INEQUALITIES OF GRÜSS TYPE INVOLVING THE p-HH-NORMS IN THE CARTESIAN PRODUCT SPACE

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Abstract. Inequalities in estimating a type of Čebyšev functional involving the p-HH-norms are obtained by applying the known results by Grüss, Ostrowski, Čebyšev and Lupaş. Some of these inequalities are proven to be sharp. In 1998, Dragomir and Fedotov considered a generalised Čebyšev functional, in order to approximate the Riemann-Stieltjes integral. In this paper, some sharp bounds for the generalised Čebyšev functional with convex integrand and monotonically increasing integrator, are established as well. An application for the Čebyšev functional involving the p-HH-norms is also considered, and the bounds are proven to be sharp.

1. Introduction

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and consider the Cartesian product space $\mathbf{X}^2 = \{(x,y) : x, y \in \mathbf{X}\}$, where the addition and scalar multiplication are defined in the usual way. This space is a normed space together with any of the following p-norms (see [2, p. 397–398], [11, p. 36], and [13, p. 142]):

$$\|(x,y)\|_p := \begin{cases} (\|x\|^p + \|y\|^p)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \max\{\|x\|, \|y\|\}, & p = \infty, \end{cases}$$

for any $(x,y) \in \mathbf{X}^2$. Kikianty and Dragomir in [8] introduced another norm on $\mathbf{X}^2$ which is called the p-HH-norm, and is defined as follows:

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p dt\right)^{\frac{1}{p}}, \tag{1.1}$$

for any $1 \le p < \infty$ and $(x,y) \in \mathbf{X}^2$. For fundamental properties of this norm, see [8]. We note that the p-norms and the p-HH-norms are equivalent in $\mathbf{X}^2$.

Some inequalities of Ostrowski type, which involve the p-HH-norms and the p-norms, have been considered in [9] and [10]. Continuing these works, we are interested in obtaining some new inequalities involving the p-HH-norms.

In this paper, we consider bounds in estimating the difference of $\|(\cdot,\cdot)\|_{p+q-HH}^{p+q}$ and the product $\|(\cdot,\cdot)\|_{p-HH}^{p}\|(\cdot,\cdot)\|_{q-HH}^{q}$ for any $p, q \ge 1$. This difference, however, is a particular type of *Čebyšev functional*. In the following, we recall some classical facts concerning this functional.

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$, the Čebyšev functional is defined by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt.$$

In 1935, Grüss proved the following inequality which bounds the Čebyšev functional [14, p. 295–296]:

$$|T(f,g)| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{1.2}$$

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provided that f and g satisfy the condition $\phi \le f(t) \le \Phi$ and $\gamma \le g(t) \le \Gamma$ for all $t \in [a,b]$. The constant $\frac{1}{4}$ is best possible and is achieved for $f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$. Some related results regarding the sharp upper bounds for this functional can be summarised as follows:

(1) Čebyšev (1882): If f, g are continuously differentiable functions on $[a,b]$, then

$$|T(f,g)| \le \frac{1}{12}\|f'\|_{L^\infty}\|g'\|_{L^\infty}(b-a)^2, \tag{1.3}$$

where $\|f'\|_{L^\infty} := \sup_{t\in[a,b]}|f'(t)|$; and equality holds iff f' and g' are constants [14, p. 297] (it is also valid for absolutely continuous functions f, g where $f', g' \in L^\infty[a,b]$);

(2) Ostrowski (1970): If f is Lebesgue integrable on $[a,b]$, $m, M \in \mathbb{R}$ such that $-\infty \le m \le f \le M \le \infty$, g is absolutely continuous and $g' \in L^\infty[a,b]$, then

$$|T(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_{L^\infty}, \tag{1.4}$$

and the constant $\frac{1}{8}$ is the best possible [14, p. 300];

(3) Lupaş (1973): If f, g are absolutely continuous, $f', g' \in L^2[a,b]$, then

$$|T(f,g)| \le \frac{1}{\pi^2}(b-a)\|f'\|_{L^2}\|g'\|_{L^2}; \tag{1.5}$$

(note that $\|f'\|_{L^2} = \int_a^b |f'(t)|^2 dt$) with equality valid iff

$$f(x) = A + B\sin\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right] \text{ and } g(x) = C + D\sin\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right],$$

where A, B, C, and D are constants [14, p. 301].

In Section 3, we apply these results to obtain upper bounds in estimating the difference of $\|(x,y)\|_{p+q-HH}^{p+q}$ and $\|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q}$ $(p,q \ge 1)$. Some of these inequalities are proven to be sharp.

More results regarding the Čebyšev functional were pointed out by Dragomir and Fedotov in [4]. In order to approximate the Riemann-Stieltjes integral, they considered a generalised Čebyšev functional

$$D(f,u) := \int_a^b f(t)du(t) - \frac{1}{b-a}[u(b)-u(a)]\int_a^b f(s)ds,$$

where f is Riemann integrable and Stieltjes integrable with respect to a function u. Some bounds for D, when u is monotonically nondecreasing, were obtained by Dragomir in [3], are summarised as follows:

(1) If $f : [a,b] \to \mathbb{R}$ is L-Lipschitzian on $[a,b]$, then

$$|D(f,u;a,b)| \le \frac{1}{2}L(b-a)\left[u(b)-u(a) - \frac{4}{(b-a)^2}\int_a^b u(t)\left(t - \frac{a+b}{2}\right)dt\right] \le \frac{1}{2}L(b-a)[u(b)-u(a)],$$

and the constant $\frac{1}{2}$ is best possible in both inequalities.

