A RELATION OF WEAK MAJORIZATION AND ITS APPLICATIONS TO CERTAIN INEQUALITIES FOR MEANS

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ABSTRACT. A relation of weak majorization for n-dimensional real vectors is established, the result is then used to derive some inequalities involving the power mean, the arithmetic mean and the geometric mean in n variables.

1. Introduction

Over the years, the theory of majorization as a powerful tool has widely been applied to the related research areas of pure mathematics and the applied mathematics (see [1]). A good survey on the theory of majorization was given by Marshall and Olkin in [2]. Recently, the authors have given considerable attention to the applications of majorization in the field of inequalities, for details, we refer the reader to our papers [3–18].

In this paper, we shall establish a weak majorization relation for positive real numbers $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \cdots x_n \geq 1$, and discuss the Schur-convexity of the elementary symmetric function. In Section 4, the result is used to derive some inequalities involving the power mean, the arithmetic mean and the geometric mean in n variables.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ denotes n-tuple (n-dimensional real vector), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n\},$$
$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i \geq 0, i = 1, \ldots, n\},$$
$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i > 0, i = 1, \ldots, n\}.$$

Definition 1 ([1, 2])**.** Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the kth elementary symmetric function is defined as follows:

$$E_k(\boldsymbol{x}) = E_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}, \ k = 1, \ldots, n.$$

The dual form of the elementary symmetric function is defined by

$$E^*_k(\boldsymbol{x}) = E^*_k(x_1, \ldots, x_n) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{j=1}^{k} x_{i_j}, \ k = 1, \ldots, n.$$

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Definition 2 ([1, 2])**.** Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\dots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$; $\boldsymbol{x}$ is said to be weakly submajorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec_w\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\dots,n$, where $x_{[1]}\ge\cdots\ge x_{[n]}$ and $y_{[1]}\ge\cdots\ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(2) $\boldsymbol{x}\ge\boldsymbol{y}$ means $x_i\ge y_i$ for all $i=1,2,\dots,n$. Let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\ge\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\ge\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

(3) let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ be said to be a Schur-convex function on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\le\varphi(\boldsymbol{y})$. φ is said to be the Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

2. Lemmas

To prove the main results stated in Sections 3 and 4, we need the following lemmas.

Lemma 1 ([1])**.** *Let $\boldsymbol{x}\in\mathbb{R}^n_+$, $\boldsymbol{y}\in\mathbb{R}^n$ and $\delta=\sum_{i=1}^n(y_i-x_i)$. If $\boldsymbol{x}\prec_w\boldsymbol{y}$, then*

$$\Bigg(\boldsymbol{x},\underbrace{\frac{\delta}{n},\dots,\frac{\delta}{n}}_{n}\Bigg)\prec\Bigg(\boldsymbol{y},\underbrace{0,\dots,0}_{n}\Bigg).\tag{1}$$

Lemma 2 ([2])**.** *Let $\boldsymbol{x}$, $\boldsymbol{y}\in\mathbb{R}^n$. If $\boldsymbol{x}\prec_w\boldsymbol{y}$, then*

$$(\boldsymbol{x},\ x_{n+1})\prec(\boldsymbol{y},\ y_{n+1}),\tag{2}$$

where $x_{n+1}=\min\{x_1,\cdots,x_n,\ y_1,\cdots,y_n\}$, $y_{n+1}=\sum\limits_{i=1}^{n+1}x_i-\sum\limits_{i=1}^{n}y_i$.

