthogonality, i.e. x and y is said to be *HH-I-orthogonal* if and only if they satisfy (4.1), and we denote it by $x\perp_{HH-I} y$. The HH-I-orthogonality is equivalent to the usual orthogonality, if the space is equipped with an inner product (and we omit the proof).

Lemma 3. *The HH-I-orthogonality satisfies the nondegeneracy, simplification, continuity and symmetry properties.*

Proof. If $x\perp_{HH-I} x$, then $\|x\|^2=\int_0^1\|(1-t)x+tx\|^2dt=\int_0^1\|(1-t)x-tx\|^2dt=\frac{1}{3}\|x\|^2$, which implies that $\|x\|=0$, i.e., $x=0$, which proves the nondegeneracy property. If $x\perp_{HH-I} y$, then for any $\lambda\in\mathbb{R}$,

$$\begin{aligned}\int_0^1\|(1-t)\lambda x+t\lambda y\|^2dt &= |\lambda|^2\int_0^1\|(1-t)x+ty\|^2dt\\ &= |\lambda|^2\int_0^1\|(1-t)x-ty\|^2dt\\ &= \int_0^1\|(1-t)\lambda x-t\lambda y\|^2dt,\end{aligned}$$

i.e., $\lambda x\perp_{HH-I}\lambda y$, for any $\lambda\in\mathbb{R}$. If $x\perp_{HH-I} y$, then

$$\begin{aligned}\int_0^1\|(1-t)y+tx\|^2dt &= \int_0^1\|(1-t)x+ty\|^2dt\\ &= \int_0^1\|(1-t)x-ty\|^2dt=\int_0^1\|(1-t)y-tx\|^2dt,\end{aligned}$$

i.e., the HH-I-orthogonality is symmetric. If $x_n\to x$, $y_n\to y$ and $x_n\perp_{HH-I} y_n$ for any $n\in\mathbb{N}$, then by the continuity of norm,

$$\begin{aligned}\int_0^1\|(1-t)x+ty\|^2dt &= \int_0^1\lim_{n\to\infty}\|(1-t)x_n+ty_n\|^2dt\\ &= \lim_{n\to\infty}\int_0^1\|(1-t)x_n+ty_n\|^2dt\\ &= \lim_{n\to\infty}\int_0^1\|(1-t)x_n-ty_n\|^2dt=\int_0^1\|(1-t)x-ty\|^2dt,\end{aligned}$$

which shows the continuity. □

Remark 5. In general, the HH-I-orthogonality is not right-additive nor homogeneous. For example, in $\mathbb{R}^2$, with l^1-norm, $x=(2,1)$ is HH-I-orthogonal to $y=(1,-2)$, but $x\not\perp_{HH-I} 2y$.

The following lemma will be used in proving the existence of HH-I-orthogonality, and we refer to [4] for the proof.

Lemma 4 (James [4])**.** *Let $x, y \in \mathbf{X}$. Then*

$$\lim_{\alpha\to\infty} \|(\alpha + a)x + y\| - \|\alpha x + y\| = a\|x\|.$$

Theorem 5. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, the HH-I-orthogonality is existent, i.e. for any $x, y \in \mathbf{X}$, there exists an $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp_{HH-I} x$.*

Proof. We will prove this theorem by the similar continuity argument and the intermediate value theorem to that of James' in [4, 296–297]. Let $x, y \in \mathbf{X}$, where $x \neq 0$ (as the proof is trivial for $x = 0$), $h : \mathbb{R} \times (0, 1) \to \mathbb{R}$ be a function defined by

$$\begin{aligned} h(\alpha, t) &:= \|(1-t)(\alpha x + y) + tx\| - \|(1-t)(\alpha x + y) - tx\| \\ &= \|[(1-t)\alpha + t]x + (1-t)y\| - \|[(1-t)\alpha - t]x + (1-t)y\|, \end{aligned}$$

and associated to h, a function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(\alpha) := \int_0^1 h(\alpha, t)dt.$$