(2) If $f : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$, and $\int_a^b f(t)du(t)$ exists, then

$$|D(f,u;a,b)| \le \left[u(b)-u(a) - \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right)dt\right]\bigvee_a^b(f) \le [u(b)-u(a)]\bigvee_a^b(f),$$

and the first inequality is sharp.

In Section 4, we establish some sharp bounds for the generalised Čebyšev functional D in order to approximate the Riemann-Stieltjes integral for differentiable convex integrand and monotonically increasing integrator. The result follows by utilising an Ostrowski type inequality. Then in Section 5, we apply this result for the Čebyšev functional T, and the obtained bounds are sharp. A similar result is established for a general convex function, and the obtained bounds are also sharp. By applying the result for the p-HH-norms, we also obtain some upper and lower bounds for the difference between $\|(x,y)\|_{p+q-HH}^{p+q}$ and $\|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q}$ $(p,q \ge 1)$. These bounds are proven to be sharp.

2. Definitions and notation

Throughout this paper, we assume that all vector spaces are over the field of real numbers and the measure that we consider is the Lebesgue measure.

Let $x, y \in \mathbf{X},\ x \neq y$ and define the segment $[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}$. Let $f : [x, y] \to \mathbb{R}$ and the associated function $h : [0,1] \to \mathbb{R}$, defined by $h(t) := f[(1-t)x + ty],\ t \in [0,1]$. It is well known that the function h is convex on $[0,1]$ if and only if f is convex on $[x, y]$.

In any normed space $\mathbf{X}$, the norm $\|\cdot\|$ is right-(left)-Gâteaux differentiable at $x \in \mathbf{X} \setminus \{0\}$, i.e. the following limits

$$(\nabla_{+(-)}\|\cdot\|(x))(y) := \lim_{t \to 0^{+(-)}} \frac{\|x+ty\| - \|x\|}{t}$$

exist for all $y \in \mathbf{X}$ (see [12, p. 483–485] for the proof). The norm $\|\cdot\|$ is Gâteaux differentiable at $x \in \mathbf{X} \setminus \{0\}$ if and only if $(\nabla_{+}\|\cdot\|(x))(y) = (\nabla_{-}\|\cdot\|(x))(y)$, for all $y \in \mathbf{X}$. The function $f_0(x) = \frac{1}{2}\|x\|^2$ $(x \in X)$ is convex and the following limits

$$(x, y)_{s(i)} \quad := \quad (\nabla_{+(-)} f_0(y))(x) = \lim_{t \to 0^{+(-)}} \frac{\|y+tx\|^2 - \|y\|^2}{2t},$$

exist for any $x, y \in X$. They are called the superior (inferior) semi-inner products (s.i.p.) associated with the norm $\|\cdot\|$ (see [6, p. 27–39] for further properties).

3. Grüss type inequality involving the p-HH-norms

In this section, we obtain some inequalities involving the p-HH-norms in the Cartesian product space $\mathbf{X}^2$ from the results due to Grüss, Čebyšev, Ostrowski and Lupaş which have been stated in Section 1.

Lemma 1. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $x, y \in \mathbf{X}$, and $p, q \geq 1$. Then,*

$$\|(x,y)\|_{p+q-HH}^{p+q} \geq \|(x,y)\|_{p-HH}^{p} \|(x,y)\|_{q-HH}^{q}. \tag{3.1}$$

Equality holds in (3.1) for $x = y$.

Proof. Define $f_p(t) := \|(1-t)x + ty\|^p$, where $t \in [0,1]$. We claim that for any $p, q \geq 1$, f_p and f_q are synchronous (similarly ordered, see [7, p. 43]) on $[0,1]$. The proof is as follows: let $t, s \in [0,1]$ and assume that $f_1(t) \leq f_1(s)$ (as for the other case, the proof follows similarly). Since $f_1(t) \geq 0$ for any $t \in [0,1]$, it implies that $f_p(t) \leq f_p(s)$, for any $p \geq 1$. Thus, for any $t, s \in [0,1]$ and $p, q \geq 1$, we have

$$[f_p(t) - f_p(s)]\,[f_q(t) - f_q(s)] \geq 0.$$

Since f and g are synchronous, the Čebyšev inequality holds (see [7, p. 43])), i.e.,

$$\int_0^1 f_p(t) f_q(t)dt \geq \int_0^1 f_p(t)dt \int_0^1 f_q(t)dt,$$

or, equivalently,

$$\|(x,y)\|_{p+q-HH}^{p+q} \geq \|(x,y)\|_{p-HH}^{p} \|(x,y)\|_{q-HH}^{q},$$

as desired. It is easily shown that equality holds for $x = y$. □

Theorem 1. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space, $x, y \in \mathbf{X}$, $p, q \geq 1$, and set*

$$T_{p,q}(x,y) := \|(x,y)\|_{p+q-HH}^{p+q} - \|(x,y)\|_{p-HH}^{p} \|(x,y)\|_{q-HH}^{q} \geq 0.$$

Then,

$$0 \leq T_{p,q}(x,y) \leq \frac{1}{12} pq \|y - x\|^2 \max\{\|x\|, \|y\|\}^{p+q-2} =: C_{p,q}(x,y). \tag{3.2}$$

The constant $\frac{1}{12}$ in (3.2) is sharp.

