On Trigonometrical Proofs of the Steiner-Lehmus Theorem

Róbert Oláh-Gál and József Sándor

Dedicated to the memory of Professor Ferenc Radó (1921-1990)

Abstract

We offer a survey of some less known or new trigonometrical proofs of the Steiner-Lehmus theorem. A new proof of a recent refined variant is pointed out, too.

1 Introduction

The famous Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. For a recent survey of the Steiner-Lehmus theorem, see M. Hajja [6]. From the References in [6] one can find many methods of proof of this theorem, including pure geometrical, trigonometrical, etc. proofs. One aim of this note is also to add some new references; to call the attention to some little or not-known proofs, especially trigonometrical ones. On the other hand, we will obtain also a new trigonometric proof of a refined version of the Steiner-Lehmus theorem, published recently [7].

For the first aim, we want to point out some classical geometrical proofs published in 1967 by A. Froda [4], attributed to W.T Williams and G.T. Savage. Another interesting proof by A. Froda appears in his book [5] (see also the book of the second author [12]). Another pure-geometrical proof was published in 1973 by M. K. Sathya Narayama [13]. Other papers are by K. Seydel and C. Newman [14], or the more recents by D. Beran [1] or D. Rüthing [10]. None of the recent extensive surveys connected with the Steiner-Lehmus theorem mentions the use of complex numbers in the proof. Such a method appears in the paper by C.I. Lubin [8] from 1959. Related to the question, first posed by Sylvester (and mentioned in [6], too) whether there is a direct proof of the Steiner-Lehmus theorem, recently J. Conway (see [2]) has given an intriguing argument that there is no such proof. However, there are discussions on the valubility of this proof, as perhaps we should formulate in a completely precise manner this proposition: "the Steiner-Lehmus theorem has no direct proof" (by using e.g., concept of intuitionistic logic)

2 Trigonometric proofs of the Steiner-Lehmus theorem

Perhaps one of the shortest trigonometric proofs of the Steiner-Lehmus theorem one can find in a forgotten paper (written in Romanian) from 1916 by V. Cristescu [3]. Let BB' and CC' denote two angle bisectors of the triangle ABC (see fig. 1).

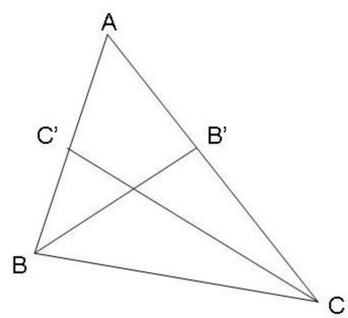

Figure 1:

By using the sinus theorem in triangle $BB'C$, one gets $\dfrac{BB'}{\sin C} = \dfrac{BC}{\sin\left(C + \dfrac{B}{2}\right)}$.

As $C + \dfrac{B}{2} = C + \dfrac{180° - C - A}{2} = 90° - \dfrac{A - C}{2}$, one has

$$BB' = a \cdot \frac{\sin C}{\cos \dfrac{A - C}{2}}. \tag{1}$$

One can obtain in a similar manner the relation

$$CC' = a \cdot \frac{\sin B}{\cos \dfrac{A - B}{2}}. \tag{2}$$

Assuming $BB' = CC'$, and remarking that $\sin C = 2 \sin \dfrac{C}{2} \cos \dfrac{C}{2}$, and $\sin \dfrac{C}{2} = \cos \dfrac{A + B}{2}$, $\sin \dfrac{B}{2} = \cos \dfrac{A + C}{2}$, we get the equality

$$\cos \frac{C}{2} \cdot \cos \frac{A + B}{2} \cos \frac{A - B}{2} = \cos \frac{B}{2} \cos \frac{A + C}{2} \cos \frac{A - C}{2}. \tag{3}$$

Now from the identity

$$\cos(x + y) \cdot \cos(x - y) = \cos^2 x + \cos^2 y - 1, \tag{4}$$

relation (4) becomes

$$\cos\frac{C}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-1\right)=\cos\frac{B}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{C}{2}-1\right). \qquad (5)$$

A simple remark shows that (5) can be rewritten also as

$$\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\cdot\left(\sin^2\frac{A}{2}+\cos\frac{B}{2}\cos\frac{C}{2}\right)=0. \qquad (6)$$

As the second paranthesis of (6) is strictly positive, this implies $\cos\frac{B}{2}-\cos\frac{C}{2}=0$, so $B=C$.

