ON A RESULT OF LEVIN AND STEČKIN

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Abstract. We extend a result of Levin and Stečkin concerning an inequality analogous to Hardy's inequality.

1. Introduction

Let $p > 1$ and l^p be the Banach space of all complex sequences $\mathbf{a} = (a_n)_{n\geq 1}$. The celebrated Hardy's inequality [7, Theorem 326] asserts that for $p > 1$ and any $\mathbf{a} \in l^p$,

$$\sum_{n=1}^{\infty} \Big|\frac{1}{n}\sum_{k=1}^{n} a_k\Big|^p \leq \Big(\frac{p}{p-1}\Big)^p \sum_{k=1}^{\infty} |a_k|^p. \tag{1.1}$$

As an analogue of Hardy's inequality, Theorem 345 of [7] asserts that the following inequality holds for $0 < p < 1$ and $a_n \geq 0$ with $c_p = p^p$:

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n}\sum_{k=n}^{\infty} a_k\Big)^p \geq c_p \sum_{n=1}^{\infty} a_n^p. \tag{1.2}$$

It is noted in [7] that the constant $c_p = p^p$ may not be best possible and a better constant was indeed obtained by Levin and Stečkin [9, Theorem 61]. Their result is more general as they proved, among other things, the following inequality ([9, Theorem 62]), valid for $0 < r \leq p \leq 1/3$ or $1/3 < p < 1, r \leq (1-p)^2/(1+p)$ with $a_n \geq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^r}\Big(\sum_{k=n}^{\infty} a_k\Big)^p \geq \Big(\frac{p}{1-r}\Big)^p \sum_{n=1}^{\infty} \frac{a_n^p}{n^{r-p}}. \tag{1.3}$$

We note here the constant $(p/(1-r))^p$ is best possible, as shown in [9] by setting $a_n = n^{-1-(1-r)/p-\epsilon}$ and letting $\epsilon \to 0^+$. This implies inequality (1.2) for $0 < p \leq 1/3$ with the best possible constant $c_p = (p/(1-p))^p$. On the other hand, it is also easy to see that inequality (1.2) fails to hold with $c_p = (p/(1-p))^p$ for $p \geq 1/2$. The point is that in these cases $p/(1-p) \geq 1$ so one can easily construct counter examples.

A simpler proof of Levin and Stečkin's result (for $0 < r = p \leq 1/3$) is given in [3]. It is also pointed out there that using a different approach, one may be able to extend their result to p slightly larger than $1/3$, an example is given for $p = 0.34$. The calculation however is more involved and therefore it is desirable to have a simpler approach. For this, we let q be the number defined by $1/p+1/q = 1$ and note that by the duality principle (see [10, Lemma 2] but note that our situation is slightly different since we have $0 < p < 1$ with an reversed inequality), the case $0 < r < 1, 0 < p < 1$ of inequality (1.3) is equivalent to the following one for $a_n > 0$:

$$\sum_{n=1}^{\infty} \Big(n^{(r-p)/p}\sum_{k=1}^{n} \frac{a_k}{k^{r/p}}\Big)^q \leq \Big(\frac{p}{1-r}\Big)^q \sum_{n=1}^{\infty} a_n^q. \tag{1.4}$$

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The above inequality can be regarded as an analogue of a result of Knopp [8], which asserts that Hardy's inequality (1.1) is still valid for $p < 0$ if we assume $a_n > 0$. We may also regard inequality (1.4) as an inequality concerning the factorable matrix with entries $n^{(r-p)/p}k^{-r/p}$ for $k \leq n$ and 0 otherwise. Here we recall that a matrix $A = (a_{nk})$ is factorable if it is a lower triangular matrix with $a_{nk} = a_n b_k$ for $1 \leq k \leq n$. We note that the approach in [5] for the l^p norms of weighted mean matrices can also be easily adopted to treat the l^p norms of factorable matrices and it is our goal in this paper to use this similar approach to extend the result of Levin and Stečkin. Our main result is

Theorem 1.1. *Inequality* (1.2) *holds with the best possible constant* $c_p = (p/(1-p))^p$ *for any* $1/3 < p < 1/2$ *satisfying*

$$2^{p/(1-p)}\Big(\Big(\frac{1-p}{p}\Big)^{1/(1-p)} - \frac{1-p}{p}\Big) - (1 + \frac{3-1/p}{2})^{1/(1-p)} \geq 0. \tag{1.5}$$

In particular, inequality (1.2) *holds for* $0 < p \leq 0.346$.