Proof. Define $f(t) = \|(1-t)x+ty\|^p$ and $g(t) = \|(1-t)x+ty\|^q$, for $t \in [0,1]$. Since both f and g are convex, they are absolutely continuous; also f' and g' exist a.e. on $[0,1]$. Therefore,

$$f'(t) = \nabla_{\pm}\|\cdot\|^p[(1-t)x+ty](y-x) = p\|(1-t)x+ty\|^{p-2}(y-x,(1-t)x+ty)_{s(i)}$$

(note that $(\cdot,\cdot)_{s(i)}$ is the superior (inferior) s.i.p.), and

$$\begin{aligned} \|f'\|_{L^\infty} &= \sup_{t\in[0,1]} p\|(1-t)x+ty\|^{p-2}|(y-x,(1-t)x+ty)_{s(i)}| \\ &\leq p\|y-x\| \sup_{t\in[0,1]} \|(1-t)x+ty\|^{p-1} = p\|y-x\| \max\{\|x\|,\|y\|\}^{p-1}. \end{aligned}$$

Similarly for g, we have $\|g'\|_{L^\infty} \leq q\|y-x\| \max\{\|x\|,\|y\|\}^{q-1}$. Due to Čebyšev's result (1.3), we have

$$T_{p,q}(x,y) \leq \frac{1}{12}\|f'\|_{L^\infty}\|g'\|_{L^\infty} \leq \frac{1}{12}pq\|y-x\|^2 \max\{\|x\|,\|y\|\}^{p+q-2}.$$

Now, we will prove the sharpness of the constant. Assume that the inequality holds for a constant $A > 0$ instead of $\frac{1}{12}$, i.e.

$$\|(x,y)\|_{p+q-HH}^{p+q} - \|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q} \leq A\ pq\|y-x\|^2 \max\{\|x\|,\|y\|\}^{p+q-2}.$$

Choose $p=1$, $q=1$, $\mathbf{X} = \mathbb{R}$, and $0 < x < y$, to obtain

$$\frac{1}{3}\left(x^2+xy+y^2\right) - \left(\frac{y+x}{2}\right)^2 = \frac{1}{12}(y-x)^2 \leq A(y-x)^2.$$

Since $x \neq y$, then $A \geq \frac{1}{12}$, and the proof is completed. □

For any x and y in the normed space $(\mathbf{X},\|\cdot\|)$, we set the following quantities for $p,q \geq 1$

$$G_{p,q}(x,y) := \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q}, \quad O_{p,q}(x,y) := \frac{1}{8}q\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1},$$

$$\text{and} \quad L_{p,q}(x,y) := \frac{1}{\pi^2}pq\|y-x\|^2\|(x,y)\|_{(2p-2)-HH}^{p-1}\|(x,y)\|_{(2q-2)-HH}^{q-1}.$$

The following proposition is due to the results by Grüss, Ostrowski and Lupaş. However, these upper bounds are not yet proven to be sharp.

Proposition 1. *Under the assumptions of Theorem 1 and the above notation, we have*

$$0 \leq T_{p,q}(x,y) \leq G_{p,q}(x,y); \quad 0 \leq T_{p,q}(x,y) \leq O_{p,q}(x,y); \quad \text{and} \quad 0 \leq T_{p,q}(x,y) \leq L_{p,q}(x,y),$$

for any $p,q \geq 1$ and $x,y \in \mathbf{X}$.

Proof. Define $f(t) = \|(1-t)x+ty\|^p$, and $g(t) = \|(1-t)x+ty\|^q$, for $t \in [0,1]$. Since $p,q \geq 1$, we have $0 \leq f(t) \leq \max\{\|x\|,\|y\|\}^p$ and $0 \leq g(t) \leq \max\{\|x\|,\|y\|\}^q$. Then, due to Grüss' result (1.2), we have

$$T_{p,q}(x,y) \leq \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q} = G_{p,q}(x,y).$$

Since g is absolutely continuous, $\|g'\|_{L^\infty} \leq q\|y-x\|\max\{\|x\|,\|y\|\}^{q-1}$ (see the proof of Theorem 1). Then, due to Ostrowski's result (1.4), we have

$$T_{p,q}(x,y) \leq \frac{1}{8}\max\{\|x\|,\|y\|\}^p\|g'\|_{L^\infty} \leq \frac{1}{8}q\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1} = O_{p,q}(x,y).$$