Lemma 3 ([1])**.** *Let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$, and let $I\subset\mathbb{R}$ be an interval, $g:I\to\mathbb{R}$. Then*

(1) $\boldsymbol{x}\prec\boldsymbol{y}$ *if and only if*

$$\sum_{i=1}^n g(x_i)\le\sum_{i=1}^n g(y_i)\tag{3}$$

holds for all convex functions g;

(2) $\boldsymbol{x}\prec\boldsymbol{y}$ *if and only if the reverse inequality of* (3) *holds for all concave functions g.*

Lemma 4 ([1])**.** *Let $I\subset\mathbb{R}$, $g:I\to B$, $\varphi:B^n\to\mathbb{R}$, $\psi(\boldsymbol{x})=\varphi(g(x_1),\cdots,g(x_n))$. If g is concave on I, φ is increasing and Schur-concave on B^n, then ψ is Schur-concave on I^n.*

Lemma 5 ([1, 2])**.** *Let $\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{R}^n_+$, $1\le k\le n$, then the elementary symmetric function $E_k(\boldsymbol{x})$ and its dual version $E^*_k(\boldsymbol{x})$ are increasing and Schur-concave on $\mathbb{R}^n_+$.*

3. Main results and their proofs

Our main results are given in the Theorem 1 and Corollary 2 below.

Theorem 1. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}, n \geq 2$ *and* $\prod_{i=1}^n x_i \geq 1$. *Then*

$$\Bigg(\underbrace{1, \ldots, 1}_{n}\Bigg) \prec_w (x_1, \ldots, x_n). \tag{4}$$

Proof. We show the validity of majorization relation (4) by induction.

When $n = 2$, without loss of generality, we may assume that $x_1 \geq x_2$. From $x_1, x_2 > 0$ and $x_1 x_2 \geq 1$, it follows that $x_1 \geq 1$ and $x_1 + x_2 \geq 2\sqrt{x_1 x_2} \geq 2 = 1 + 1$. This means that $(1, 1) \prec_w (x_1, x_2)$.

We now assume that (4) holds true for $n = k$. In the following, we need to prove that (4) holds true for $n = k + 1$.

Let $\boldsymbol{x} = (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}_{++}$ and $\prod_{i=1}^{k+1} x_i \geq 1$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \ldots \geq x_{k+1} > 0$.

If $x_{k+1} > 1$, then $x_i > 1$ for $i = 1, \ldots, k + 1$. It is clear that

$$\Bigg(\underbrace{1, \ldots, 1}_{k+1}\Bigg) \prec_w (x_1, \ldots, x_{k+1}).$$

If $x_{k+1} \leq 1$, then $x_1 \geq x_2 \geq \ldots \geq x_{k-1} \geq x_k x_{k+1}$. By using the above assumption, we have

$$\Bigg(\underbrace{1, \ldots, 1}_{k}\Bigg) \prec_w (x_1, \ldots, x_{k-1}, x_k x_{k+1}).$$

It follows that

$$\sum_{i=1}^{t} x_i \geq t \text{ for } t = 1, \ldots k - 1$$

and

$$\sum_{i=1}^{k-1} x_i + x_k x_{ki+1} \geq k.$$

Thus, we have

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k-1} x_i + x_k x_{k+1} \geq k$$

and

$$\sum_{i=1}^{k+1} x_i \geq (k + 1) \sqrt[k+1]{x_1 \ldots x_{k+1}} \geq k + 1.$$

This proves that (4) holds true for $n = k + 1$, hence the proof of Theorem 1 is completed. □

Remark 1. As a direct consequence of Theorem 1, we obtain the following weak majorization relations.

Corollary 1. *Let x_1, x_2, x_3 be positive real numbers. Then*

$$(1,1,1) \prec_w \left(\frac{x_2+x_3}{x_3+x_1}, \frac{x_3+x_1}{x_1+x_2}, \frac{x_1+x_2}{x_2+x_3}\right), \tag{5}$$

$$(1,1,1) \prec_w \left(\frac{x_1}{\sqrt{x_2x_3}}, \frac{x_2}{\sqrt{x_3x_1}}, \frac{x_3}{\sqrt{x_1x_2}}\right), \tag{6}$$

$$(1,1,1) \prec_w \left(\frac{\sqrt{x_2x_3}}{x_1}, \frac{\sqrt{x_3x_1}}{x_2}, \frac{\sqrt{x_1x_2}}{x_3}\right). \tag{7}$$

Corollary 2. *Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ and $\prod_{i=1}^n x_i \geq 1$. Then*

$$\Big(\underbrace{1, \dots, 1}_{n}, \underbrace{A-1, \dots, A-1}_{n}\Big) \prec \Big(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{n}\Big), \tag{8}$$

$$\Big(\underbrace{1, \dots, 1}_{n}, a\Big) \prec (x_1, \dots, x_n, x_{n+1}), \tag{9}$$

where $A = \frac{1}{n}\sum_{i=1}^n x_i$, $a = \min\{x_1, \dots, x_n, 1\}$, $x_{n+1} = n + a - \sum_{i=1}^n x_i$.