It readily follows from Theorem 1.1 and our discussions above that we have the following dual version of Theorem 1.1:

Corollary 1.1. *Inequality* (1.4) *holds with* $r = p$ *for any* $1/3 < p < 1/2$ *satisfying* (1.5) *and the constant is best possible. In particular, inequality* (1.4) *holds with* $r = p$ *for* $0 < p \leq 0.346$.

In Section 3, we shall study some inequalities which can be regarded as generalizations of (1.2). Motivations for considerations for such inequalities come both from their integral analogues as well as from their counterparts in the l^p spaces. As an example, we consider the following inequality for $0 < p < 1, 0 < \alpha < 1/p$:

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{\alpha k^{\alpha-1}}{n^{\alpha}}a_k\Big)^p \geq \Big(\frac{\alpha p}{1-\alpha p}\Big)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.6}$$

As in the case of (1.2), the above inequality doesn't hold for all $0 < p < 1, 0 < \alpha < 1/p$. In Section 3, we will however prove a result concerning the validity of (1.6) that can be regarded as an analogue to that of Levin and Stečkin's concerning the validity of (1.2).

Inequality (1.6) is motivated partially by integral analogues of (1.2), as we shall explain in Section 3. It is also motivated by the following inequality for $p > 1, \alpha p > 1$, $a_n \geq 0$:

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{\alpha k^{\alpha-1}}{n^{\alpha}}a_k\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.7}$$

The above inequality is in turn motivated by the following inequality

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{k^{\alpha-1}}{\sum_{i=1}^{n}i^{\alpha-1}}a_k\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.8}$$

Inequality (1.8) was first suggested by Bennett [2, p. 40-41], see [6] and the references therein for recent progress on this. We point out here that it is easy to see that inequality (1.7) implies (1.8) when $\alpha > 1$, hence it is interesting to know for what α's, inequality (1.7) is valid. We first note that on setting $a_1 = 1$ and $a_n = 0$ for $n \geq 2$ in (1.7) that it is impossible for it to hold when α is large for fixed p. On the other hand, when $\alpha = 1$, inequality (1.7) becomes Hardy's inequality and hence one may expect it to hold for α close to 1 and we shall establish such a result in Section 4.

2. Proof of Theorem 1.1

First we need a lemma:

Lemma 2.1. *Let $1/3 < p < 1/2$, then the following inequality holds for $0 \leq y \leq 1$ and $1/2 < t < 1$:*

$$(1+y/(2t))^{1+t} - (1+y)^{-t}(1+(2t-1)y/(2t))^{1+t} - y/t \geq 0. \tag{2.1}$$

Proof. We set $x = y/(2t)$ so that $0 \leq x \leq 1$ and we recast the above inequality as

$$f(x,t) := (1+x)^{1+t} - (1+2tx)^{-t}(1+(2t-1)x)^{1+t} - 2x \geq 0.$$

Direct calculation shows that $f(0,t) = \frac{\partial f}{\partial x}(0,t) = 0$ and

$$\frac{\partial^2 f}{\partial x^2}(x,t) = t(1+t)(1+x)^{t-1}\Big(1-(1+2tx)^{-t-2}(1+(2t-1)x)^{t-1}(1+x)^{1-t}\Big) := t(1+t)(1+x)^{t-1}g(x,t).$$

Note that

$$\frac{\partial g}{\partial x}(x,t) = (1+2tx)^{-t-3}(1+(2t-1)x)^{t-2}(1+x)^{-t}\Big(2(4t-1)+4t(4t-1)x+2t(2t-1)(t+2)x^2\Big) \geq 0.$$

As $g(0,t) = 0$, it follows that $g(x,t) \geq 0$ for $0 \leq x \leq 1$ which in turn implies the assertion of the lemma. □

We now describe a general approach towards establishing inequality (1.3) for $0 < r < 1, 0 < p < 1$. A modification from the approach in Section 3 of [3] shows that in order for (1.3) to hold for any given p with $c_{p,r}(= (p/(1-r))^p)$, it suffices to find a sequence $\mathbf{w}$ of positive terms for each $0 < r < 1$ and $0 < p < 1$, such that for any integer $n \geq 1$,

$$n^{(p-r)/(1-p)}(w_1 + \cdots + w_n)^{-1/(1-p)} \leq c_{p,r}^{-1/(1-p)}\Big(\frac{w_n^{-1/(1-p)}}{n^{r/(1-p)}} - \frac{w_{n+1}^{-1/(1-p)}}{(n+1)^{r/(1-p)}}\Big).$$