Note that for any $p \geq 1$, we have

$$\begin{aligned} \|f'\|_{L^2} &= \left[\int_0^1 |p\|(1-t)x+ty\|^{p-2}(y-x,(1-t)x+ty)_{s(i)}|^2 dt\right]^{\frac{1}{2}} \\ &\leq p\ \|y-x\| \left[\int_0^1 \|(1-t)x+ty\|^{2p-2}dt\right]^{\frac{1}{2}} = p\ \|y-x\|\ \|(x,y)\|_{(p-2)-HH}^{p-1} \end{aligned}$$

by the Cauchy-Schwarz inequality; and similarly for $q \geq 1$, we have $\|g'\|_{L^2} \leq q\|y-x\|\ \|(x,y)\|_{(2q-2)-HH}^{q-1}$. Therefore, by Lupaş' result (1.5), we obtain

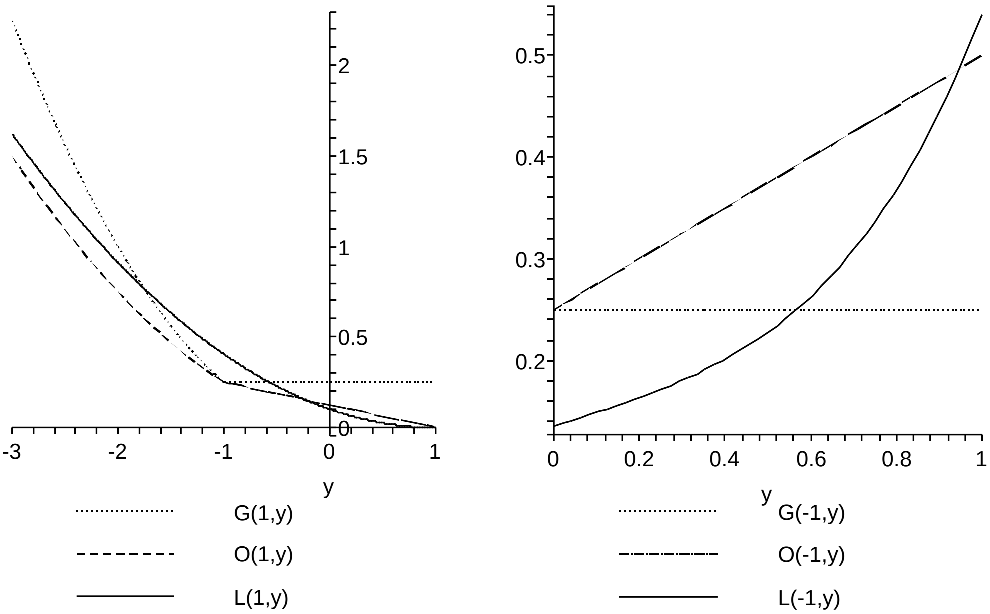
$$T_{p,q}(x,y) \leq \frac{1}{\pi^2}\|f'\|_{L^2}\|g'\|_{L^2} \leq \frac{1}{\pi^2}pq\|y-x\|^2\|(x,y)\|_{(2p-2)-HH}^{p-1}\|(x,y)\|_{(2q-2)-HH}^{q-1} = L_{p,q}(x,y). \quad □$$

Remark 1. We note that none of the upper bounds for $T_{p,q}(x,y)$ that we have obtained in Proposition 1 is better than the other ones, for each $x, y \in \mathbf{X}$. For example, choose $\mathbf{X} = \mathbb{R}$, $p = q = 1$, and $x = 1$. By utilising MAPLE, we obtain (see Figure 1(a))

$$\begin{aligned} G(1,y) &\geq O(1,y) \geq L(1,y), \quad y \in [0,1], \\ G(1,y) &\geq L(1,y) \geq O(1,y), \quad y \in [-3,-2], \\ L(1,y)) &\geq G(1,y) \geq O(1,y), \quad y \in [-\tfrac{3}{2},-1]. \end{aligned}$$

Again, by utilising MAPLE, for $p = q = 2$, and $x = -1$, we have (see Figure 1(b))

$$\begin{aligned} O(-1,y) &\geq L(-1,y) \geq G(-1,y), \quad y \in [\tfrac{3}{5},\tfrac{4}{5}], \\ O(-1,y) &\geq G(-1,y)(x,y) \geq L(-1,y), \quad y \in [0,\tfrac{2}{5}], \\ L(-1,y) &\geq O(-1,y)(x,y) \geq G(-1,y), \quad y \in [\tfrac{19}{20},1]. \end{aligned}$$



(a) Case of $p = q = 1$

(b) Case of $p = q = 2$

FIGURE 1.

Open Problem 1. Are the constants $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{\pi^2}$ in Proposition 1 the best possible?