Proof. By using Theorem 1, Lemma 1 and Lemma 2, the majorization relations (8) and (9) follow respectively. □

4. Some Applications

In this section, we show that our results can be used to establish some new inequalities for means.

As in [19], the power mean, the arithmetic mean and the geometric mean for positive numbers $x_1, x_2, \dots, x_n$ are defined respectively by

$$M_\alpha = \left(\frac{1}{n}\sum_{i=1}^n x_i^\alpha\right)^{1/\alpha}, \quad A = \frac{1}{n}\sum_{i=1}^n x_i, \quad G = \left(\prod_{i=1}^n x_i\right)^{1/n}.$$

Theorem 2. *Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ and $\prod_{i=1}^n x_i \geq 1$.*

If $\alpha \geq 1$, then

$$M_\alpha \geq \left(1 + \left(\frac{1}{n}\sum\nolimits_{i=1}^n (x_i - 1)\right)^\alpha\right)^{1/\alpha}. \tag{10}$$

If $\alpha \geq 1$ and $\sum_{i=1}^n x_i \leq n + a$, then

$$M_\alpha \geq \left(1 + \frac{a^\alpha - (n + a - \sum_{i=1}^n x_i)^\alpha}{n}\right)^{1/\alpha}, \tag{11}$$

where $a = \min\{x_1, \dots, x_n, 1\}$.

Furthermore, the inequalities (10) *and* (11) *are reversed for $0 < \alpha < 1$.*

Proof. When $\alpha \geq 1$, the function $f(x) = x^\alpha$ is convex on $(0, +\infty)$.

By using Lemma 3, we deduce from (8) and (9) that

$$\sum_{i=1}^n f(x_i) + nf(0) \geq nf(1) + nf(A-1) \tag{12}$$

and

$$\sum_{i=1}^{n} f(x_i) + f\left(n + a - \sum_{i=1}^{n} x_i\right) \geq nf(1) + f(a). \tag{13}$$

After a simple calculation, the inequalities (12) and (13) can be transformed to the inequalities (10) and (11) respectively.

When $0 < \alpha < 1$, the function $f(x) = x^\alpha$ is concave on $(0, +\infty)$. By using Lemma 3 and the majorization relations (8) and (9), we obtain the reverse inequalities of (10) and (11). Theorem 2 is proved. □

Corollary 3. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++},\ n \geq 2$.

If $\alpha \geq 1$, *then*

$$M_\alpha \geq (G^\alpha + (A-G)^\alpha)^{1/\alpha} \geq G. \tag{14}$$

If $\alpha \geq 1$ *and* $b \geq n(A-G)$, *then*

$$M_\alpha \geq \left(G^\alpha + \frac{b^\alpha - (b - n(A-G))^\alpha}{n}\right)^{1/\alpha} \geq G, \tag{15}$$

where $b = \min\{x_1, \ldots, x_n, G\}$.

Proof. For positive numbers $x_1/G,\ x_2/G,\ \ldots,\ x_n/G$, we have

$$\prod_{i=1}^{n} \frac{x_i}{G} = 1, \quad \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{G} = \frac{A}{G}, \quad \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{G}\right)^\alpha\right)^{\frac{1}{\alpha}} = \frac{M_\alpha}{G},$$

$$\min\left\{\frac{x_1}{G}, \ldots, \frac{x_n}{G}, 1\right\} = \frac{b}{G}.$$

In (10) and (11), replacing $x_1, x_2 \ldots, x_n$ by $x_1/G,\ x_2/G,\ \ldots,\ x_n/G$, respectively, we obtain

$$\frac{M_\alpha}{G} \geq \left(1 + \left(\frac{A}{G} - 1\right)^\alpha\right)^{1/\alpha} \tag{16}$$

and

$$\frac{M_\alpha}{G} \geq \left(1 + \frac{\left(\frac{b}{G}\right)^\alpha - \left(n + \frac{b}{G} - \sum_{i=1}^{n} \frac{x_i}{G}\right)^\alpha}{n}\right)^{1/\alpha}. \tag{17}$$