We note here that if we study the equivalent inequality (1.4) instead, then we can also obtain the above inequality from inequality (2.2) of [3], on setting $\Lambda_n = n^{-(r-p)/p}, \lambda_n = n^{-r/p}$ there. For the moment, we assume $c_{p,r}$ is an arbitrary fixed positive number and on setting $b_n^{p-1} = w_n/w_{n+1}$, we can recast the above inequality as

$$\Big(\sum_{k=1}^n\prod_{i=k}^n b_i^{p-1}\Big)^{-1/(1-p)} \leq c_{p,r}^{-1/(1-p)}n^{(r-p)/(1-p)}\Big(\frac{b_n}{n^{r/(1-p)}} - \frac{1}{(n+1)^{r/(1-p)}}\Big).$$

The choice of b_n in Section 3 of [3] suggests that for optimal choices of the b_n's, we may have asymptotically $b_n \sim 1 + c/n$ as $n \to +\infty$ for some positive constant c (depending on p). This observation implies that $n^{1/(1-p)}$ times the right-hand side expression above should be asymptotically a constant. To take the advantage of possible contributions of higher order terms, we now further recast the above inequality as

$$\Big(\frac{1}{n+a}\sum_{k=1}^n\prod_{i=k}^n b_i^{p-1}\Big)^{-1/(1-p)} \leq c_{p,r}^{-1/(1-p)}n^{(r-p)/(1-p)}(n+a)^{1/(1-p)}\Big(\frac{b_n}{n^{r/(1-p)}} - \frac{1}{(n+1)^{r/(1-p)}}\Big), \tag{2.2}$$

where a is a constant (may depend on p) to be chosen later. It will also be clear from our arguments below that the choice of a will not affect the asymptotic behavior of b_n to the first order of magnitude. We now choose b_n so that

$$n^{(r-p)/(1-p)}(n+a)^{1/(1-p)}\Big(\frac{b_n}{n^{r/(1-p)}} - \frac{1}{(n+1)^{r/(1-p)}}\Big) = c_{p,r}^{-\alpha/(1-p)}, \tag{2.3}$$

where α is a parameter to be chosen later. This implies that

$$b_n = c_{p,r}^{-\alpha/(1-p)}\frac{n^{p/(1-p)}}{(n+a)^{1/(1-p)}} + \frac{n^{r/(1-p)}}{(n+1)^{r/(1-p)}}.$$

For the so chosen b_n's, inequality (2.2) becomes

$$\sum_{k=1}^{n}\prod_{i=k}^{n} b_i^{p-1} \geq (n+a)c_{p,r}^{1+\alpha}. \tag{2.4}$$

We first assume the above inequality holds for $n=1$. Then induction shows it holds for all n as long as

$$b_n^{1-p} \leq \frac{n+a+c_{p,r}^{-(1+\alpha)}-1}{n+a}.$$

Taking account into the value of b_n, the above becomes (for $0 \leq y \leq 1$ with $y=1/n$)

$$(1+(a+c_{p,r}^{-(1+\alpha)}-1)y)^{1/(1-p)}-(1+y)^{-r/(1-p)}(1+ay)^{1/(1-p)}-c_{p,r}^{-\alpha/(1-p)}y \geq 0. \tag{2.5}$$

The first order term of the Taylor expansion of the left-hand side expression above implies that it is necessary to have

$$c_{p,r}^{-(1+\alpha)}-(1-p)c_{p,r}^{-\alpha/(1-p)}+r-1 \geq 0.$$

For fixed $c_{p,r}$, the left-hand side expression above is maximized when $\alpha = 1/p-1$ with value $pc_{p,r}^{-1/p}+r-1$. This suggests us to take $c_{p,r}=(p/(1-r))^p$. From now on we fix $c_{p,r}=(p/(1-r))^p$ and note that in this case (2.5) becomes

$$(1+(a+(1-r)/p-1)y)^{1/(1-p)}-(1+y)^{-r/(1-p)}(1+ay)^{1/(1-p)}-\frac{1-r}{p}y \geq 0. \tag{2.6}$$

We note that the choice of $a=0$ in (2.6) with $r=p$ reduces to that considered in Section 3 of [3] (in which case the case $n=1$ of (2.4) is also included in (2.6)). Moreover, with $a=0$ in the above inequality and following the treatment in Section 3 of [3], one is able to improve some cases of the above mentioned result of Levin and Stečkin concerning the validity of (1.3). We shall postpone the discussion of this to the next section and focus now on the proof of Theorem 1.1. Since the cases $0<p\leq 1/3$ of the assertion of the theorem are known, we may assume $1/3<p<1/2$ from now on. In this case we set $r=p$ in (2.6) and Taylor expansion shows that it is necessary to have $a \geq (3-1/p)/2$ in order for inequality (2.6) to hold. We now take $a=(3-1/p)/2$ and write $t=p/(1-p)$ to see that inequality (2.6) is reduced to (2.1) and Lemma 2.1 now implies that inequality (2.6) holds in this case. Inequality (1.2) with the best possible constant $c_p=(p/(1-p))^p$ thus follows for any $1/3<p<1/2$ as long as the case $n=1$ of (2.4) is satisfied, which is just inequality (1.5) and this proves the first assertion of Theorem 1.1. For the second assertion, we note that inequality (1.5) can be rewritten as