4. NEW BOUNDS FOR THE GENERALISED ČEBYŠEV FUNCTIONAL D

The following result gives upper and lower bounds for the generalised Čebyšev functional $D(\cdot,\cdot)$ in order to approximate the Riemann-Stieltjes integral.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function, and $u : [a,b] \to \mathbb{R}$ be a monotonically increasing function. Then,*

$$\frac{(b-a)}{2}[f'(a)u(b) + f'(b)u(a)] - \int_0^1 u(t)\left[\left(\frac{t-a}{b-a}\right)f'(a) + \left(\frac{b-t}{b-a}\right)f'(b)\right]dt$$

$$\leq D(f,u) \leq \int_a^b \left(t - \frac{b+a}{2}\right) f'(t)du(t). \tag{4.1}$$

The constants $\frac{1}{2}$ and 1 in (4.1) are sharp.

Proof. Since f is a differentiable convex function on $[a,b]$, then we have the following Ostrowski type inequality (see [5])

$$\frac{1}{2}\left[(b-t)^2-(t-a)^2\right]f'(t)\le\int_a^b f(s)ds-(b-a)f(t)\le\frac{1}{2}\left[(b-t)^2f'(b)-(t-a)^2f'(a)\right],\tag{4.2}$$

for any $t\in[a,b]$. Since u is a monotonic increasing function on $[a,b]$, then we may integrate the inequality (4.2) (in the Riemann-Stieltjes sense) with respect to u, i.e.,

$$\begin{aligned}\frac{1}{2}\int_a^b\left[(b-t)^2-(t-a)^2\right]f'(t)du(t)&\le\int_a^b\left[\int_a^b f(s)ds-(b-a)f(t)\right]du(t)\\&\le\frac{1}{2}\int_a^b\left[(b-t)^2f'(b)-(t-a)^2f'(a)\right]du(t).\end{aligned}\tag{4.3}$$

Note that

$$\frac{1}{2}\int_a^b\left[(b-t)^2-(t-a)^2\right]f'(t)du(t)=(b-a)\int_a^b\left(\frac{b+a}{2}-t\right)f'(t)du(t),$$

and

$$\begin{aligned}\int_a^b\left[\int_a^b f(s)ds-(b-a)f(t)\right]du(t)&=\int_a^b f(s)ds\int_a^b du(t)-(b-a)\int_a^b f(t)du(t)\\&=[u(b)-u(a)]\int_a^b f(s)ds-(b-a)\int_a^b f(t)du(t),\end{aligned}$$

and, using integration by parts

$$\begin{aligned}&\frac{1}{2}\int_a^b\left[(b-t)^2f'(b)-(t-a)^2f'(a)\right]du(t)\\=&\frac{1}{2}(b-a)^2[-f'(b)u(a)-f'(a)u(b)]+\int_a^b u(t)[(b-t)f'(b)+(t-a)f'(a)]dt.\end{aligned}$$

Therefore, by (4.3) we get

$$\begin{aligned}&(b-a)\int_a^b\left(\frac{b+a}{2}-t\right)f'(t)du(t)\\\le&[u(b)-u(a)]\int_a^b f(s)ds-(b-a)\int_a^b f(t)du(t)\\\le&\frac{1}{2}(b-a)^2[-f'(b)u(a)-f'(a)u(b)]+\int_a^b u(t)[(b-t)f'(b)+(t-a)f'(a)]dt,\end{aligned}\tag{4.4}$$

and the proof follows by multiplying inequality (4.4) by $\left(-\frac{1}{b-a}\right)$.

The sharpness of the constants follows by a particular case which will be stated in Corollary 3. □

Corollary 1. *Under the assumptions of Theorem 2, if $f'(b)=-f'(a)$, then*

$$\begin{aligned}&\frac{f'(a)(b-a)[u(b)-u(a)]}{2}-\frac{f'(a)}{(b-a)}\int_a^b u(t)\left(2t-(a+b)\right)dt\\\le&\ D(f,u)\le\int_a^b\left(t-\frac{b+a}{2}\right)f'(t)du(t).\end{aligned}\tag{4.5}$$

Remark 2. A common example of such function is the function defined on interval $[a,b]$ which is symmetric with respect to the midpoint $\frac{a+b}{2}$, e.g., $f(t)=\left|t-\frac{a+b}{2}\right|^p$, where $p\ge1$.

Corollary 2. *Under the assumptions of Theorem 2, if $f'(a) = -f'(b)$ and f'' exists, then*

$$\begin{aligned}(4.6)\qquad & \frac{f'(a)(b-a)[u(b)-u(a)]}{2} - \frac{f'(a)}{(b-a)}\int_a^b u(t)\,(2t-(a+b))\,dt \\ \le\ & D(f,u) \le \left(\frac{b-a}{2}\right) f'(b)[u(b)-u(a)] - \int_a^b u(t)\left[f'(t) + \left(t - \frac{b+a}{2}\right) f''(t)\right] dt.\end{aligned}$$

Proof. This is a particular case of Corollary 1. Note that

$$\int_a^b \left(t - \frac{b+a}{2}\right) f'(t)du(t) = \left(\frac{b-a}{2}\right) f'(b)[u(b)-u(a)] - \int_a^b u(t)\left[f'(t) + \left(t - \frac{b+a}{2}\right) f''(t)\right] dt,$$

and the details are omitted. □

Open Problem 2. Are the inequalities in Corollaries 1 and 2 sharp?