In 2000, resp. 2001 the German mathematicians D. Plachky [9], and D. Rüthing [11] have given other direct trigonometric proofs of the Steiner-Lehmus theorem, based on area considerations.

We will present here shortly the method by D. Plachky [9]. Denote the angles from B and C resp. by β and γ, and the angle bisectors BB' and AA' by w_b and w_a (see fig.2).

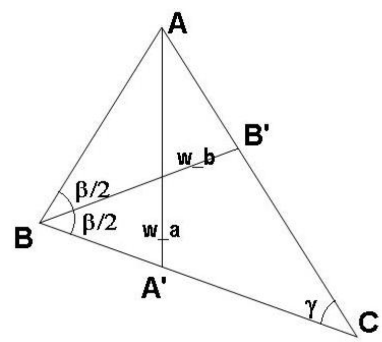

Figure 2:

By using the trigonometric form of the area of a triangle ABC as $\frac{1}{2}ab\sin\gamma$, and decomposing the initial triangle in two triangles, we get

$$\frac{1}{2}aw_\beta\sin\frac{\beta}{2}+\frac{1}{2}cw_\beta\sin\frac{\beta}{2}=\frac{1}{2}bw_\alpha\sin\frac{\alpha}{2}+\frac{1}{2}cw_\alpha\sin\frac{\alpha}{2}.$$

By the sinus-law we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin(\pi-(\alpha+\beta))}{c},$$

so assuming $w_\alpha=w_\beta$, we obtain

$$c\frac{\sin\alpha}{\sin(\alpha+\beta)}\sin\frac{\beta}{2}+c\sin\frac{\beta}{2}=c\frac{\sin\beta}{\sin(\alpha+\beta)}\sin\frac{\alpha}{2}+c\sin\frac{\alpha}{2},$$

or

$$\sin(\alpha+\beta)\left(\sin\frac{\alpha}{2}-\sin\frac{\beta}{2}\right)+\sin\frac{\alpha}{2}\sin\beta-\sin\alpha\sin\frac{\beta}{2}=0. \tag{7}$$

Writing $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$, etc; and using also the formulae

$$\sin u-\sin v=2\sin\frac{u-v}{2}\cos\frac{u+v}{2},\quad \cos u-\cos v=-2\sin\frac{u-v}{2}\sin\frac{u+v}{2}, \tag{8}$$

we get from (7)

$$2\sin\frac{\alpha-\beta}{4}[\sin(\alpha+\beta)\cos\frac{\alpha+\beta}{4}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\alpha+\beta}{2}]=0. \tag{9}$$

As in (9) the paranthesis is strictly positive (by $0<\frac{\alpha+\beta}{2}<\frac{\pi}{4}$, $0<\alpha+\beta<\pi$), (9) implies $\alpha=\beta$.

The following trigonometric proof (due to the authors) seems to be much simpler. Writing the area of triangle ABC in two distinct ways (using triangles ABB' and $BB'C$) we get immediately

$$w_b=\frac{2ac}{a+c}\cos\frac{\beta}{2}. \tag{10}$$

Similarly,

$$w_a=\frac{2bc}{b+c}\cos\frac{\alpha}{2}. \tag{11}$$

Suppose now that, $a>b$. Then $\alpha>\beta$, so $\frac{\alpha}{2}>\frac{\beta}{2}$. As $\frac{\alpha}{2},\frac{\beta}{2}\in(0,\frac{\pi}{2})$, one gets $\cos\frac{\alpha}{2}<\cos\frac{\beta}{2}$. Remark also that $\frac{bc}{b+c}<\frac{ac}{a+c}$ is equivalent to $b<a$. Thus (10) and (11) imply $w_a>w_b$. This is indeed a proof of the Steiner-Lehmus theorem, as supposing $w_a=w_b$ and letting $a>b$, we would obtain $w_a>w_b$, a contradiction; and if $a<b$, then $w_a<w_b$, again a contradiction.

3 A new trigonometric proof of a refined version

Recently, M. Hajja [7] proved the following stronger version of the Steiner-Lehmus theorem. Let BY and CZ be the angle bisectors and denote $BY=y$, $CZ=z$, $YC=v$, $BZ=V$ (see fig. 3).