$$\frac{2^t}{t}(t^{-t}-1) \geq (1+a)^{1/(1-p)}, \tag{2.7}$$

where t is defined as above. Note that $1/2<t<1$ for $1/3<p<1/2$ and both $2^t/t$ and $t^{-t}-1$ are decreasing functions of t. It follows that the left-hand side expression of (2.7) is a decreasing function of p. Note also that for fixed a, the right-hand side expression of (2.7) is an increasing function of $p<1$. As $a=(3-1/p)/2$ in our case, it follows that one just need to check the above inequality for $p=0.346$ and the assertion of the theorem now follows easily.

We remark here that in the proof of Theorem 1.1, if instead of choosing b_n to satisfy (2.3) (with $r=p$ and $c_{p,p}=(p/(1-p))^p$ there), we choose b_n for $n \geq 2$ so that

$$(n+c)^{1/(1-p)}\Big(\frac{1}{n^{p/(1-p)}}-\frac{1}{(n+1)^{p/(1-p)}b_n}\Big)=(1-p)/p.$$

Moreover, note that we can also rewrite (2.2) for $n \geq 2$ as (with a replaced by c and $r = p$, $c_{p,p} = (p/(1-p))^p$)

$$\Big(\frac{1}{n+c}\Big(\sum_{k=1}^{n-1}\prod_{i=k}^{n-1} b_i^{p-1}+1\Big)\Big)^{-1/(1-p)} \leq \Big(\frac{1-p}{p}\Big)^{p/(1-p)}(n+c)^{1/(1-p)}\Big(\frac{1}{n^{p/(1-p)}}-\frac{1}{(n+1)^{p/(1-p)}b_n}\Big).$$

If we further choose b_1 so that

$$1=\Big(\frac{1-p}{p}\Big)^{p/(1-p)}\Big(1-\frac{1}{2^{p/(1-p)}b_1}\Big).$$

Then repeating the same process as in the proof of Theorem 1.1, we find that the induction part (with $c = (1/p-1)/2$ here) leads back to inequality (2.6) (with $r = p$ and $a = (3-1/p)/2$ there) while the initial case (corresponding to $n = 2$ here) is just (2.7), so this approach gives another proof of Theorem 1.1.

We end this section by pointing out the relation between the treatment in Sections 3 and 4 in [3] on inequality (1.2). We note that it is shown in Section 3 of [3] that for any $N \geq 1$ and any positive sequence $\mathbf{w}$, we have

$$\sum_{n=1}^{N} a_n^p \leq \sum_{n=1}^{N} w_n\Big(\sum_{k=n}^{N}\Big(\sum_{i=1}^{k} w_i\Big)^{-1/(1-p)}\Big)^{1-p}\Big(\sum_{k=n}^{N} a_k\Big)^p. \tag{2.8}$$

We now use $w_n = \sum_{k=1}^{n} w_k - \sum_{k=1}^{n-1} w_k$ and set (with $\nu_N = 0$)

$$\nu_n = \frac{\sum_{k=n+1}^{N}\Big(\sum_{i=1}^{k} w_i\Big)^{1/(p-1)}}{\Big(\sum_{i=1}^{n} w_i\Big)^{1/(p-1)}}$$

to see that inequality (2.8) leads to (with $\nu_0 = 0$)

$$\sum_{n=1}^{N} a_n^p \leq \sum_{n=1}^{N}\Big((1+\nu_n)^{1-p}-\nu_{n-1}^{1-p}\Big)\Big(\sum_{k=n}^{N} a_k\Big)^p.$$

The above inequality is essentially what's used in Section 4 of [3].

3. A generalization of Theorem 1.1

For $0 < p < 1$, let $f(x) \geq 0$ and α be a real number such that $\alpha < 1/p$, we have the following identity:

$$\int_0^\infty \Big(\frac{1}{x^\alpha}\int_x^\infty f(t)t^{\alpha-1}dt\Big)^p dx = \Big(\frac{p}{1-\alpha p}\Big)\int_0^\infty \Big(\frac{1}{x^\alpha}\int_x^\infty f(t)t^{\alpha-1}dt\Big)^{p-1} f(x)dx. \tag{3.1}$$