5. Application for the Čebyšev functional

In this section, we apply the result of Section 4 to obtain bounds for the classical Čebyšev functional.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function, and $g : [a,b] \to \mathbb{R}$ be a nonnegative Lebesgue integrable function. Then,*

$$(5.1)\qquad \frac{1}{2}\int_a^b \left[\left(\frac{t-a}{b-a}\right)^2 f'(a) - \left(\frac{b-t}{b-a}\right)^2 f'(b)\right] g(t)dt \le T(f,g) \le \frac{1}{b-a}\int_a^b \left(t - \frac{b+a}{2}\right) f'(t)g(t)dt.$$

The constants $\frac{1}{2}$ and 1 in (5.1) are sharp.

Proof. Recall that Theorem 2 gives us

$$(5.2)\qquad \frac{1}{2(b-a)}\int_a^b [(t-a)^2 f'(a) - (b-t)^2 f'(b)]du(t) \le D(f,u) \le \int_a^b \left(t - \frac{b+a}{2}\right) f'(t)du(t).$$

Since g is positive on $[a,b]$, $u(t) = \int_a^t g(s)ds$ is monotonically increasing on $[a,b]$. Thus, inequality (5.1) follows by applying (5.2) to u and multiply the obtained inequality by $\frac{1}{b-a}$. The sharpness of the constants in (5.1) is demonstrated by choosing $f(t) = g(t) = t$ on $[0,1]$; the details are omitted. □

Example 1. Let $f(t) = g(t) = \frac{1}{t}$ defined on $[x,y]$, where $x,y > 0$. Then by Corollary 3, we obtain

$$(5.3)\qquad 0 \le \left(\frac{1}{G(x,y)}\right)^2 - \left(\frac{1}{L(x,y)}\right)^2 \le \left(\frac{y-x}{2xy}\right)^2,$$

where $G(x,y)$ and $L(x,y)$ are the geometric mean and logarithmic mean of x and y, respectively (note that $G(x,y) = \sqrt{xy}$ and $L(x,y) = \frac{x-y}{\log x - \log y}$). We note that we do not consider the lower bound in this case, since it is not always positive.

Example 2. Let $f(t) = t^p$ and $g(t) = t^q$ defined on $[x,y]$, where $x,y > 0$ and. If $p,q \in \mathbb{R}\setminus\{0\}$ such that $p \ne \pm 1, 2$, $q \ne -3,-2,-1$, and $p+q \ne 0, \pm 1$, then we obtain

$$\begin{aligned} & -\frac{p(p+1)}{2}(\mathfrak{L}^{[p]}(x,y))^p(\mathfrak{L}^{[q]}(x,y))^q + p^2(\mathfrak{L}^{[p-1]}(x,y))^{p-1}(\mathfrak{L}^{[q+1]}(x,y))^{q+1} \\ & \quad -\frac{p(p-1)}{2}(\mathfrak{L}^{[p-2]}(x,y))^{p-2}(\mathfrak{L}^{[q+2]}(x,y))^{q+2} \\ \le\ & (\mathfrak{L}^{[p+q]}(x,y))^{p+q} - (\mathfrak{L}^{[p]}(x,y))^p(\mathfrak{L}^{[q]}(x,y))^q \\ \le\ & p\left[(\mathfrak{L}^{[p+q]}(x,y))^{p+q} - A(x,y)(\mathfrak{L}^{[p+q-1]}(x,y))^{p+q-1}\right],\end{aligned}$$

where $\mathfrak{L}^{[p]}$ is the generalised logarithmic mean of order p of two positive numbers, defined by (see [1, p. 385]):

$$\mathfrak{L}^{[p]}(x,y) = \begin{cases} \left[\frac{1}{p+1}\left(\frac{y^{p+1}-x^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } p \neq -1, 0, \pm\infty; \\ \frac{y-x}{\log y - \log x}, & \text{if } p = -1; \\ \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}, & \text{if } p = 0; \\ \max\{x,y\}, & \text{if } p = +\infty; \\ \min\{x,y\}, & \text{if } p = -\infty, \end{cases} \tag{5.4}$$

when $x \neq y$, and $\mathfrak{L}^{[p]}(x,x) = x$.

5.1. **Čebyšev functional for convex functions.** In Corollary 3, we assume that f is a differentiable convex function. However, we can 'drop' the assumption of differentiability, and get a similar result for general convex functions, where the derivative exists almost everywhere.

Proposition 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, and $g : [a,b] \to \mathbb{R}$ be a nonnegative Lebesgue integrable function. Then,*

$$\frac{1}{2}\int_a^b \left[\left(\frac{t-a}{b-a}\right)^2 f'(a) - \left(\frac{b-t}{b-a}\right)^2 f'(b)\right] g(t)dt \leq T(f,g) \leq \frac{1}{b-a}\int_a^b \left(t - \frac{b+a}{2}\right) f'(t)g(t)dt. \tag{5.5}$$

The constants 1 and $\frac{1}{2}$ in (5.5) are sharp.