Then

$$c>b\Rightarrow y+v>z+V. \tag{12}$$

As $V=\frac{ac}{a+b}$, $v=\frac{ab}{a+c}$, it is immediate that $c>b\Rightarrow V>v$. Thus, assuming $c>b$, on base of (12) we get $y>z$, i.e. the Steiner-Lehmus theorem (see the last proof of paragraph 2). In the proof of (12), in [7] a strong lemma by R. Breuch is applied.

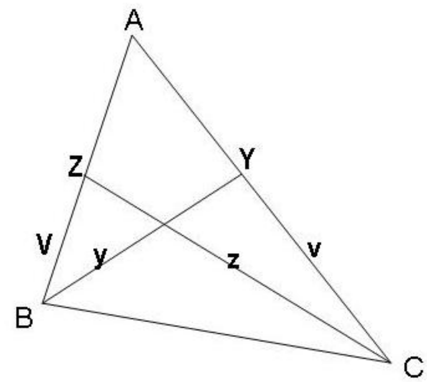

Figure 3:

Our aim here is to offer a new trigonometric proof of (12), based only on the sinus-law, and simple trigonometrical facts.

In triangle BCY one can write

$$\frac{a}{\sin\left(C+\frac{B}{2}\right)}=\frac{CY}{\sin\frac{B}{2}}=\frac{BY}{\sin C},\ \text{so}\ \frac{y+v}{\sin C+\sin\frac{B}{2}}=\frac{a}{\sin\left(C+\frac{B}{2}\right)}\ \text{implying,}$$

$$y+v=\frac{a\left(\sin C+\sin\frac{B}{2}\right)}{\sin\left(C+\frac{B}{2}\right)}. \tag{13}$$

In completely similar manner one gets

$$z+V=\frac{a\left(\sin B+\sin\frac{C}{2}\right)}{\sin\left(B+\frac{C}{2}\right)}. \tag{14}$$

Assume now that $y+v>z+V$. Applying $\sin u+\sin v=2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$ and remarking that $\cos\left(\frac{C}{2}+\frac{B}{4}\right)>0$, $\cos\left(\frac{B}{2}+\frac{C}{4}\right)>0$, after simplification, from (13)–(14) we get the inequality

$$\cos\left(\frac{C}{2}-\frac{B}{4}\right)\cos\left(\frac{B}{2}+\frac{C}{4}\right)>\cos\left(\frac{B}{2}-\frac{C}{4}\right)\cos\left(\frac{C}{2}+\frac{B}{4}\right). \tag{15}$$

Using $2\cos u\cos v=\cos\frac{u+v}{2}+\cos\frac{u-v}{2}$, this implies

$$\cos\left(\frac{3C}{4}+\frac{B}{4}\right)+\cos\left(\frac{C}{4}-\frac{3B}{4}\right)>\cos\left(\frac{3B}{4}+\frac{C}{4}\right)+\cos\left(\frac{B}{4}-\frac{3C}{4}\right),$$

or

$$\cos\left(\frac{3C}{4}+\frac{B}{4}\right)-\cos\left(\frac{3B}{4}+\frac{C}{4}\right)>\cos\left(\frac{B}{4}-\frac{3C}{4}\right)-\cos\left(\frac{C}{4}-\frac{3B}{4}\right). \quad (16)$$

Now applying the second formula of (8), we get

$$-\sin\frac{B}{2}\sin\frac{3C}{2}>-\sin\frac{C}{2}\sin\frac{3B}{2}. \quad (17)$$

By $\sin 3u = 3\sin u - 4\sin^3 u$ we get immediately from (17) that

$$-3+4\sin^2\frac{C}{2}>-3+4\sin^2\frac{B}{2}. \quad (18)$$

Remark now that the function $x \mapsto \sin^2 x$ is strictly increasing in $x \in \left(0, \frac{\pi}{2}\right)$, so as (18) gives $\sin^2\frac{C}{2} > \sin^2\frac{B}{2}$, this is possible only if

$$C > B. \quad (19)$$

This finishes the proof of (12), as if the implication in (12) would not be true, then the argument above would imply $C \leq B$, contrary to $c > b$.

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Róbert Oláh-Gál and József Sándor:
Department of Mathematics and Informatics, Babeş–Bolyai University,
Extension of Miercurea–Ciuc, Romania
Str. Topliţa Nr. 20. Miercurea–Ciuc.
E-mail addresses:
olah_gal@topnet.ro
jsandor@math.ubbcluj.ro