In the above expression, we assume f is taken so that all the integrals converge. The case of $\alpha = 1$ is given in the proof of Theorem 337 of [7] and the general case is obtained by some changes of variables. As in the proof of Theorem 337 of [7], we then deduce the following inequality (with the same assumptions as above):

$$\int_0^\infty \Big(\frac{1}{x^\alpha}\int_x^\infty f(t)t^{\alpha-1}dt\Big)^p dx \geq \Big(\frac{p}{1-\alpha p}\Big)^p\int_0^\infty f^p(x)dx.$$

The above inequality can also be deduced from Theorem 347 of [7] (see also [4, (2.4)]). Following the way how Theorem 338 is deduced from Theorem 337 of [7], we deduce similarly from (3.1) the following inequality for $0 < p < 1, 0 < \alpha < 1/p$ and $a_n \geq 0$:

$$\sum_{n=1}^{\infty}{}' \Big(\frac{1}{n^\alpha}\sum_{k=n}^{\infty}\Big((k+1)^\alpha-k^\alpha\Big)a_k\Big)^p \geq \Big(\frac{\alpha p}{1-\alpha p}\Big)\sum_{n=1}^{\infty}\Big(\frac{1}{n^\alpha}\sum_{k=n}^{\infty}\Big((k+1)^\alpha-k^\alpha\Big)a_k\Big)^{p-1}a_n.$$

The dash over the summation on the left-hand side expression above (and in what follows) means that the term for which $n=1$ is to be multiplied by $1+1/(1-\alpha p)$ and the constant is best possible (on taking $a_n = n^{-1/p-\epsilon}$ and letting $\epsilon \to 0^+$). The above inequality readily implies the following one by Hölder's inequality:

$$\sum_{n=1}^{\infty}{}' \Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}\Big((k+1)^{\alpha}-k^{\alpha}\Big)a_k\Big)^p \geq \Big(\frac{\alpha p}{1-\alpha p}\Big)^p \sum_{n=1}^{\infty} a_n^p.$$

We are thus motivated to consider the above inequality with the dash sign removed and this can be regarded as an analogue of inequality (1.2) with $c_p = (p/(1-p))^p$, which corresponds to the case $\alpha = 1$ here. As in the case of (1.2), such an inequality does not hold for all α and p satisfying $0 < p < 1$ and $0 < \alpha < 1/p$. However, on setting $a_n = n^{-1/p-\epsilon}$ and letting $\epsilon \to 0^+$, one sees easily that if such an inequality holds for certain α and p, then the constant is best possible. More generally, we can consider the following inequality:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\sum_{i=1}^n L_{\beta}^{\alpha-1}(i,i-1)}\sum_{k=n}^{\infty}L_{\beta}^{\alpha-1}(k\pm 1,k)a_k\Big)^p \geq \Big(\frac{\alpha p}{1-\alpha p}\Big)^p \sum_{n=1}^{\infty} a_n^p, \tag{3.2}$$

where the function $L_r(a,b)$ for $a>0, b>0, a\neq b$ and $r \neq 0,1$ (the only case we shall concern here) is defined as $L_r^{r-1}(a,b) = (a^r-b^r)/(r(a-b))$. It is known [1, Lemma 2.1] that the function $r \mapsto L_r(a,b)$ is strictly increasing on $\mathbb{R}$. Here we restrict our attention to the plus sign in (3.2) for the case $\beta > 0, \max(1,\beta) \leq \alpha$ and to the minus sign in (3.2) for the case $0 < \alpha < 1$ and $\beta \geq \alpha$. Our remark above implies that in either case (note that $L_{\beta}(1,0)$ is meaningful)

$$\sum_{i=1}^{n} L_{\beta}^{\alpha-1}(i,i-1) \leq \sum_{i=1}^{n} L_{\alpha}^{\alpha-1}(i,i-1) = n^{\alpha}/\alpha.$$

As we also have $L_{\beta}^{\alpha-1}(k\pm 1,k) \geq k^{\alpha-1}$, we see that the validity of (3.2) follows from that of (1.6). We therefore focus on (1.6) from now on and we proceed as in Section 3 of [3] to see that in order for inequality (1.6) to hold, it suffices to find a sequence $\mathbf{w}$ of positive terms for each $0<p<1$, such that for any integer $n \geq 1$,

$$\Big(\sum_{k=1}^{n} w_k\Big)^{1/(p-1)} \leq \Big(\frac{\alpha p}{1-\alpha p}\Big)^{p/(p-1)}\Big(\alpha n^{\alpha-1}\Big)^{p/(1-p)}\Big(\frac{w_n^{1/(p-1)}}{n^{\alpha p/(1-p)}} - \frac{w_{n+1}^{1/(p-1)}}{(n+1)^{\alpha p/(1-p)}}\Big). \tag{3.3}$$