Note that, for any $t \in (0, 1)$,

$$\begin{aligned} \lim_{\alpha\to\infty} h(\alpha, t) &= \lim_{\alpha\to\infty} \left[\|[(1-t)\alpha + t]x + (1-t)y\| - \|[(1-t)\alpha - t]x + (1-t)y\|\right] \\ &= (1-t)\lim_{\alpha\to\infty}\left[\left\|\left(\alpha + \frac{t}{(1-t)}\right)x + y\right\| - \left\|\left(\alpha - \frac{t}{(1-t)}\right)x + y\right\|\right] \\ &= (1-t)\lim_{\alpha\to\infty}\left[\left\|\left(\alpha + \frac{2t}{(1-t)}\right)x + y\right\| - \|\alpha x + y\|\right] \\ &= (1-t)\frac{2t}{(1-t)}\|x\| = 2t\|x\|, \end{aligned}$$

by Lemma 4, and that

$$\lim_{\alpha\to\infty} H(\alpha) = \lim_{\alpha\to\infty}\int_0^1 h(\alpha, t)\, dt = \int_0^1 \lim_{\alpha\to\infty} h(\alpha, t)\, dt,$$

by the continuity of h. Therefore,

$$\lim_{\alpha\to\infty} H(\alpha) = \int_0^1 2t\|x\|\, dt = \|x\| > 0.$$

We also note that for any $t \in (0, 1)$

$$\begin{aligned} &\lim_{\alpha\to\infty} h(-\alpha, t) \\ =\ &\lim_{\alpha\to\infty}\left[\|[(1-t)(-\alpha) + t]x + (1-t)y\| - \|[(1-t)(-\alpha) - t]x + (1-t)y\|\right] \\ =\ &\lim_{\alpha\to\infty}\left[\|[(1-t)\alpha - t]x - (1-t)y\| - \|[(1-t)\alpha + t]x - (1-t)y\|\right] \\ =\ &(1-t)\lim_{\alpha\to\infty}\left[\left\|\left(\alpha - \frac{t}{1-t}\right)x - y\right\| - \left\|\left(\alpha + \frac{t}{1-t}\right)x - y\right\|\right] \\ =\ &(1-t)\lim_{\alpha\to\infty}\left[\left\|\left(\alpha - \frac{2t}{1-t}\right)x - y\right\| - \|\alpha x - y\|\right] \\ =\ &(1-t)\left(-\frac{2t}{1-t}\right)\|x\| = -2t\|x\|, \end{aligned}$$

again by Lemma 4, and by the continuity of h,

$$\lim_{\alpha\to\infty} H(-\alpha) = \lim_{\alpha\to\infty}\int_0^1 h(-\alpha, t)\, dt = \int_0^1 \lim_{\alpha\to\infty} h(-\alpha, t)\, dt.$$

Therefore,

$$\lim_{\alpha\to\infty} H(-\alpha) = \int_0^1 (-2t)\|x\|\ dt = -\|x\| < 0.$$

Now, we have shown that there exist $\alpha_1 > 0$ such that $H(\alpha_1) > 0$ and $\alpha_2 < 0$ such that $H(\alpha_1) < 0$. By continuity of H, we conclude that there exists an α_0 such that $H(\alpha_0) = 0$, and therefore

$$\int_0^1 \|(1-t)(\alpha_0 x + y) + tx\|^2 dt = \int_0^1 \|(1-t)(\alpha_0 x + y) - tx\|^2 dt,$$

as required. □

The following lemma will be used to prove the uniqueness property of the HH-I-orthogonality.

Lemma 5. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a strictly convex normed space,* $x, y \in \mathbf{X}$ *and* $t \in (0,1)$. *Let* g *be a function on* $\mathbb{R}$ *defined by*

$$g(k) := \int_0^1 \|(1-t)y + k(tx)\|^2\ dt.$$

Then, g *is a strictly convex function on* $\mathbb{R}$.

The proof follows readily from the fact that $\mathbf{X}$ is strictly convex and the details are left to the reader.

Theorem 6. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. Then, HH-I-orthogonality is unique if and only if* $\mathbf{X}$ *is strictly convex.*

Proof. The proof is inspired by that of Kapoor and Prasad in [5, p. 405]. Suppose that $\mathbf{X}$ is strictly convex and HH-I-orthogonality is not unique. Then, there exist $x, y \in \mathbf{X}$, where $x \neq 0$, and $\alpha > 0$ such that

$$y \perp_{HH-I} x, \tag{4.2}$$

and

$$\alpha x + y \perp_{HH-I} x. \tag{4.3}$$

Recall the strictly convex function g as defined in Lemma 5, for the given x, y and $t \in (0,1)$:

$$g(k) := \int_0^1 \|(1-t)y + k(tx)\|^2\ dt.$$

Note that (4.2) gives us

$$g(1) = \int_0^1 \|(1-t)y + tx\|^2\ dt = \int_0^1 \|(1-t)y - tx)\|^2\ dt = g(-1).$$