Proof. Since f is a convex function on $[a,b]$, we have the following Ostrowski type inequality for any $t \in [a,b]$ (see [5])

$$\frac{1}{2}\left[(b-t)^2 f'_+(t) - (t-a)^2 f'_-(t)\right] \leq \int_a^b f(s)ds - (b-a)f(t) \leq \frac{1}{2}\left[(b-t)^2 f'_-(b) - (t-a)^2 f'_+(a)\right] \tag{5.6}$$

We multiply the (5.6) by $g(t)$, take the integral over $[a,b]$ and multiply it by $-\frac{1}{(b-a)^2}$ to obtain

$$\begin{aligned} \frac{1}{2}\int_a^b \left[\left(\frac{t-a}{b-a}\right)^2 f'_+(a) - \left(\frac{b-t}{b-a}\right)^2 f'_-(b)\right] g(t)dt &\leq T(f,g) \\ &\leq \frac{1}{2}\int_a^b \left[\left(\frac{t-a}{b-a}\right)^2 f'_-(t) - \left(\frac{b-t}{b-a}\right)^2 f'_+(t)\right] g(t)dt \end{aligned}$$

Since f is convex, then f' exists almost everywhere and we may write $f'(t) = f'_\pm(t)$, for a.e. $t \in [a,b]$, and the details are omitted. The sharpness of the constants follows by Remark 3. □

Corollary 4. *Let $\mathbf{X}$ be a linear space and x, y be two distinct vectors in $\mathbf{X}$. Let g be a nonnegative functional on $[x,y]$ such that $\int_0^1 g[(1-t)x+ty]dt < \infty$. Then, for any convex function f defined on the segment $[x,y]$ and $t \in (0,1)$, we have*

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left[t^2(\nabla f(x))(y-x) - (1-t)^2(\nabla f(y))(y-x)\right] g[(1-t)x+ty]dt \\ &\leq \int_0^1 f[(1-t)x+ty]g[(1-t)x+ty]dt - \int_0^1 f[(1-t)x+ty]dt \int_0^1 g[(1-t)x+ty]dt \\ &\leq \int_0^1 \left(t - \frac{1}{2}\right)(\nabla f[(1-t)x+ty])(y-x)g[(1-t)x+ty]dt. \end{aligned} \tag{5.7}$$

The constants $\frac{1}{2}$ and 1 in (5.7) are sharp.

Proof. Consider the functions h, k defined on $[0,1]$ by $h(t) = f[(1-t)x+ty]$ and $k(t) = g[(1-t)x+ty]$. Since f is convex on the segment $[x,y]$, then h is also convex on $[0,1]$. Thus we may apply Proposition 2 to h and k. Note that $h'_\pm(t) = (\nabla_\pm f[(1-t)x+ty])(y-x)$, by the chain rule; and since h is convex,

$$h'(t) := h'_\pm(t) = (\nabla_\pm f[(1-t)x+ty])(y-x) =: (\nabla f[(1-t)x+ty])(y-x)$$

exists almost everywhere on $[0,1]$ (we get a similar identity for k). The proof for the sharpness follows by the particular case given later in Corollary 5. $\square$

5.2. **Application to the p-HH-norms.** Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Recall from Lemma 1 that

$$T_{p,q}(x,y) := \|(x,y)\|_{p+q-HH}^{p+q} - \|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q} \geq 0,$$

for any $x, y \in \mathbf{X}$ and $p, q \geq 1$.

Corollary 5. *Under the above notation and assumptions, we have*

$$\begin{aligned} &\frac{1}{2}p\int_0^1 \left[t^2\|x\|^{p-2}(y-x,x) - (1-t)^2\|y\|^{p-2}(y-x,y)\right]\|(1-t)x+ty\|^q dt \\ \leq\ & T_{p,q}(x,y) \leq p\int_0^1 \left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}(y-x,(1-t)x+ty)dt, \end{aligned} \tag{5.8}$$

for any $x, y \in \mathbf{X}$ whenever $p \geq 2$. If $1 \leq p < 2$, then the inequality (5.8) holds for any nonzero $x, y \in \mathbf{X}$ (here $(\cdot,\cdot) := (\cdot,\cdot)_{s(i)}$ is the superior (inferior) s.i.p. associated to the norm $\|\cdot\|$ on $\mathbf{X}$).
The constants $\frac{1}{2}$ and 1 are sharp in (5.8).

Proof. Define $f(t) = \|(1-t)x+ty\|^p$ and $g(t) = \|(1-t)x+ty\|^q$ for $t \in [0,1]$. Note that for any $x, y \in \mathbf{X}$,

$$(\nabla_{\pm}\|\cdot\|^p[(1-t)x+ty])(y-x) = p\|(1-t)x+ty\|^{p-2}(y-x,(1-t)x+ty)_{s(i)},$$

provided that $p \geq 2$; otherwise, it holds for any linearly independent x and y. Since $(\nabla\|\cdot\|^p[(1-\cdot)x+\cdot y])(y-x)$ exist a.e. on $[0,1]$, and by denoting $(\cdot,\cdot) := (\cdot,\cdot)_{s(i)}$, we have