We now choose $\mathbf{w}$ inductively by setting $w_1 = 1$ and for $n \geq 1$,

$$w_{n+1} = \frac{n+1/p-\alpha-1}{n}w_n.$$

The above relation implies that

$$\sum_{k=1}^{n} w_k = \frac{n+1/p-\alpha-1}{1/p-\alpha}w_n.$$

We now assume $0 < p < 1/2$ and note that for the so chosen $\mathbf{w}$, inequality (3.3) follows (with $x = 1/n$) from $f(x) \geq 0$ for $0 \leq x \leq 1$, where

$$f(x) = \Big(1+(1/p-\alpha-1)x\Big)^{1/(1-p)} - (1+x)^{-\alpha p/(1-p)} - \frac{1-\alpha p}{p}x. \tag{3.4}$$

As $f(0) = f'(0) = 0$, it suffices to show $f''(x) \geq 0$, which is equivalent to showing $g(x) \geq 0$ where

$$g(x) = \Big(\frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)}\Big)^{(1-p)/(1-2p)}(1+x)^{(2+(\alpha-2)p)/(1-2p)} - (1+(1/p-\alpha-1)x).$$

Now
(3.5)

$$g'(x) = \Big(\frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)}\Big)^{(1-p)/(1-2p)}\Big(\frac{2+(\alpha-2)p}{1-2p}\Big)(1+x)^{(2+(\alpha-2)p)/(1-2p)-1}-(1/p-\alpha-1)$$
$$\geq \Big(\frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)}\Big)^{(1-p)/(1-2p)}\Big(\frac{2+(\alpha-2)p}{1-2p}\Big)-(1/p-\alpha-1) := h(\alpha,p).$$

Suppose now $\alpha \geq 1$, then when $1/p \geq (\alpha+2)(\alpha+1)/2$, we have $1/p \geq \alpha(\alpha-1)p+2\alpha+1$ since $p<1/2$ so that both inequalities $1/p-\alpha-1 \geq 1$ and $1/p-\alpha-1 \geq \alpha((\alpha-1)p+1)$ are satisfied. In this case we have

$$h(\alpha,p) \geq \Big(\frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)}\Big)^{(1-p)/(1-2p)}-(1/p-\alpha-1) \geq \frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)}-(1/p-\alpha-1) \geq 0.$$

It follows $g'(x) \geq 0$ and as $g(0) \geq 0$, we conclude that $g(x) \geq 0$ and hence $f(x) \geq 0$. Similar discussion leads to the same conclusion for $0<\alpha<1$ when $p \leq 1/(\alpha+2)$. We now summarize our discussions above in the following

Theorem 3.1. *Let $0<p<1/2$ and $0<\alpha<1/p$. Let $h(\alpha,p)$ be defined as in (3.5). Inequality (1.6) holds for α,p satisfying $h(\alpha,p) \geq 0$. In particular, when $\alpha \geq 1$, inequality (1.6) holds for $0<p \leq 2/((\alpha+2)(\alpha+1))$. When $0<\alpha \leq 1$, inequality (1.6) holds for $0<p\leq 1/(\alpha+2)$.*

Corollary 3.1. *Let $0<p<1/2$ and $0<\alpha<1/p$. Let $h(\alpha,p)$ be defined as in (3.5). When $\beta > 0, \max(1,\beta) \leq \alpha$, inequality (3.2) holds (where we take the plus sign) for α,p satisfying $h(\alpha,p)\geq 0$. In particular, inequality (3.2) holds for $0<p\leq 2/((\alpha+2)(\alpha+1))$. When $0<\alpha<1, \beta \geq \alpha$, inequality (3.2) holds (where we take the minus sign) for α,p satisfying $h(\alpha,p)\geq 0$. In particular, inequality (3.2) holds for $0<p\leq 1/(\alpha+2)$.*

We note here a special case of the above corollary, the case $0<\alpha<1$ and $\beta \to +\infty$ leads to the following inequality, valid for $0<p\leq 1/(\alpha+2)$:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\sum_{i=1}^n i^{\alpha-1}}\sum_{k=n}^{\infty}k^{\alpha-1}a_k\Big)^p \geq \Big(\frac{\alpha p}{1-\alpha p}\Big)^p\sum_{n=1}^{\infty}a_n^p.$$

We further note here if we set $r=\alpha p$ and $a=0$ in inequality (2.6), then it is reduced to $f(x)\geq 0$ for $f(x)$ defined as in (3.4). Since the case $0<r<p\leq 1/3$ is known, we need only concern the case $\alpha \geq 1$ here and we now have the following improvement of the result of Levin and Stečkin [9, Theorem 62]:

Corollary 3.2. *Let $0<p<1/2$ and $1\leq \alpha<1/p$. Let $h(\alpha,p)$ be defined as in (3.5). Inequality (1.3) holds for $r=\alpha p$ for α,p satisfying $h(\alpha,p)\geq 0$. In particular, inequality (1.3) holds for $r=\alpha p$ for α,p satisfying $0<p\leq 2/((\alpha+2)(\alpha+1))$.*

Just as Theorem 1.1 and Corollary 1.1 are dual versions to each other, our results above can also be stated in terms of their dual versions and we shall leave the formulation of the corresponding ones to the reader.