Set $\alpha'(t) = \frac{(1-t)\alpha}{t}$, $t \in (0,1)$, then

$$\begin{aligned} g(\alpha'(t)-1) &= \int_0^1 \|(1-t)y + (\alpha'(t)-1)(tx)\|^2\ dt \\ &= \int_0^1 \|(1-t)(\alpha x + y) - tx\|^2\ dt \\ &= \int_0^1 \|(1-t)(\alpha x + y) + tx\|^2\ dt \\ &= \int_0^1 \|(1-t)y + (\alpha'(t)+1)(tx)\|^2\ dt = g(\alpha'(t)+1), \end{aligned}$$

from (4.3). Consider the case where $0 < \alpha'(t) \leq 2$. We have

$$\begin{aligned} g(\alpha'(t)-1) &= g\left[\left(1-\frac{\alpha'(t)}{2}\right)(-1)+\frac{\alpha'(t)}{2}(1)\right] \\ &< \left(1-\frac{\alpha'(t)}{2}\right)g(-1)+\frac{\alpha'(t)}{2}\,g(1) \\ &= g(1) \\ &= g\left[\frac{\alpha'(t)}{2}(\alpha'(t)-1)+\left(1-\frac{\alpha'(t)}{2}\right)(\alpha'(t)+1)\right] \\ &< \frac{\alpha'(t)}{2}\,g\left(\alpha'(t)-1\right)+\left(1-\frac{\alpha'(t)}{2}\right)g\left(\alpha'(t)+1\right)=g(\alpha'(t)-1), \end{aligned}$$

which leads us to a contradiction. Now consider the case where $\alpha'(t) > 2$. The intervals $[-1,1]$ and $[\alpha'(t)-1, \alpha'(t)+1]$ are disjoint, and therefore, we have two distinct local minimum, one on each of these intervals. But, g is strictly convex, and thus can only have one (global) minimum, which yields a contradiction.

Conversely, let us assume that $\mathbf{X}$ is not strictly convex. Let $x, y \in \mathbf{X}$, $x \neq y$, such that $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$. Then, the quantities

$$\int_0^1 \left\|(1-t)\frac{x+y}{1-t}\right\|^2 \, dt = \|x+y\|^2,$$

$$\int_0^1 \left\|(1-t)\frac{x+y}{1-t}+t\left(\frac{x-y}{t}\right)\right\|^2 \, dt = 4\|x\|^2,$$

and

$$\int_0^1 \left\|(1-t)\frac{x+y}{1-t}-t\left(\frac{x-y}{t}\right)\right\|^2 \, dt = 4\|y\|^2,$$

are all equal by our assumption. Set $x' = \frac{x+y}{1-t}$ and $y' = \frac{x-y}{t}$, so we have

$$\int_0^1 \|(1-t)x'\|^2 \, dt = \int_0^1 \|(1-t)x't+y'\|^2 \, dt = \int_0^1 \|(1-t)x'-ty'\|^2 \, dt. \tag{4.4}$$

Note that by the first equality in (4.4), we have

$$\begin{aligned} &\int_0^1 \left\|(1-t)\left(x'+\frac{ty'}{2(1-t)}\right)+t\left(\frac{y'}{2}\right)\right\|^2 \, dt \\ = &\int_0^1 \|(1-t)x'+ty'\|^2 \, dt \\ = &\int_0^1 \|(1-t)x'\|^2 \, dt \\ = &\int_0^1 \left\|(1-t)\left(x'+\frac{ty'}{2(1-t)}\right)-t\left(\frac{y'}{2}\right)\right\|^2 \, dt, \end{aligned}$$

that is,

$$\left[x'+\frac{t}{1-t}\left(\frac{y'}{2}\right)\right] \perp_{HH-I} \left(\frac{y'}{2}\right). \tag{4.5}$$

Also, by the second equality in (4.4), we have

$$\begin{aligned}
& \int_0^1 \left\|(1-t)\left(x' - \frac{ty'}{2(1-t)}\right) - t\left(\frac{y'}{2}\right)\right\|^2 \, dt \\
= & \int_0^1 \|(1-t)x' - ty'\|^2 \, dt \\
= & \int_0^1 \|(1-t)x'\|^2 \, dt \\
= & \int_0^1 \left\|(1-t)\left(x' - \frac{ty'}{2(1-t)}\right) + t\left(\frac{y'}{2}\right)\right\|^2 \, dt,
\end{aligned}$$

that is,

$$\left[x' - \frac{t}{1-t}\left(\frac{y'}{2}\right)\right] \perp_{HH-I} \left(\frac{y'}{2}\right). \tag{4.6}$$

By (4.5) and (4.6), we conclude that the HH-I-orthogonality is not unique. □

The following lemma is due to Ficken [3], in characterizing an inner product space.