$$(\nabla\|\cdot\|^p[(1-t)x+ty])(y-x) = p\|(1-t)x+ty\|^{p-2}(y-x,(1-t)x+ty),$$

and we obtain the similar identity for g. Therefore, by Corollary 4,

$$\begin{aligned} &\frac{1}{2}p\int_0^1 \left[t^2\|x\|^{p-2}(y-x,x) - (1-t)^2\|y\|^{p-2}(y-x,y)\right]\|(1-t)x+ty\|^q dt \\ \leq\ & T_{p,q}(x,y) \leq p\int_0^1 \left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}(y-x,(1-t)x+ty)dt, \end{aligned}$$

for any $x, y \in \mathbf{X}$ whenever $p \geq 2$; otherwise, it holds for any nonzero $x, y \in \mathbf{X}$ (by Corollary 4). The proof for the sharpness of the constants follows by a particular case which will be stated in Remark 3. $\square$

Remark 3 (Case of inner product space)**.** Let $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ be an inner product space and x, y be two distinct vectors in $\mathbf{X}$. Then, for any $p, q \geq 1$, we have

$$\begin{aligned} &\frac{1}{2}p\int_0^1 \langle y-x, t^2\|x\|^{p-2}x - (1-t)^2\|y\|^{p-2}y\rangle\|(1-t)x+ty\|^q dt \\ \leq\ & T_{p,q}(x,y) \leq p\int_0^1 \left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}\langle y-x,(1-t)x+ty\rangle dt. \end{aligned} \tag{5.9}$$

If $p = q = 1$, then

$$\begin{aligned} &\frac{1}{2}\int_0^1 \|(1-t)x+ty\|\left\langle y-x, \frac{t^2}{\|x\|}x - \frac{(1-t)^2}{\|y\|}y\right\rangle dt \\ \leq\ & \|(x,y)\|_{2-HH}^2 - \|(x,y)\|_{1-HH}^2 \leq \frac{1}{12}\|y-x\|^2. \end{aligned} \tag{5.10}$$

Note that when $\mathbf{X} = \mathbb{R}$, and $x, y > 0$ (some details are omitted),

$$\begin{aligned} \frac{1}{2}\int_0^1 ((1-t)x+ty)(y-x)\left(t^2-(1-t)^2\right)dt &= \frac{y-x}{2}\int_0^1 \left(t^2(1-t)-(1-t)^3\right)x + \left(t^3 - t(1-t)^2\right)y\ dt \\ &= \frac{y-x}{2}\left(\frac{y-x}{6}\right) = \frac{1}{12}(y-x)^2, \end{aligned}$$

and

$$\|(x,y)\|_{2-HH}^2 - \|(x,y)\|_{1-HH}^2 = \frac{y^3-x^3}{3(y-x)} - \left(\frac{y+x}{2}\right)^2 = \frac{1}{12}(y-x)^2.$$

Therefore, we obtain equality in (5.10).

Remark 4. Although the inequality that we obtain in Corollary 5 is sharp, the bounds are complicated to compute. We remark that the lower bound is not always positive, e.g., take $\mathbf{X} = \mathbb{R}$, $p = q = 1$, $x = -1, y = 1$, we have

$$\frac{1}{2}p\int_0^1 \left(t^2 |x|^{p-2}(y-x)x - (1-t)^2 |y|^{p-2}(y-x)y\right)(|(1-t)x+ty|)^q\,dt = -\frac{3}{8}.$$

In this case, the lower bound cannot be used to improve the Čebyšev inequality. We obtain coarser but simpler upper bounds for $T_{p,q}(x,y)$, as follows:

$$\begin{aligned} 0 \le T_{p,q}(x,y) &\le p\int_0^1 \left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}(y-x,(1-t)x+ty)dt, \\ &\le p\|y-x\|\int_0^1 \left|t-\frac{1}{2}\right| \|(1-t)x+ty\|^{p+q-1}dt, \\ &\le p\|y-x\| \begin{cases} \frac{1}{2}\|(x,y)\|^{p+q-1}_{(p+q-1)-HH} \\ \left(\frac{1}{2^{s'}(s'+1)}\right)^{\frac{1}{s'}} \|(x,y)\|^{p+q-1}_{(p+q-1)s}, & s>1,\ \frac{1}{s}+\frac{1}{s'}=1; \\ \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q-1}. \end{cases} \end{aligned} \tag{5.11}$$

Remark 5. Although, in general, these upper bounds are not always better than those obtained in Section 3, we remark that under certain conditions, they are better. For example, when $p \le \frac{1}{2}q$, we have

$$\frac{1}{4}p\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1} \le O_{p,q}(x,y)$$

(recall that $O_{p,q}(x,y) := \frac{1}{8}q\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1}$). Also, when $p \le 1$ and $\|y-x\| \le \max\{\|x\|,\|y\|\}$, we have

$$\frac{1}{4}p\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1} \le G_{p,q}(x,y)$$

(recall that $G_{p,q}(x,y) := \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q}$).

Open Problem 3. Are the constants in $\frac{1}{2}$, $\left(\frac{1}{2^{s'}(s'+1)}\right)^{\frac{1}{s'}}$, $\frac{1}{4}$ and (5.11) the best possible?

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