4. Some results on l^p norms of factorable matrices

In this section we first state some results concerning the l^p norms of factorable matrices. In order to compare our result to that of weighted mean matrices, we consider the following type of inequalities:

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_n}a_k\Big)^p \leq U_p\sum_{n=1}^{\infty}a_n^p, \tag{4.1}$$

where $p>1, U_p$ is a constant depending on p. Here we assume the two positive sequences (λ_n) and (Λ_n) are independent (in particular, unlike in the weighted mean matrices case, we do not have $\Lambda_n=\sum_{k=1}^n\lambda_k$ in general). We begin with the following result concerning the bound for U_p:

Theorem 4.1. *Let $1<p<\infty$ be fixed in (4.1). Let a be a constant such that $\Lambda_n+a\lambda_n>0$ for all $n\geq 1$. Let $0<L<p$ be a positive constant and let*

$$b_n=(\frac{p-L}{p})(1+a\frac{\lambda_n}{\Lambda_n})^{p-1}\frac{\lambda_n}{\Lambda_n}+\frac{\lambda_n}{\lambda_{n+1}}.$$

If for any integer $n\geq 1$, we have

$$\sum_{k=1}^n\lambda_k\prod_{i=k}^n b_i^{1/(p-1)}\leq\frac{p}{p-L}(\Lambda_n+a\lambda_n),$$

then inequality (4.1) holds with $U_p\leq(p/(p-L))^p$.

We point out that the proof of the above theorem is analogue to that of Theorem 3.1 of [5], except instead of choosing b_n to satisfy the equation (3.4) in [5], we choose b_n so that

$$\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Lambda_n^p=(\frac{p-L}{p})(\Lambda_n+a\lambda_n)^{p-1}.$$

We shall leave the details to the reader and we point out that as in the case of weighted mean matrices in [5], we deduce from Theorem 4.1 the following

Corollary 4.1. *Let $1<p<\infty$ be fixed in (4.1). Let a be a constant such that $\Lambda_n+a\lambda_n>0$ for all $n\geq 1$. Let $0<L<p$ be a positive constant such that the following inequality is satisfied for all $n\geq 1$ (with $\Lambda_0=\lambda_0=0$):*

$$(\frac{p-L}{p})(1+a\frac{\lambda_n}{\Lambda_n})^{p-1}+\frac{\Lambda_n}{\lambda_{n+1}}\leq\frac{\Lambda_n}{\lambda_n}(1+a\frac{\lambda_n}{\Lambda_n})^{p-1}((1-\frac{L}{p})\frac{\lambda_n}{\Lambda_n}+\frac{\Lambda_{n-1}}{\Lambda_n}+a\frac{\lambda_{n-1}}{\Lambda_n})^{1-p}.$$

Then inequality (4.1) holds with $U_p\leq(p/(p-L))^p$.

We now apply the above corollary to the special case of (4.1) with $\lambda_n=\alpha n^{\alpha-1}, \Lambda_n=n^\alpha$ for some $\alpha>1$. On taking $L=1/\alpha$ and $a=0$ in Corollary 4.1 and setting $y=1/n$, we see that inequality (1.7) holds as long as we can show for $0\leq y\leq 1$,

$$\Big((1-\frac{1}{p\alpha})\alpha y+(1-y)^\alpha\Big)^{p-1}\Big((1-\frac{1}{p\alpha})\alpha y+(1+y)^{1-\alpha}\Big)\leq 1. \tag{4.2}$$

We note first that as $(1-\frac{1}{p\alpha})\alpha y+(1-y)^\alpha\leq(1-\frac{1}{p\alpha})\alpha y+(1+y)^{1-\alpha}$, we need to have $(1-\frac{1}{p\alpha})\alpha y+(1-y)^\alpha\leq 1$ in order for the above inequality to hold. Taking $y=1$ shows that it is necessary to have $\alpha\leq 1+1/p$. In particular, we may assume $1<\alpha\leq 2$ from now on and it then follows from Taylor expansion that in order for (4.2) to hold, it suffices to show

$$\Big(1-\frac{1}{p}y+\frac{\alpha(\alpha-1)}{2}y^2\Big)^{p-1}\Big(1+(1-\frac{1}{p})y+\frac{\alpha(\alpha-1)}{2}y^2\Big)\leq 1. \tag{4.3}$$