After a simple calculation, the inequalities (16) and (17) reduce to the inequalities (14) and (15) respectively. □

Theorem 3. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++},\ n \geq 2,\ \ 0 < \alpha \leq 1$ *and* $\prod_{i=1}^{n} x_i \geq 1$.

If $1 \leq k \leq n$, *then*

$$E_k(x^\alpha) \leq \sum_{i=0}^{k} C_n^i C_n^{k-i} (A-1)^{(k-i)\alpha}. \tag{18}$$

If $n+1 \leq k \leq 2n$, *then*

$$\prod_{l=k-n}^{n} (E_l^*(x^\alpha))^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} (l + (k-l)(A-1)^\alpha)^{C_n^l C_n^{k-l}}. \tag{19}$$

Proof. By Lemma 4 and Lemma 5, we conclude that $E_k(x^\alpha)$ and $E_k^*(x^\alpha)$ are Schur-concave on $\mathbb{R}^n_{++}$. Using the majorization relation (8) with the definition of Schur-concavity leads us to the desired inequalities (18) and (19). □

Corollary 4. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $0 < \alpha \leq 1$.

If $1 \leq k \leq n$, *then*

$$E_k(x^\alpha) \leq \sum_{i=0}^{k} C_n^i C_n^{k-i} G^{(i-k+C_n^k)\alpha} (A - G)^{(k-i)\alpha}. \tag{20}$$

If $n + 1 \leq k \leq 2n$, *then*

$$\prod_{l=k-n}^{n} (E_l^*(x^\alpha))^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} (lG^\alpha + (k-l)(A-G)^\alpha)^{C_n^l C_n^{k-l}}. \tag{21}$$

Proof. Using a substitution: $x_1 \longmapsto x_1/G, \quad x_2 \longmapsto x_2/G, \ \ldots, \ x_n \longmapsto x_n/G$ in (18) and (19), respectively, we obtain

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod\nolimits_{j=1}^{k} \left(\frac{x_{i_j}}{G}\right)^\alpha \leq \sum_{i=0}^{k} C_n^i C_n^{k-i} \left(\frac{A}{G} - 1\right)^{(k-i)\alpha} \tag{22}$$

and

$$\prod_{l=k-n}^{n} \left(E_l^*\left(\frac{x^\alpha}{G^\alpha}\right)\right)^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} \left(l + (k-l)\left(\frac{A-G}{G}\right)^\alpha\right)^{C_n^l C_n^{k-l}}. \tag{23}$$

By a simple calculation, the inequalities (22) and (23) can be simplified to the inequalities (20) and (21) respectively. □

Theorem 4. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}, n \geq 2$, $\prod_{i=1}^n x_i \geq 1$ *and* $\sum_{i=1}^n x_i \leq n + a$.

If $1 \leq k \leq n$ *and* $0 < \alpha \leq 1$, *then*

$$E_k(x^\alpha) + \left(n + a - \sum_{i=1}^{n} x_i\right)^\alpha E_{k-1}(x^\alpha) \leq C_n^k + C_n^{k-1} a^\alpha \tag{24}$$

and

$$E_k^*(x^\alpha) \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(\left(n + a - \sum_{i=1}^{n} x_i\right)^\alpha + \sum_{j=1}^{k-1} x_{i_j}^\alpha\right) \leq k^{C_n^k} (a^\alpha + k - 1)^{C_n^{k-1}}, \tag{25}$$

where $a = \min\{x_1, \ldots, x_n, 1\}$.