Lemma 6 (Ficken [3])**.** *A normed space* $(\mathbf{X}, \|\cdot\|)$ *is an inner product space, if and only if*

$$\|kx + y\| = \|x + ky\|,$$

for any $k \in \mathbb{R}$ *and* $x, y \in \mathbf{X}$*, with* $\|x\| = \|y\|$*.*

Theorem 7. *If HH-I-orthogonality is homogeneous in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space.*

Proof. The proof is inspired by that of James [4, p. 298]. Assume that the homogeneity property of HH-orthogonality holds, and let $x, y \in \mathbf{X}$, where $\|x\| = \|y\|$. For any $t \in (0,1)$, set

$$A(t) = \frac{x+y}{(1-t)}, \quad \text{and} \quad B(t) = \frac{x-y}{t}.$$

Note that

$$\int_0^1 \|(1-t)A(t) + tB(t)\|^2 dt = \int_0^1 \|x + y + x - y\|^2 dt = 4\|x\|^2,$$

and

$$\int_0^1 \|(1-t)A(t) - tB(t)\|^2 dt = \int_0^1 \|x + y - (x-y)\|^2 dt = 4\|y\|^2.$$

Since $\|x\| = \|y\|$,

$$\int_0^1 \|(1-t)A(t) + tB(t)\|^2 dt = \int_0^1 \|(1-t)A(t) - tB(t)\|^2 dt,$$

i.e, $A(t) \perp_{HH-I} B(t)$, for all $t \in (0,1)$. Since we are assuming the homogeneity of HH-I-orthogonality, then for any $k \in \mathbb{R}$, we have $\frac{k+1}{2}A(t) \perp_{HH-I} \frac{k-1}{2}B(t)$,

$$\int_0^1 \left\|(1-t)\left(\frac{k+1}{2}\right)A(t) + t\left(\frac{k-1}{2}\right)B(t)\right\|^2 dt = \|kx + y\|^2,$$

and

$$\int_0^1 \left\|(1-t)\left(\frac{k+1}{2}\right)A(t) - t\left(\frac{k-1}{2}\right)B(t)\right\|^2 dt = \|x + ky\|^2.$$

Thus,

$$\|kx + y\| = \|x + ky\|,$$

for all $k \in \mathbb{R}$; and by Lemma 6, we conclude that $\mathbf{X}$ is an inner product space. □

Theorem 8. *The property of homogeneity and additivity of HH-I-orthogonality are equivalent.*

The proof is similar to that of Theorem 4, and the details are omitted.

Corollary 2. *If HH-I-orthogonality is additive in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space.*

Remark 6.
(1) Note that if $x, y \in \mathbf{X}$ such that $(1-t)x \perp ty\ (I)$ for almost every $t \in [0,1]$, then by the continuity of I-orthogonality, $(1-t)x \perp ty\ (I)$ for every $t \in [0,1]$; and furthermore, $\alpha x \perp \beta y\ (I)$ for any $\alpha, \beta \in \mathbb{R}$.
(2) If $x \perp y\ (I)$ implies that $(1-t)x \perp ty\ (I)$, then the I-orthogonality is homogeneous, and therefore the underlying space is an inner product space. Thus, $x \perp_{HH-I} y$.
(3) Note that $x \perp y\ (I)$ does not imply $x \perp_{HH-I} y$. For example, in $\mathbb{R}^2$ equipped with l^1-norm, $x = (2, -1)$ is I-orthogonal to $y = (1, 1)$, but $x \not\perp_{HH-I} y$.
(4) Note that $x \perp_{HH-I} y$ does not imply $x \perp y\ (I)$. For example, in $\mathbb{R}^2$ equipped with l^1-norm, $x = (-\frac{1}{8} + \frac{\sqrt{129}}{8}, 1)$ is HH-I-orthogonal to $y = (-\frac{1}{8} + \frac{\sqrt{129}}{8}, -2)$, but $x \not\perp y\ (P)$.

Acknowledgments. The author would like to thank Professor Hendra Gunawan (Head of Analysis and Geometry Group, Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology) for the useful comments and discussions.

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Research Group in Mathematical Inequalities and Applications, School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: eder.kikianty@vu.edu.au

E-mail address: sever.dragomir@vu.edu.au