We first assume $1<p\leq 2$ and in this case we use

$$\Big(1-\frac{1}{p}y+\frac{\alpha(\alpha-1)}{2}y^2\Big)^{p-1}\leq 1+(p-1)(-\frac{1}{p}y+\frac{\alpha(\alpha-1)}{2}y^2)$$

to see that (4.3) follows from

$$h_{1,\alpha,p}(y):=\frac{\alpha(\alpha-1)p}{2}-(1-1/p)^2+\frac{\alpha(\alpha-1)(p-1)}{2p}(p-2)y+\frac{\alpha^2(\alpha-1)^2}{4}(p-1)y^2\leq 0.$$

We now denote $\alpha_1(p)>1$ as the unique number satisfying $h_{1,\alpha_1,p}(0)=0$ and $\alpha_2(p)>1$ the unique number satisfying $h_{1,\alpha_2,p}(1)=0$ and let $\alpha_0(p)=\min(\alpha_1(p),\alpha_2(p))$. It is easy to see that both $\alpha_1(p)$ and $\alpha_2(p)$ are $\leq 1+1/p$ and that for $1<\alpha\leq\alpha_0$, we have $h_{1,\alpha,p}(y)\leq 0$ for $0\leq y\leq 1$.

Now suppose that $p > 2$, we recast (4.3) as

$$1+(1-\frac{1}{p})y+\frac{\alpha(\alpha-1)}{2}y^2 \leq \Big(1-\frac{1}{p}y+\frac{\alpha(\alpha-1)}{2}y^2\Big)^{1-p}. \tag{4.4}$$

In order for the above inequality to hold for all $0 \leq y \leq 1$, we must have $\alpha(\alpha-1)y^2/2 \leq y/p$. Therefore, we may from now on assume $\alpha(\alpha-1) \leq 2/p$. Applying Taylor expansion again, we see that (4.4) follows from the following inequality:

$$1+(1-\frac{1}{p})y+\frac{\alpha(\alpha-1)}{2}y^2 \leq 1+(1-p)(-\frac{1}{p}y+\frac{\alpha(\alpha-1)}{2}y^2)+p(p-1)(-\frac{1}{p}y+\frac{\alpha(\alpha-1)}{2}y^2)^2/2.$$

We can recast the above inequality as

$$h_{2,\alpha,p}(y) := \frac{\alpha(\alpha-1)p}{2}-\frac{1-1/p}{2}+\frac{(p-1)\alpha(\alpha-1)y}{2}-\frac{p(p-1)\alpha^2(\alpha-1)^2y^2}{8} \leq 0.$$

We now denote $\alpha_0(p) > 1$ as the unique number satisfying $\alpha(\alpha-1) \leq 2/p$ and $h_{2,\alpha_0,p}(1) = 0$. It is easy to see that for $1 < \alpha \leq \alpha_0$, we have $h_{2,\alpha,p}(y) \leq 0$ for $0 \leq y \leq 1$. We now summarize our result in the following

Theorem 4.2. *Let $p > 1$ be fixed and let $\alpha_0(p)$ be defined as above, then inequality (1.7) holds for $1 < \alpha \leq \alpha_0(p)$.*

As we have explained in Section 1, the study of (1.7) is motivated by (1.8). As (1.7) implies (1.8) and the constant $(\alpha p/(\alpha p-1))^p$ there is best possible (see [6]), we see that the constant $(\alpha p/(\alpha p-1))^p$ in (1.7) is also best possible. More generally, we note that inequality (4.7) in [6] proposes to determine the best possible constant $U_p(\alpha,\beta)$ in the following inequality ($\mathbf{a} \in l^p, p > 1, \beta \geq \alpha \geq 1$):

$$\sum_{n=1}^{\infty}\Big|\frac{1}{\sum_{k=1}^{n}L_{\beta}^{\alpha-1}(k,k-1)}\sum_{i=1}^{n}L_{\beta}^{\alpha-1}(i,i-1)a_i\Big|^p \leq U_p(\alpha,\beta)\sum_{n=1}^{\infty}|a_n|^p. \tag{4.5}$$

We easily deduce from Theorem 4.2 the following

Corollary 4.2. *Keep the notations in the statement of Theorem 4.2. For fixed $p > 1$ and $1 < \alpha \leq \alpha_0(p)$, inequality (4.5) holds with $U_p(\alpha,\beta) = (\alpha p/(\alpha p-1))^p$ for any $\beta \geq \alpha$.*

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