Proof. From Lemma 4 and Lemma 5, it is easy to find that $E_k(x^\alpha)$ and $E_k^*(x^\alpha)$ are Schur-concave on $\mathbb{R}^n_{++}$. Using the majorization relation (9) with the definition of Schur-concavity, inequalities (24) and (25) follow immediately. □

Corollary 5. *Let* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $b \geq n(A - G)$.

If $1 \leq k \leq n$ *and* $0 < \alpha \leq 1$, *then*

$$E_k(x^\alpha) + (b - n(A-G))^\alpha E_{k-1}(x^\alpha) \leq C_n^k G^{k\alpha} + C_n^{k-1} b^\alpha G^{(k-1)\alpha}, \tag{26}$$

and

$$E_k^*(x^\alpha) \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left((b - n(A-G))^\alpha + \sum_{j=1}^{k-1} x_{i_j}^\alpha\right) \leq k^{C_n^k} G^{\alpha C_n^k} (b^\alpha + (k-1)G^\alpha)^{C_n^{k-1}}, \tag{27}$$

where $b = \min\{x_1, \ldots, x_n, G\}$.

Proof. Using a substitution: $x_1 \longmapsto x_1/G, \quad x_2 \longmapsto x_2/G, \ \ldots, \ x_n \longmapsto x_n/G$ in (24) and (25), respectively, it follows that

$$\begin{aligned} & E_k\left(\left(\frac{x}{G}\right)^\alpha\right) + \left(\frac{b}{G} + n - \sum_{j=1}^{n} \frac{x_j}{G}\right)^\alpha E_{k-1}\left(\left(\frac{x}{G}\right)^\alpha\right) \\ & \leq C_n^k + C_n^{k-1}\left(\frac{b}{G}\right)^\alpha \end{aligned} \tag{28}$$

and

$$\begin{aligned} & E_k^*\left(\left(\frac{x}{G}\right)^\alpha\right) \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(\left(\frac{b}{G} + n - \sum_{i=1}^{n} \frac{x_i}{G}\right)^\alpha + \sum_{j=1}^{k-1}\left(\frac{x_{i_j}}{G}\right)^\alpha\right) \\ & \leq k^{C_n^k}\left(\left(\frac{b}{G}\right)^\alpha + k - 1\right)^{C_n^{k-1}}, \end{aligned} \tag{29}$$

which leads to the desired inequalities (26) and (27). □

Remark 2. Theorems 2,3,4 and their corollaries enable us to obtain a large number of inequalities by assigning appropriate values to the parameters α, n and k. For example, if we take $n = 3, k = 2$ in (20) and take $n = 3, k = 5$ in (21), respectively, we get the following interesting inequalities:

$$\left(x_1^\alpha x_2^\alpha + x_2^\alpha x_3^\alpha + x_3^\alpha x_1^\alpha\right)/3 \leq G^\alpha (A-G)^{2\alpha} + 3G^{2\alpha}(A-G)^\alpha + G^{3\alpha}, \tag{30}$$

$$\begin{aligned} & \left(x_1^\alpha + x_2^\alpha + x_3^\alpha\right)\sqrt[3]{\left(x_1^\alpha + x_2^\alpha\right)\left(x_2^\alpha + x_3^\alpha\right)\left(x_3^\alpha + x_1^\alpha\right)} \\ & \leq \left(2G^\alpha + 3(A-G)^\alpha\right)\left(3G^\alpha + 2(A-G)^\alpha\right), \end{aligned} \tag{31}$$

where $x_i > 0$ $(i = 1, 2, 3)$ and $0 < \alpha \leq 1$.

In particular, putting $\alpha = 1$ in (30) and (31), respectively, gives

$$\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right)/3 \leq G(A^2 + AG - G^2), \tag{32}$$

$$\left(x_1 + x_2 + x_3\right)\sqrt[3]{\left(x_1 + x_2\right)\left(x_2 + x_3\right)\left(x_3 + x_1\right)} \leq (3A - G)(G + 2A). \tag{33}$$

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